

From Continuous to Fully Discrete Carleman Estimates: A Transfer Principle for Parabolic Schemes

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Abstract

Direct approaches to discrete parabolic Carleman estimates often rely on weighted identities tailored to the discrete operator. We prove an abstract transfer theorem that derives such estimates for fully discrete backward Euler schemes with conforming spatial reconstructions from a continuous Carleman inequality together with suitable weighted approximation and compatibility properties. For a class of time-independent positive self-adjoint operators, the key construction assigns to each discrete state a shifted continuous auxiliary problem whose discrete solution is exactly the prescribed state. Weighted space-time error estimates, uniform over the full discrete load space, then transfer the inequality while preserving the residual and observation terms of the original scheme. This separates the PDE-specific Carleman analysis from the numerical verification required for each discretization. We verify the hypotheses for the heat equation discretized by consistent-mass P_1 finite elements on locally graded meshes in two and three dimensions and by standard Cartesian finite differences in any fixed dimension. For the heat equation, the transferred estimate is valid for Carleman parameters up to the scale $\min\{h^{-4/5}, (\delta t)^{-2/5}\}$. For bounded real-valued space-time potentials, the resulting estimates yield relaxed observability and approximate null controls with uniformly bounded cost and exponentially small terminal errors under independent spatial and temporal refinement; weak accumulation points of the reconstructed controls are continuous null controls.

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1 Introduction

Partial differential equations with distributed controls arise in many applications where the state is to be steered to a prescribed target by acting only on a small subregion of the domain. For parabolic equations, controllability and the cost of the control are classically characterized by observability estimates for the adjoint equation, which in turn follow from global Carleman estimates. When the problem is discretized for numerical computation, however, these properties need not survive uniformly: a scheme that is consistent and stable may still lose uniform controllability as the mesh and the time step are refined. The present paper addresses this issue for fully discrete parabolic schemes by transferring a continuous Carleman estimate directly to the discrete setting.

This introduction is organized as follows. In Subsection 1.1 we formulate the numerical control problem and state the discrete control bounds that motivate the paper. In Subsection 1.2, we recall the continuous observability and Carleman estimates from which these bounds will be derived, and explain the obstructions to a direct perturbation argument. The transfer mechanism is then illustrated on an elliptic model in Subsection 1.3, and the relations to previous work are summarized in subsection 1.4.

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1.1 The numerical control problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded convex polytope, and let $\omega \Subset \Omega$ be a nonempty open set. Fix $T > 0$, $M \geq 0$, and $b \in L^\infty((0, T) \times \Omega; \mathbb{R})$ with $\|b\|_{L^\infty} \leq M$. Consider the controlled heat equation:

$$\begin{cases} \partial_t y - \Delta y + b(t, x)y = \mathbf{1}_\omega u, & \text{in } (0, T) \times \Omega, \\ y = 0, & \text{on } (0, T) \times \partial\Omega, \\ y(0) = y_0 \in L^2(\Omega). \end{cases} \quad (1.1)$$

The null controllability result [12] provides, for every y_0 , a control $u \in L^2((0, T) \times \omega)$ such that $y(T) = 0$. The control satisfies the cost bound

$$\|u\|_{L^2((0, T) \times \omega)} \leq C \|y_0\|_{L^2(\Omega)}. \quad (1.2)$$

Here C depends only on Ω, ω, T, M . Thus the target is reached with a control cost bounded linearly in the size of the initial datum.

Given the above null controllability result for (1.1), a natural question is how to numerically construct a control that drives the system state to zero. To address this question, one can follow two approaches: either formulate a control construction for the continuous equation and then discretize it, or first discretize the state equation and then construct controls for the resulting discrete system; see [21, Section 1]. In this paper, we adopt the latter, discretize-then-control approach. A key issue, however, is that convergence of the forward scheme alone does not guarantee that the associated controls remain bounded under refinement. Therefore, the numerical control problem is to keep the control cost bounded while simultaneously driving the terminal error to zero as the spatial mesh and the time step are refined.

To make this discussion concrete, we focus on dimensions $d = 2, 3$ and let $V_h \subset H_0^1(\Omega)$ be the conforming P_1 space on a uniformly shape-regular mesh satisfying the grading condition (6.1), with maximum element diameter h . Define $A_h : V_h \rightarrow V_h$ by

$$(A_h z_h, v_h)_{L^2(\Omega)} = (\nabla z_h, \nabla v_h)_{L^2(\Omega)}, \quad z_h, v_h \in V_h.$$

For an integer $N \geq 2$, write $\delta t = T/N$, and set $b^n = (\delta t)^{-1} \int_{(n-1)\delta t}^{n\delta t} b(t, \cdot) dt$. Given $y^0 = y_{0,h} \in V_h$ and controls $u^n \in L^2(\omega)$, the backward-Euler scheme seeks $y^n \in V_h$ satisfying

$$\left(\frac{y^n - y^{n-1}}{\delta t}, v_h \right)_{L^2(\Omega)} + (A_h y^n, v_h)_{L^2(\Omega)} + (b^n y^n, v_h)_{L^2(\Omega)} = \int_\omega u^n v_h dx, \quad v_h \in V_h, \quad (1.3)$$

for $1 \leq n \leq N$.

For sufficiently small h and δt , the results in Corollary 2.3 and Theorem 7.3 provide controls satisfying

$$\delta t \sum_{n=1}^N \|u^n\|_{L^2(\omega)}^2 \leq C \|y_{0,h}\|_{L^2(\Omega)}^2. \quad (1.4)$$

Here and in the terminal estimate below, the constants may also depend on the fixed mesh regularity and grading bounds, but are independent of h , δt , and $y_{0,h}$. This is the discrete analogue of the control-cost bound in (1.2): the control norm is bounded linearly by the norm of the initial datum, uniformly under spatial and temporal refinement. Thus the control cost remains uniformly bounded.

Moreover, the same controls also satisfy the terminal error estimate

$$\|y^N\|_{L^2(\Omega)} \leq C \exp[-c \min\{h^{-4/5}, (\delta t)^{-2/5}\}] \|y_{0,h}\|_{L^2(\Omega)}. \quad (1.5)$$

Here $c > 0$ is also independent of h and δt . Thus, for a family of initial data bounded in $L^2(\Omega)$, the terminal estimate gives

$$\lim_{h, \delta t \rightarrow 0} \|y^N\|_{L^2(\Omega)} = 0, \quad N = T/\delta t,$$

with an exponentially small bound under independent spatial and temporal refinement.

As a consequence of (1.4), for approximations of a fixed initial datum, the piecewise constant control reconstructions remain bounded in $L^2((0, T) \times \omega)$ and therefore admit weakly convergent subsequences as $h, \delta t \rightarrow 0$. Together with consistency of the scheme, (1.5) ensures that every such weak limit is a null control for (1.1) (see Corollary 7.9 for more details).

1.2 From continuous to discrete Carleman estimates

To derive control bounds for the discrete problem, it is natural to work with the adjoint equation and to seek a discrete counterpart of the continuous observability inequality. Classically, the continuous inequality is deduced from a global parabolic Carleman estimate. A natural attempt is therefore to apply the continuous Carleman estimate directly to a reconstruction of the discrete state and to absorb the consistency error as a small perturbation. However, for fully discrete schemes this direct perturbation argument is not sufficient. The obstruction is twofold: the continuous estimate does not control the highest discrete modes uniformly, and consistency alone does not make the perturbation small over the full discrete space. These difficulties motivate the transfer principle developed in this paper, in which each discrete state is embedded into a shifted continuous auxiliary problem and the continuous Carleman inequality is transferred back to the original scheme through weighted numerical error estimates.

Write $Pp = -\partial_t p - \Delta p + b(t, x)p$ for the continuous adjoint operator. By duality [18], the null controllability of (1.1) and the control cost (1.2) are equivalent to an observability estimate for the backward equation

$$\begin{cases} Pp = 0, & \text{in } (0, T) \times \Omega, \\ p = 0, & \text{on } (0, T) \times \partial\Omega, \\ p(T) = p_T \in L^2(\Omega). \end{cases} \quad (1.6)$$

Specifically, every solution satisfies

$$\|p(0)\|_{L^2(\Omega)}^2 \leq C \int_0^T \|p(t)\|_{L^2(\omega)}^2 dt. \quad (1.7)$$

At the continuous level, observability is classically obtained from a global parabolic Carleman estimate [12]. Let $\alpha < 0$ and $\xi > 0$ be standard parabolic weight functions. For all sufficiently large s and all sufficiently regular functions p vanishing on $(0, T) \times \partial\Omega$, the gradient and zeroth-order terms satisfy

$$\int_0^T \int_{\Omega} e^{2s\alpha} (s\xi |\nabla p|^2 + s^3 \xi^3 |p|^2) dx dt \leq C \int_0^T \int_{\Omega} e^{2s\alpha} |Pp|^2 dx dt + Cs^3 \int_0^T \int_{\omega} e^{2s\alpha} \xi^3 |p|^2 dx dt. \quad (1.8)$$

This, together with an energy estimate, yields (1.7) and hence null controllability by duality.

These observations suggest a direct route to numerical control. A consistent discretization approximates the continuous operator on smooth functions, so one might try to apply (1.8) to a reconstruction of a discrete state and absorb the consistency error as a small perturbation. To recover (1.4) and (1.5), the error must be absorbable over the full discrete space and for a range of s growing as the mesh is refined. As already indicated, two obstructions to this approach are explained in Subsection 1.3.

To overcome these obstructions, we prove an abstract transfer theorem, Theorem 2.1, for fully discrete schemes satisfying weighted approximation and compatibility assumptions. It yields a Carleman estimate over the full discrete space and applies, in particular, to the finite-element and finite-difference schemes verified in this paper.

To illustrate this general result concretely, we return to the finite-element scheme in Subsection 1.1 and define the discrete adjoint operator by

$$(P_{h,\delta t} p)^n := \frac{p^n - p^{n+1}}{\delta t} + A_h p^n + \Pi_h(b^{n+1} p^n), \quad 0 \leq n < N,$$

where Π_h is the $L^2(\Omega)$ projection onto V_h . We use regularized weight functions with matching time endpoints and write $\alpha^n(x) = \alpha(n\delta t, x)$ and $\xi^n = \xi(n\delta t)$, with $\max_{[0,T]} \xi = 1$. The weight conditions and normalization are specified in (2.1). For sufficiently small $h, \delta t$, Theorem 2.1 and Corollary 2.3 give, for every sequence $p^n \in V_h$ with $p^N = p^0$,

$$\delta t \sum_{n=0}^{N-1} \int_{\Omega} e^{2s\alpha^n} (s\xi^n |\nabla p^n|^2 + s^3 (\xi^n)^3 |p^n|^2) dx$$

$$\leq C\delta t \sum_{n=0}^{N-1} \int_{\Omega} e^{2s\alpha^n} |(P_{h,\delta t} p)^n|^2 dx + Cs^3\delta t \sum_{n=0}^{N-1} \int_{\omega} e^{2s\alpha^n} (\xi^n)^3 |p^n|^2 dx, \quad (1.9)$$

provided $s \min_{[0,T]} \xi \geq s_0$ and $s_0 \leq s \leq \kappa \min\{h^{-4/5}, (\delta t)^{-2/5}\}$. Here $C, \kappa, s_0 > 0$ are independent of $h, \delta t, s$ and the allowed weights; s_0 is chosen sufficiently large depending on M to absorb the potential. The two terms on the right are the residual and observation of the original scheme, just as in (1.8).

For a prescribed terminal state $p_T \in V_h$, the discrete adjoint of (1.3) reads

$$(P_{h,\delta t} p)^n = 0, \quad 0 \leq n < N, \quad p^N = p_T. \quad (1.10)$$

To apply (1.9), we cut off p near the time endpoints. Since $P_{h,\delta t} p = 0$, the residual of the cutoff sequence contains only cutoff differences. Choosing the weights at the admissible mesh scale and combining (1.9) with a discrete energy estimate yields

$$\|p^0\|_{L^2(\Omega)}^2 \leq C \left(\delta t \sum_{n=0}^{N-1} \|p^n\|_{L^2(\omega)}^2 + e^{-2c \min\{h^{-4/5}, (\delta t)^{-2/5}\}} \|p^N\|_{L^2(\Omega)}^2 \right), \quad (1.11)$$

where C, c are independent of h and δt . Compared with (1.7), this estimate has an exponentially small terminal term arising from the endpoint cutoff, hence the name *relaxed observability*. Minimizing a quadratic functional of the terminal adjoint datum then gives the control bounds in (1.4) and (1.5) (see Theorems 7.2 and 7.3). With suitable filtering of the terminal adjoint data, the remainder can be absorbed, yielding uniform observability (see Theorem 7.5).

The heat-equation example above illustrates the general transfer framework developed in this paper. In summary, our main contributions are as follows.

1. We establish an abstract framework for transferring continuous Carleman estimates to fully discrete schemes through weighted approximation estimates uniform over the full discrete space.
2. The transferred inequality retains the residual and observation of the original scheme and allows distinct physical and reconstruction masses and a small weighted stiffness defect.
3. We verify the heat-equation framework for locally graded consistent-mass P_1 finite elements in dimensions two and three and for standard Cartesian finite differences in any fixed dimension.
4. The resulting relaxed observability estimates yield approximate null controls with uniformly bounded cost, exponentially small terminal errors, and continuous null controls as weak accumulation points, with no coupling between the spatial and temporal refinement parameters.

1.3 The transfer mechanism: an elliptic model

We use the Poisson operator as a model to isolate the two obstructions to a direct perturbation argument and to motivate the construction that overcomes them.

Fix $1 < p < \infty$ in the Dirichlet regularity range of the Laplacian. The classical $W^{1,p}$ perturbation argument associated with Meyers' theory [19] allows $\operatorname{div}(\epsilon \nabla v)$ to be absorbed whenever $C(p)\|\epsilon\|_{L^\infty}$ is sufficiently small. In the Carleman setting, however, the analogous threshold depends on the large parameter: the argument in [23, Section 6.1, (6.5)] requires $Cs\|\epsilon\|_{L^\infty}^2 < 1$. Thus a growing range of s requires a correspondingly smaller perturbation.

Consistency alone, however, does not provide such smallness uniformly over a discrete space. A simple example illustrates this obstruction. For centered differences on $(0, 1)$ with $h = 1/J$, the continuous and discrete eigenvalues associated with the sine mode $v_j(x) = \sin(j\pi x)$ are (see [21, Section 4.2])

$$\lambda_j = (j\pi)^2, \quad \lambda_{h,j} = 4h^{-2} \sin^2(j\pi h/2), \quad 1 \leq j < J.$$

Writing $\epsilon_{h,j} = \lambda_{h,j}/\lambda_j - 1$ for the relative error, we have

$$-\Delta v_j = \lambda_{h,j} v_j + \operatorname{div}(\epsilon_{h,j} \nabla v_j).$$

For each fixed j , $\lim_{j \rightarrow \infty} \epsilon_{1/J,j} = 0$, whereas the highest mode satisfies $\lim_{j \rightarrow \infty} |\epsilon_{1/J,j-1}| = 1 - 4/\pi^2 > 0$. Hence the preceding absorption argument cannot yield an admissible Carleman range that grows under mesh refinement.

This obstruction motivates a different comparison. Take $\Omega = (0,1)^2$ and the finite-element space V_h , operator A_h , and mesh assumptions of Subsection 1.1. Let ρ be the spatial weight supplied by Proposition A.3 for a nonempty open set $\omega_0 \Subset \omega$, and set $\theta = e^{s\rho}$. Given any $u_h \in V_h$, put $f_h = A_h u_h$ and solve

$$(-\Delta + Ks^2)U = f_h + Ks^2 u_h \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega. \quad (1.12)$$

The constant K is fixed sufficiently large, independently of h and s , to provide weighted coercivity. By the definition of A_h , the prescribed state u_h is exactly the Galerkin solution of this auxiliary problem.

With $e = U - u_h$, the static case of Lemma 4.1, with $m = 1$ and $q \equiv s$, gives

$$\|\theta \nabla e\|_{L^2(\Omega)} + h^{-1} \|\theta e\|_{L^2(\Omega)} \leq Ch \|\theta(f_h + Ks^2 u_h)\|_{L^2(\Omega)}, \quad (1.13)$$

provided hs is sufficiently small. The constant is uniform over all $u_h \in V_h$, including mesh-scale oscillations. The essential cancellation is

$$-\Delta U = f_h - Ks^2 e. \quad (1.14)$$

Consequently, applying the continuous Carleman estimate to U introduces the original discrete residual f_h and a term controlled by the comparison error. Moreover, estimate (1.13) controls the difference between the continuous and discrete energy and observation terms. The most restrictive state-error contribution is of order $h^4 s^5$, giving the sufficient range $s_0 \leq s \leq ch^{-4/5}$. For $h \leq 1$, choosing $c > 0$ sufficiently small after fixing K and the estimate constants absorbs the state error and bounds the residual coefficient by Cs^{-1} , yielding

$$\|\theta \nabla u_h\|_{L^2(\Omega)}^2 + s^2 \|\theta u_h\|_{L^2(\Omega)}^2 \leq C \left(s^{-1} \|\theta A_h u_h\|_{L^2(\Omega)}^2 + \|\theta \nabla u_h\|_{L^2(\omega)}^2 + s^2 \|\theta u_h\|_{L^2(\omega)}^2 \right), \quad (1.15)$$

where $C, c, s_0 > 0$ are independent of h, s, u_h , and c is sufficiently small. The continuous elliptic Carleman estimate follows from Proposition A.3 with $q \equiv s$ and a time-independent function, after division by s ; its observation on ω_0 is bounded by the displayed observation on ω .

The role of the auxiliary problem is therefore to embed each discrete state as the Galerkin solution of a continuous problem to which the Carleman estimate applies. Weighted numerical error estimates then transfer the inequality to the original scheme. The parabolic argument combines the temporal and spatial comparisons in Sections 3 and 4 with the absorption argument in Section 5.

1.4 Relation to previous work

A first approach to uniform numerical controllability consists in deriving Carleman estimates directly at the discrete level, by developing weighted identities and estimating remainders within the discrete calculus. Elliptic difference estimates were established in [4, 5], followed by semi-discrete parabolic, time-discrete, and fully discrete estimates in [2, 7, 13]. By contrast, we derive the discrete Carleman estimate from its continuous counterpart through weighted approximation estimates that hold uniformly over the full discrete space. This separation of the PDE-specific Carleman analysis from the numerical estimates required for each discretization yields a common transfer mechanism for the finite-element and finite-difference realizations considered here.

A second line of work, rather than deriving discrete Carleman estimates directly, addresses the high-frequency obstruction through spectral truncation, filtering, and multilevel constructions (see [22]). In particular, Boyer, Hubert, and Le Rousseau [6] developed an abstract framework for spatial Galerkin approximations and temporal discretization based on a uniform partial spectral inequality and the Lebeau–Robbiano strategy. Under their spectral and discretization assumptions, [6, Theorem 3.5] provides uniformly bounded controls that annihilate a low-frequency spectral projection exactly and leave an exponentially small full terminal state. The resulting control conclusions are therefore closely related, but the underlying estimates differ: their argument uses a partial spectral inequality, whereas ours transfers the Carleman inequality itself

through weighted numerical approximation properties. In the autonomous backward-Euler setting, their results also allow for independent refinement after reducing the usable spectral range (see Remark 7.5).

In contrast to the spectral approach, in our framework the transferred estimate itself determines a sufficient threshold for filtering. In Subsection 7.3, the remainder in (1.11) is absorbed on terminal spaces whose discrete Rayleigh quotient is sufficiently small relative to the available Carleman scale. This restriction can be realized by compatible nested P_1 coarse spaces and Cartesian coarse-grid interpolation, without a discrete eigendecomposition. The corresponding controls simultaneously annihilate the retained terminal projection, have cost bounded uniformly in the discretization parameters, and leave an exponentially small full terminal state for the bounded space-time potentials considered here.

Beyond these Carleman-based and spectral approaches, complementary methods draw on other continuous control and stability arguments. Boyer and Hernández-Santamaría [3] adapt the moments method to time-discrete boundary-controlled parabolic systems by constructing and estimating discrete biorthogonal families. The penalized Hilbert Uniqueness Method (HUM), which turns relaxed observability estimates into numerically computable controls, has also been studied systematically in the discrete setting (see [6, Sections 4–6] and [1]).

Discrete Carleman estimates have also found applications in numerical inverse problems. For semi-discrete parabolic equations, they yield stability estimates for inverse source and coefficient problems [17].

Finally, we outline the structure of the paper. The abstract framework and transfer theorem are stated in Section 2. Weighted temporal and spatial comparisons are proved in Sections 3 and 4, and the transfer is completed in Section 5. The finite-element and finite-difference realizations are verified in Section 6. Relaxed and filtered observability, control, and weak convergence are treated in Section 7. Numerical illustration follows in Section 8. The appendices verify the continuous analytic assumptions, the difference interfaces, the consistency statements used in the control limit, and the auxiliary filtering estimates.

2 Abstract framework and transfer theorem

This section develops the abstract framework that allows us to transfer a continuous Carleman estimate to a fully discrete scheme. We first collect the continuous analytic assumptions, then introduce the general fully discrete scheme together with the reconstruction and transfer operators that link it to the continuous problem, and finally state the numerical assumptions ensuring weighted approximation, stiffness compatibility, and observation compatibility. Under these hypotheses, the main transfer theorem yields a Carleman estimate over the full discrete space while preserving the residual and observation terms of the original scheme. Its proof is given in Section 5, after the necessary temporal and spatial comparison estimates have been established in Sections 3 and 4.

2.1 Continuous analytic assumptions and Carleman estimate

Fix $d \geq 1$, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, set $H = L^2(\Omega)$, and fix an integer $m \geq 1$.

Fix $T > 0$. In the auxiliary estimates we impose matching time endpoints in order to cancel temporal boundary terms and to solve both the forward and adjoint auxiliary problems. The original control and adjoint systems have prescribed initial or terminal data. The auxiliary estimate is applied to the adjoint only after a time cutoff. Endpoint values of functions in $H^1(0, T; H)$ are understood as traces in H . Set $\mathcal{Q} = (0, T) \times \Omega$ and, for a measurable subset $E \subset \Omega$, write $\mathcal{Q}_E = (0, T) \times E$.

Fix $\beta > 0$, $q_c \geq 1$, and $\rho \in C^\infty(\bar{\Omega})$ with $\rho \leq -\rho_*$ on $\bar{\Omega}$ for some fixed $\rho_* > 0$. Let $q \in C^2([0, T])$ satisfy $q^{(j)}(0) = q^{(j)}(T)$ for $j = 0, 1, 2$, and extend q periodically when needed. Assume

$$1 \leq q_c \leq q(t) \leq q_{\max}, \quad |q'| \leq \beta q^2, \quad |q''| \leq \beta^2 q^3. \quad (2.1)$$

One may take $q_{\max} = \max_{t \in [0, T]} q(t)$. In (2.1), it need only be an upper bound. Set $\theta(t, x) = e^{q(t)\rho(x)}$. The regularized weights in (1.9) are $\xi = q/s$ and $\alpha = \xi\rho$, with $s = \max_{[0, T]} q$. Thus $\max_{[0, T]} \xi = 1$, $s\xi = q$, and $e^{s\alpha} = \theta$. Fix $\tau_* > 0$. An allowed weight has the form $\sigma = q^\tau \theta$ or $\sigma = q^\tau \theta^{-1}$, where τ is a fixed real exponent with $|\tau| \leq \tau_*$.

Write $D^j v$ for the collection of all spatial derivatives of v of order j , with the Euclidean norm on its components. At each fixed time t , define the spatial norms

$$\begin{aligned}\mathcal{N}_{\sigma,\Omega}(v)^2 &= \sum_{j=0}^m \|q(t)^{m-j}\sigma(t,\cdot)D^j v\|_{L^2(\Omega)}^2, & v \in H^m(\Omega), \\ \mathcal{R}_{\sigma,\Omega}(v) &= \sum_{j=0}^{2m} \|q(t)^{2m-j}\sigma(t,\cdot)D^j v\|_{L^2(\Omega)}, & v \in H^{2m}(\Omega).\end{aligned}\tag{2.2}$$

For fixed t and weights, these norms are equivalent to the usual $H^m(\Omega)$ and $H^{2m}(\Omega)$ norms, respectively. For time-dependent functions, define the space-time norms by

$$\mathcal{N}_{\sigma,\mathcal{Q}}(v)^2 = \sum_{j=0}^m \|q^{m-j}\sigma D^j v\|_{L^2(\mathcal{Q})}^2, \quad \mathcal{R}_{\sigma,\mathcal{Q}}(v) = \sum_{j=0}^{2m} \|q^{2m-j}\sigma D^j v\|_{L^2(\mathcal{Q})}.\tag{2.3}$$

These are norms on $L^2(0,T;H^m(\Omega))$ and $L^2(0,T;H^{2m}(\Omega))$, respectively. All these definitions also apply with $\sigma = 1$.

We now collect the continuous analytic assumptions used throughout the paper. They are required to hold with uniform positive constants for every $T > 0$ and every allowed weight. The symbols $c, C > 0$ denote generic constants that may change from line to line.

(A1) Form and graph regularity. The space V is a closed subspace of $H^m(\Omega)$, dense in H . The real symmetric form $a : V \times V \rightarrow \mathbb{R}$ satisfies the boundedness estimate

$$|a(v, w)| \leq C \|v\|_{H^m(\Omega)} \|w\|_{H^m(\Omega)}$$

and the coercivity estimate

$$a(v, v) \geq c \|v\|_{H^m(\Omega)}^2.$$

Its associated positive self-adjoint operator A has $D(A) \hookrightarrow H^{2m}(\Omega)$ and satisfies

$$\|v\|_{H^{2m}(\Omega)} \leq C \|Av\|_{L^2(\Omega)}, \quad v \in D(A).\tag{2.4}$$

In particular, $D(A^{1/2}) = V$ and $\|A^{1/2}v\|_{L^2(\Omega)}^2 = a(v, v)$.

(A2) Weight compatibility and commutator bound. Multiplication by every allowed weight σ and its inverse σ^{-1} preserves both V and $D(A)$. Write $[A, \sigma]u = A(\sigma u) - \sigma Au$. At each fixed time t , for every allowed weight σ and every $v \in D(A)$,

$$\|[A, \sigma]\sigma^{-1}v\|_{L^2(\Omega)} \leq C \sum_{j=0}^{2m-1} q^{2m-j} \|D^j v\|_{L^2(\Omega)}.\tag{2.5}$$

(A3) Weighted form bounds. At each fixed time, for every allowed weight σ and all $v, w \in V$,

$$|a(v, w)| \leq C \mathcal{N}_{\sigma,\Omega}(v) \mathcal{N}_{\sigma^{-1},\Omega}(w),\tag{2.6}$$

and

$$a(v, \sigma^2 v) \geq c \mathcal{N}_{\sigma,\Omega}(v)^2 - C \|q^m \sigma v\|_{L^2(\Omega)}^2.\tag{2.7}$$

(A4) Continuous Carleman estimate. Write $P_\epsilon = \epsilon \partial_t + A$, $\epsilon \in \{-1, 1\}$, and fix a nonempty open set $\omega_0 \Subset \Omega$. Fix $c_0 > 0$, $c_j \geq 0$ for $1 \leq j \leq m$, and $o \geq 0$. Let $\mathcal{D}_j$ be fixed spatial differential expressions of order at most j , with bounded coefficients. For each $\epsilon \in \{-1, 1\}$ and every $v \in H^1(0, T; H) \cap L^2(0, T; D(A))$ with $v(0) = v(T)$,

$$\mathcal{E}_q(v) := \|q^{c_0/2} \theta v\|_{L^2(\mathcal{Q})}^2 + \sum_{j=1}^m \|q^{c_j/2} \theta \mathcal{D}_j v\|_{L^2(\mathcal{Q})}^2 \leq C (\|\theta P_\epsilon v\|_{L^2(\mathcal{Q})}^2 + \|q^{o/2} \theta v\|_{L^2(\mathcal{Q}_{\omega_0})}^2).\tag{2.8}$$

Unless stated otherwise, all constants depend only on $d, m, \Omega, \omega_0, \rho, \beta, q_c, \tau_*$, the Carleman exponents $c_0, \dots, c_m, o$, and the fixed bounds in the hypotheses used. Additional parameter dependence is stated explicitly.

2.2 Discrete scheme and transfer operators

For each discretization parameter h , we fix the following discrete data: a finite-dimensional real Hilbert space X_h with inner product $(\cdot, \cdot)_{X_h}$ and induced norm $\|\cdot\|_{X_h}$, a linear spatial operator $A_h : X_h \rightarrow X_h$, a real normed load space $\mathfrak{F}_h$ with load map $\Gamma_h : \mathfrak{F}_h \rightarrow X_h$, and a Hilbert control space U_h with linear control operator $B_h : U_h \rightarrow X_h$. The adjoint $B_h^* : X_h \rightarrow U_h$ is taken with respect to the prescribed unweighted inner products on X_h and U_h .

To compare this discrete scheme with the continuous problem, we introduce a linear conforming reconstruction $R_h : X_h \rightarrow V$ and define its adjoint $J_h : H \rightarrow X_h$ by

$$(J_h v, z)_{X_h} = (v, R_h z)_H, \quad v \in H, \quad z \in X_h. \quad (2.9)$$

The two compositions

$$Q_h = R_h J_h : H \rightarrow V, \quad G_h = J_h R_h : X_h \rightarrow X_h \quad (2.10)$$

will play a central role in the transfer of weighted estimates: Q_h maps continuous data to the reconstructed discrete space, while G_h encodes the induced discrete mass. We emphasize that neither $Q_h^2 = Q_h$ nor $G_h = \text{Id}_{X_h}$ is assumed; the reconstruction need not be orthogonal. All scheme, reconstruction, load, and control maps are independent of time.

The load and control spaces $\mathfrak{F}_h$ and U_h need not be finite dimensional. The weighted norms on X_h , $\mathfrak{F}_h$, and U_h are part of the discretization data; their concrete definitions are given in subsections 6.1 and 6.2. At each fixed time, for every $r > 0$ and all $z \in X_h$, $f \in \mathfrak{F}_h$, $w \in U_h$, we require the homogeneity

$$\|z\|_{X_h, r\sigma} = r\|z\|_{X_h, \sigma}, \quad \|f\|_{\mathfrak{F}_h, r\sigma} = r\|f\|_{\mathfrak{F}_h, \sigma}, \quad \|w\|_{U_h, r\sigma} = r\|w\|_{U_h, \sigma},$$

so that the weighted norms scale consistently with the weight parameter.

2.3 Numerical assumptions

We write $a \asymp b$ if $cb \leq a \leq Cb$ for positive constants independent of the varying parameters.

We now state the numerical assumptions under which the transfer theorem will be proved. They are the discrete analogues of the continuous analytic assumptions and are required to hold uniformly over the prescribed family of discretizations. Specifically, we assume that there exist a fixed threshold $\delta_{\text{num}} > 0$ and uniform positive constants such that Assumptions (N1) to (N5) hold for every discretization in the prescribed family, every q satisfying (2.1) with $hq_{\max} \leq \delta_{\text{num}}$, every associated allowed weight, and every $t \in [0, T]$.

(N1) Reconstruction and stiffness compatibility. The reconstruction R_h is injective, and A_h is self-adjoint and nonnegative in the inner product. For all $z, w \in X_h$,

$$\|R_h z\|_{L^2(\Omega)} \asymp \|z\|_{X_h}, \quad \|\sigma R_h z\|_{L^2(\Omega)} \asymp \|z\|_{X_h, \sigma}. \quad (2.11)$$

The discrete stiffness satisfies

$$(A_h z, w)_{X_h} = a(R_h z, R_h w) + \mathfrak{b}_h(z, w), \quad (2.12)$$

where $\mathfrak{b}_h$ is a time-independent symmetric form satisfying, for some $\eta_A(h) \geq 0$ and every allowed weight,

$$|\mathfrak{b}_h(z, w)| \leq \eta_A(h) \mathcal{N}_{\sigma, \Omega}(R_h z) \mathcal{N}_{\sigma^{-1}, \Omega}(R_h w). \quad (2.13)$$

Exact stiffness compatibility is the case $\eta_A = 0$.

(N2) Stable load maps. There is a lift $\mathcal{L}_h : \mathfrak{F}_h \rightarrow H$ with $J_h \mathcal{L}_h = \Gamma_h$ satisfying

$$\|\sigma \mathcal{L}_h f\|_{L^2(\Omega)} \leq C \|f\|_{\mathfrak{F}_h, \sigma}. \quad (2.14)$$

The load map has a right inverse $\mathcal{I}_h : X_h \rightarrow \mathfrak{F}_h$, with $\Gamma_h \mathcal{I}_h = \text{Id}_{X_h}$, satisfying

$$\|\mathcal{I}_h z\|_{\mathfrak{F}_h, \sigma} \leq C \|z\|_{X_h, \sigma}. \quad (2.15)$$

(N3) *Weighted approximation.* For prescribed $a_E(h), a_0(h) \geq 0$, every $v \in D(A)$ satisfies

$$\mathcal{N}_{\sigma,\Omega}(v - Q_h v) \leq a_E(h) \mathcal{R}_{\sigma,\Omega}(v). \quad (2.16)$$

The weighted L^2 approximation satisfies

$$\|\sigma(v - Q_h v)\|_{L^2(\Omega)} \leq a_0(h) \mathcal{R}_{\sigma,\Omega}(v). \quad (2.17)$$

(N4) *Weighted test and superapproximation.* There is a differentiable operator family $\mathcal{T}_{\sigma,h}(t) : X_h \rightarrow X_h$, self-adjoint and positive definite in the inner product, with $\mathcal{T}_{\sigma,h}(0) = \mathcal{T}_{\sigma,h}(T)$, such that

$$(z, \mathcal{T}_{\sigma,h} z)_{X_h} \asymp \|\sigma R_h z\|_{L^2(\Omega)}^2. \quad (2.18)$$

Its temporal derivative satisfies

$$|(z, \mathcal{T}'_{\sigma,h} z)_{X_h}| \leq C \|q \sigma R_h z\|_{L^2(\Omega)}^2, \quad (2.19)$$

and the spatial superapproximation estimate is

$$\mathcal{N}_{\sigma^{-1},\Omega}(R_h \mathcal{T}_{\sigma,h} z - \sigma^2 R_h z) \leq Ch q_{\max} \mathcal{N}_{\sigma,\Omega}(R_h z). \quad (2.20)$$

(N5) *Observation compatibility.* For every $z \in X_h$,

$$\|\sigma R_h z\|_{L^2(\omega_0)} \leq C \|B_h^* z\|_{U_h, \sigma}. \quad (2.21)$$

Assumptions (A1) to (A3) provide the regularity and weighted stability needed for the shifted continuous auxiliary problem, while Assumption (A4) supplies the continuous Carleman estimate. On the numerical side, Assumptions (N1) and (N2) ensure that reconstructed states and lifted loads are compatible with the discrete scheme, whereas Assumptions (N3) and (N4) provide the weighted approximation and test estimates required for the spatial comparison. Finally, Assumption (N5) controls the continuous observation of the reconstruction by the discrete observation. These ingredients are combined in the temporal and spatial comparison estimates of sections 3 and 4, and the transfer is completed in section 5.

2.4 Fully discrete transfer theorem

We now discretize the time interval and introduce the discrete norms and difference operators needed to state the transfer theorem. Fix an integer $N \geq 2$ and discretize $[0, T]$ with uniform time step $\delta t = T/N$ and nodes $t_n = n\delta t$, $0 \leq n \leq N$. Put $q_n = q(t_n)$, $\theta_n = \theta(t_n, \cdot)$, and $\sigma_n = \sigma(t_n, \cdot)$. Superscripts on state and load sequences denote time levels; the subscript h denotes spatial discretization. Products of weights and sequences are evaluated nodewise.

For a normed space $\mathcal{H}$ and a specified finite set $\mathcal{J}$ of time indices, let $\ell_{\delta t}^2(\mathcal{H}) = \mathcal{H}^{\mathcal{J}}$ with norm

$$\|v\|_{\ell_{\delta t}^2(\mathcal{H})}^2 := \delta t \sum_{n \in \mathcal{J}} \|v^n\|_{\mathcal{H}}^2.$$

If $\mathcal{H}$ is a Hilbert space, the corresponding inner product is

$$(v, w)_{\ell_{\delta t}^2(\mathcal{H})} := \delta t \sum_{n \in \mathcal{J}} (v^n, w^n)_{\mathcal{H}}.$$

For prescribed weighted norms, set

$$\|v\|_{\ell_{\delta t}^2(\mathcal{H}), \sigma}^2 := \delta t \sum_{n \in \mathcal{J}} \|v^n\|_{\mathcal{H}, \sigma_n}^2.$$

To match the temporal endpoint condition in Assumption (A4), take $\mathcal{J} = \{0, \dots, N-1\}$ and extend sequences by $v^{n+N} = v^n$ for $n \in \mathbb{Z}$, with indices interpreted modulo N . Define the oriented difference quotients

$$D_+ v^n := \frac{v^n - v^{n-1}}{\delta t}, \quad D_- v^n := \frac{v^n - v^{n+1}}{\delta t}.$$

The notation $D_{\pm}$ denotes either choice, corresponding to $\pm\partial_t$ in the continuous equations. Periodic summation by parts gives $D_+^* = D_-$ with respect to the $\ell_{\delta t}^2(\mathcal{H})$ inner product.

Next, we introduce discrete counterparts of the continuous graph norms. For $v \in \ell_{\delta t}^2(H^m(\Omega))$ and $v \in \ell_{\delta t}^2(H^{2m}(\Omega))$, respectively, define

$$\mathcal{N}_{\sigma, \delta t}(v)^2 = \sum_{j=0}^m \|q^{m-j} \sigma D^j v\|_{\ell_{\delta t}^2(L^2(\Omega))}^2, \quad \mathcal{R}_{\sigma, \delta t}(v) = \sum_{j=0}^{2m} \|q^{2m-j} \sigma D^j v\|_{\ell_{\delta t}^2(L^2(\Omega))}.$$

These definitions also apply with $\sigma = 1$.

For $z \in \ell_{\delta t}^2(X_h)$, we define the discrete Carleman energy

$$\mathcal{E}_{q,h,\delta t}(z) := \|q^{c_0/2} z\|_{\ell_{\delta t}^2(X_h), \theta}^2 + \sum_{j=1}^m \|q^{c_j/2} \theta \mathcal{D}_j R_h z\|_{\ell_{\delta t}^2(L^2(\Omega))}^2. \quad (2.22)$$

Using the Carleman exponents in Assumption (A4), set

$$d_* = \max\{0, \max_{1 \leq j \leq m} (c_j - 2(m-j))\}, \quad b_* = \max\{c_0, o, 4m\}. \quad (2.23)$$

These exponents determine the weight powers that enter the smallness conditions below.

We are now ready to state the main transfer theorem.

Theorem 2.1. *Assume Assumptions (A1) to (A4) and Assumptions (N1) to (N5), and*

$$\tau_* \geq \max\{d_*/2, b_*/2\}. \quad (2.24)$$

Then there exist positive constants $\delta_x, \delta_{\text{time}}, \eta_*, c_*, C$, with $\delta_x \leq \delta_{\text{num}}$, such that the following holds.

Put

$$\varepsilon_E = a_E(h) + \eta_A(h) + \sqrt{\delta t}, \quad \varepsilon_0 = a_0(h) + a_E(h)^2 + \eta_A(h) + \delta t. \quad (2.25)$$

Assume that

$$a_E \leq 1, \quad \eta_A \leq \eta_*, \quad \delta t \leq 1, \quad h q_{\max} \leq \delta_x, \quad \delta t q_{\max}^2 \leq \delta_{\text{time}},$$

and that the defect quantities satisfy

$$\Lambda_f := \varepsilon_E^2 q_{\max}^{d_*} + \varepsilon_0^2 q_{\max}^{b_*} \leq 1, \quad (2.26)$$

$$\Lambda_z := \varepsilon_E^2 \max\{1, q_{\max}^{4m+d_*-c_0}\} + \varepsilon_0^2 \max\{1, q_{\max}^{4m+b_*-c_0}\} \leq c_*. \quad (2.27)$$

Then, for either choice of sign, every pair $(z, f) \in \ell_{\delta t}^2(X_h) \times \ell_{\delta t}^2(\mathfrak{F}_h)$ satisfying

$$D_{\pm} z^n + A_h z^n = \Gamma_h f^n, \quad n = 0, \dots, N-1, \quad (2.28)$$

obeys

$$\mathcal{E}_{q,h,\delta t}(z) \leq C(\|f\|_{\ell_{\delta t}^2(\mathfrak{F}_h), \theta}^2 + \|q^{o/2} B_h^* z\|_{\ell_{\delta t}^2(U_h), \theta}^2). \quad (2.29)$$

In particular, every $z \in \ell_{\delta t}^2(X_h)$ satisfies

$$\mathcal{E}_{q,h,\delta t}(z) \leq C(\|D_{\pm} z + A_h z\|_{\ell_{\delta t}^2(X_h), \theta}^2 + \|q^{o/2} B_h^* z\|_{\ell_{\delta t}^2(U_h), \theta}^2). \quad (2.30)$$

The condition $\Lambda_f \leq 1$ is a normalization. Any fixed upper bound changes only the constant in (2.29). Only the state defect Λ_z must be small.

Remark 2.1. The finite upper bound in (2.1) permits $h q_{\max} \ll 1$ and $\delta t q_{\max}^2 \ll 1$. The admissible bound grows under refinement. Thus we use a regularized temporal factor rather than an endpoint-singular one. Appendix A adapts the existing continuous weighted identities to functions with matching time endpoints, so that temporal boundary terms cancel. In section 7, the regularization is chosen at the admissible mesh scale and a cutoff gives physical-time observability; temporal endpoint terms arise in that application.

For spatial approximation orders h^m and h^{2m} , the next corollary gives an explicit sufficient bound on $q_{\max}$ in terms of $h, \delta t$ that ensures the smallness conditions in Theorem 2.1.

Corollary 2.2. *Under the analytic and numerical assumptions of Theorem 2.1, suppose $a_E(h) \leq Ch^m$ and $a_0(h) + \eta_A(h) \leq Ch^{2m}$ uniformly in h . Set*

$$\nu_E = \max\{d_*, 4m + d_* - c_0, 0\}, \quad \nu_0 = \max\{b_*, 4m + b_* - c_0, 0\}. \quad (2.31)$$

Then there exist $\kappa, h_0, \Delta_0, C > 0$, with Δ_0 an upper bound on the time step, such that (2.29) and (2.30) hold for $h \leq h_0$, $\delta t \leq \Delta_0$, and

$$q_{\max} \leq \kappa \min\{h^{-\alpha_x}, (\delta t)^{-\alpha_t}\}, \quad (2.32)$$

where $\alpha_x = \min\{1, 2m/\nu_E, 4m/\nu_0\}$, $\alpha_t = \min\{1/2, 1/\nu_E, 2/\nu_0\}$, and ratios with zero denominator are omitted from the minima.

Applying this general scale rule to the Carleman estimate for the heat equation gives the following parameter range.

Corollary 2.3. *Under the same analytic, numerical, and approximation assumptions as Corollary 2.2, take $m = 1$ and $(c_0, c_1; o) = (3, 1; 3)$. Then there exist $\kappa, h_0, \Delta_0, C > 0$ such that (2.29) and (2.30) hold whenever $h \leq h_0$, $\delta t \leq \Delta_0$, and $q_{\max} \leq S_{h, \delta t}$, where*

$$S_{h, \delta t} = \kappa \min\{h^{-4/5}, (\delta t)^{-2/5}\}. \quad (2.33)$$

Theorem 2.1 and Corollaries 2.2 and 2.3 are proved in Section 5. The two realizations in Section 6 use this Carleman estimate for the heat equation and allow $\tau_* = 4$. For other orders and continuous Carleman estimates, the range is determined by (2.32) under the analytic and numerical assumptions, with the weight powers covered by (2.24).

3 Weighted temporal estimates for the backward Euler scheme

Throughout this section, we assume Assumptions (A1) to (A3). We adapt the standard energy and parabolic duality argument for temporal discretization to the Carleman weights $\sigma = q^\tau \theta^{\pm 1}$. The auxiliary graph estimates are required to hold uniformly with respect to these weights and their reciprocals. The adjoint problem then yields a weighted $L^2(\mathcal{Q})$ error of order δt , whereas spatial interpolation yields a weighted energy error of order $\sqrt{\delta t}$. The time grid and oriented differences are those introduced in Section 2.

3.1 Regularity of the continuous auxiliary problem

We begin by introducing the shifted auxiliary problem that will be used throughout the temporal and spatial comparisons. For $L \geq 1$, define the positive scalar function $\mu(t) := (Lq(t))^{2m}$. At each fixed time, the associated bilinear form on V is

$$a_\mu(v, w) := a(v, w) + \mu(t)(v, w)_H, \quad v, w \in V. \quad (3.1)$$

The same notation is used for the nodal and cellwise constant values of the shift.

The following lemma provides the weighted regularity and coercivity for this shifted problem. These estimates are uniform in the weight and will later allow us to absorb the temporal and spatial comparison errors.

Lemma 3.1. *There exists $C > 0$ such that, for every prescribed margin $B > 0$, one can choose $L_0(B) \geq 1$ so that, for every $L \geq L_0(B)$ and $g \in L^2(0, T; H)$, the problem $(\epsilon \partial_t + A + \mu)U = g$, $U(0) = U(T)$, has a unique solution in $H^1(0, T; H) \cap L^2(0, T; D(A))$ satisfying*

$$\|\sigma U_t\|_{L^2(\mathcal{Q})} + \mathcal{R}_{\sigma, \mathcal{Q}}(U) \leq C \|\sigma g\|_{L^2(\mathcal{Q})}. \quad (3.2)$$

There is also $C(L) > 0$ such that, for all spatial slices $v, w \in V$, the shifted form satisfies

$$|a_\mu(v, w)| \leq C(L) \mathcal{N}_{\sigma, \Omega}(v) \mathcal{N}_{\sigma^{-1}, \Omega}(w). \quad (3.3)$$

The prescribed coercivity margin is retained in

$$a_\mu(v, \sigma^2 v) \geq c \mathcal{N}_{\sigma, \Omega}(v)^2 + B \|q \sigma v\|_{L^2(\Omega)}^2. \quad (3.4)$$

Proof. Step 1. In this step, we prove a graph estimate for the unweighted shifted operator.

Sobolev extension on the bounded Lipschitz domain Ω and interpolation, followed by (2.4), give, uniformly for $r \geq 1$,

$$\sum_{j=0}^{2m} r^{2m-j} \|D^j v\|_{L^2(\Omega)} \leq C(\|Av\|_{L^2(\Omega)} + r^{2m} \|v\|_{L^2(\Omega)}), \quad v \in D(A). \quad (3.5)$$

For a smooth periodic graph-domain function, time independence and self-adjointness of A give

$$\|(\epsilon \partial_t + A + \mu)v\|_{L^2(\mathcal{Q})}^2 = \|v_t\|_{L^2(\mathcal{Q})}^2 + \|Av\|_{L^2(\mathcal{Q})}^2 + \|\mu v\|_{L^2(\mathcal{Q})}^2 + 2 \int_0^T \mu a(v, v) dt - \epsilon \int_0^T \mu' \|v\|_{L^2(\Omega)}^2 dt. \quad (3.6)$$

By (2.1),

$$|\mu'|/\mu^2 \leq 2m\beta L^{-2m} q_c^{1-2m},$$

so the last term in (3.6) is absorbed for large L . Applying (3.5) with $r = Lq(t)$ then gives

$$\|v_t\|_{L^2(\mathcal{Q})} + \sum_{j=0}^{2m} \|(Lq)^{2m-j} D^j v\|_{L^2(\mathcal{Q})} \leq C\|(\epsilon \partial_t + A + \mu)v\|_{L^2(\mathcal{Q})}. \quad (3.7)$$

Existence follows by a standard spectral Galerkin argument, and uniqueness follows from (3.7).

Step 2. In this step, we transfer the graph estimate to the weighted setting. The weight definition and (2.1) give

$$|D^j \sigma^{\pm 1}| \leq C q^j \sigma^{\pm 1} \quad (0 \leq j \leq 2m), \quad |\partial_t \log \sigma| \leq C q^2. \quad (3.8)$$

Assumption (A2) ensures that multiplication by $\sigma^{\pm 1}$ preserves $D(A)$. Thus $v = \sigma U$ belongs to the graph space and satisfies

$$(\epsilon \partial_t + A + \mu)v = \sigma g + [A, \sigma] \sigma^{-1} v + \epsilon (\partial_t \log \sigma) v. \quad (3.9)$$

Before fixing L , by $q \geq 1$ and $m \geq 1$, (2.5) and (3.8) give

$$\|[A, \sigma] \sigma^{-1} v\|_{L^2(\mathcal{Q})} \leq \frac{C}{L} \sum_{j=0}^{2m} \|(Lq)^{2m-j} D^j v\|_{L^2(\mathcal{Q})}. \quad (3.10)$$

The temporal weight term satisfies

$$\|(\partial_t \log \sigma) v\|_{L^2(\mathcal{Q})} \leq \frac{C}{L^{2m}} \|\mu v\|_{L^2(\mathcal{Q})}. \quad (3.11)$$

Combining (3.9)–(3.11), increasing L absorbs both terms in (3.7) and gives

$$\|v_t\|_{L^2(\mathcal{Q})} + \sum_{j=0}^{2m} \|(Lq)^{2m-j} D^j v\|_{L^2(\mathcal{Q})} \leq C \|\sigma g\|_{L^2(\mathcal{Q})}. \quad (3.12)$$

Since $U = \sigma^{-1} v$, the product rule and (2.3), (3.7), (3.8) and (3.12) yield

$$\|\sigma U_t\|_{L^2(\mathcal{Q})} + \mathcal{R}_{\sigma, \mathcal{Q}}(U) \leq C \left(\|v_t\|_{L^2(\mathcal{Q})} + \|q^2 v\|_{L^2(\mathcal{Q})} + \sum_{j=0}^{2m} \sum_{\ell=0}^j \|q^{2m-\ell} D^\ell v\|_{L^2(\mathcal{Q})} \right)$$

$$\leq C \left(\|v_t\|_{L^2(\mathcal{Q})} + \sum_{j=0}^{2m} \|(Lq)^{2m-j} D^j v\|_{L^2(\mathcal{Q})} \right) \leq C \|\sigma g\|_{L^2(\mathcal{Q})},$$

so (3.2) follows.

Step 3. In this step, we verify the coercivity estimate.

By (2.6), (2.7) and (3.1) and $\mu = L^{2m} q^{2m}$,

$$|a_\mu(v, w)| \leq (C + L^{2m}) \mathcal{N}_{\sigma, \Omega}(v) \mathcal{N}_{\sigma^{-1}, \Omega}(w).$$

The corresponding lower bound is

$$a_\mu(v, \sigma^2 v) \geq c \mathcal{N}_{\sigma, \Omega}(v)^2 + (L^{2m} - C) \|q^m \sigma v\|_{L^2(\Omega)}^2.$$

Choosing $L^{2m} - C \geq B$ and using $q^m \geq q$, we obtain (3.4). $\square$

3.2 Backward Euler regularity and temporal error estimates

We now establish the weighted regularity of the backward Euler scheme and derive the temporal error estimates that will later be combined with the spatial comparison. Put $\mu_n = (Lq_n)^{2m}$. Define the cellwise extension by

$$\bar{v}(t) = \begin{cases} v^n, & t \in (t_{n-1}, t_n], \quad \text{for } D_+, \\ v^n, & t \in [t_n, t_{n+1}), \quad \text{for } D_-. \end{cases} \quad (3.13)$$

Write $\hat{v}$ for the continuous piecewise affine interpolant of the nodal sequence. Then $\partial_t \hat{v} = \overline{D_+ v}$ for D_+ and $-\partial_t \hat{v} = \overline{D_- v}$ for D_- , almost everywhere. The extensions $\bar{\mu}, \bar{g}$ use the same time cells as $\bar{v}$.

We first control the variation of the weights and the shift on each time cell. These bounds will be used repeatedly in the sequel.

Lemma 3.2. *There exist $\delta_{\text{time}}, C > 0$ such that, if $\delta t \leq 1$ and $\delta t q_{\max}^2 \leq \delta_{\text{time}}$, the following hold for every $L \geq 1$, n , and $x \in \Omega$.*

- (i) *For $t \in [t_{n-1}, t_n]$ and $i \in \{n-1, n\}$, $q(t) \asymp q_i$ and $\sigma(t, x) \asymp \sigma_i(x)$.*
- (ii) *For $t \in (t_{n-1}, t_n]$ with D_+ , or $t \in [t_n, t_{n+1})$ with D_- ,*

$$\left| \frac{q(t)}{q_n} - 1 \right| \leq C \delta t q_n, \quad \left| \frac{\sigma(t, x)}{\sigma_n(x)} - 1 \right| \leq C \delta t q_n^2. \quad (3.14)$$

On the same cells, the shift satisfies

$$|\mu(t) - \bar{\mu}(t)| \leq C L^{2m} \delta t q(t)^{2m+1}. \quad (3.15)$$

- (iii) *For D_+ , set*

$$H_n(x) = \frac{\sigma_n(x)/\sigma_{n-1}(x) - 1}{\delta t}.$$

Then

$$\|H_n\|_{L^\infty(\Omega)} \leq C q_n^2. \quad (3.16)$$

Moreover,

$$|\mu_{n+1} - \mu_n| \leq C L^{2m} \delta t q_n^{2m+1}. \quad (3.17)$$

Proof. Let $t \in [t_{n-1}, t_n]$ and $i \in \{n-1, n\}$. Choose δ_{time} so that $\beta \delta_{\text{time}} \leq 1/2$. By (2.1),

$$\left| \frac{1}{q(t)} - \frac{1}{q_i} \right| = \left| \int_{t_i}^t \frac{q'(s)}{q(s)^2} ds \right| \leq \beta \delta t. \quad (3.18)$$

Multiplying (3.18) by q_i gives

$$\left| \frac{q_i}{q(t)} - 1 \right| \leq \beta \delta t q_i \leq \frac{1}{2}.$$

Hence $q(t) \asymp q_i$ and $|q(t)/q_i - 1| \leq C \delta t q_i$. Using (3.8), we have

$$\left| \log \frac{\sigma(t, x)}{\sigma_i(x)} \right| = \left| \int_{t_i}^t \partial_s \log \sigma(s, x) ds \right| \leq C \delta t q_i^2, \quad (3.19)$$

and

$$\left| \frac{\sigma(t, x)}{\sigma_i(x)} - 1 \right| \leq C \delta t q_i^2 e^{C \delta t q_i^2} \leq C \delta t q_i^2. \quad (3.20)$$

By (3.19), $\sigma(t, x) \asymp \sigma_i(x)$. Taking $t = t_n$ and $i = n - 1$ in (3.20) and using $q_{n-1} \asymp q_n$ gives (3.16). By (2.1),

$$|\mu(t) - \mu_i| = 2mL^{2m} \left| \int_{t_i}^t q(s)^{2m-1} q'(s) ds \right| \leq CL^{2m} \delta t q_i^{2m+1}. \quad (3.21)$$

On either time cell associated with t_n , using (3.21) with $i = n$, $\bar{\mu}(t) = \mu_n$, and $q_n \asymp q(t)$, we obtain the bound for μ in (3.15). Taking $t = t_{n+1}$ and $i = n$ in (3.21) gives (3.17). $\square$

Unweighted step-uniform maximal parabolic regularity for backward Euler is standard (see [16, Theorem 3.1]). We prove the weighted periodic estimate below to keep the dependence on the shift and the weights explicit.

Lemma 3.3. *Let L_0 be as in Lemma 3.1. There exist $L_1 \geq L_0(1)$ and $\delta_{\text{time}}, C > 0$ such that, for every $L \geq L_1$, $\delta t \leq 1$, $\delta t q_{\max}^2 \leq \delta_{\text{time}}$, and $g \in \ell_{\delta t}^2(L^2(\Omega))$, the periodic problem*

$$D_{\pm} u^n + (A + \mu_n) u^n = g^n \quad (3.22)$$

has a unique solution $u \in \ell_{\delta t}^2(D(A))$ satisfying

$$\|\sigma D_{\pm} u\|_{\ell_{\delta t}^2(L^2(\Omega))} + \mathcal{R}_{\sigma, \delta t}(u) \leq C \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))} \quad (3.23)$$

for both orientations and all allowed weights.

Proof. We prove the lemma for D_+ . The case D_- follows by the time reflection $n \mapsto -n$, with the same constants.

Step 1. In this step, we obtain the unweighted graph estimate.

Let $v \in \ell_{\delta t}^2(D(A))$ be N -periodic. Expanding the squared norm and applying periodic summation by parts give

$$\begin{aligned} \|D_+ v + (A + \mu_n) v^n\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 &= \|D_+ v\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 + \|(A + \mu_n) v^n\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 + \delta t \|A^{1/2} D_+ v\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 \\ &\quad + \delta t \|\sqrt{\mu_n} D_+ v\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 - \sum_{n=0}^{N-1} (\mu_{n+1} - \mu_n) \|v^n\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.24)$$

By (3.17), $q_n \geq 1$, and positivity of A , we have

$$\left| \sum_{n=0}^{N-1} (\mu_{n+1} - \mu_n) \|v^n\|_{L^2(\Omega)}^2 \right| \leq CL^{2m} \delta t \sum_{n=0}^{N-1} q_n^{2m+1} \|v^n\|_{L^2(\Omega)}^2 \leq CL^{-2m} \|\mu_n v^n\|_{\ell_{\delta t}^2(L^2(\Omega))}^2, \quad (3.25)$$

and

$$\|(A + \mu_n) v^n\|_{L^2(\Omega)}^2 \geq \|A v^n\|_{L^2(\Omega)}^2 + \|\mu_n v^n\|_{L^2(\Omega)}^2. \quad (3.26)$$

For large L , combining (3.24)–(3.26), and applying (3.5) at each node yield

$$\|D_+ v\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 + \sum_{j=0}^{2m} \|(Lq)^{2m-j} D^j v\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 \leq C \|D_+ v + (A + \mu_n) v^n\|_{\ell_{\delta t}^2(L^2(\Omega))}^2. \quad (3.27)$$

Step 2. In this step, we study the conjugation of the recurrence.

By Assumption (A2), $v^n = \sigma_n u^n \in D(A)$. Conjugating (3.22) gives

$$D_+ v^n + (A + \mu_n) v^n = \sigma_n g^n + [A, \sigma_n] \sigma_n^{-1} v^n + H_n v^{n-1}. \quad (3.28)$$

By (2.5), (3.14) and (3.16) and periodic reindexing, it holds that

$$\|[A, \sigma_n] \sigma_n^{-1} v^n\|_{\ell_{\delta t}^2(L^2(\Omega))} \leq \frac{C}{L} \sum_{j=0}^{2m} \|(Lq)^{2m-j} D^j v\|_{\ell_{\delta t}^2(L^2(\Omega))}, \quad (3.29)$$

and

$$\begin{aligned} \|H_n v^{n-1}\|_{\ell_{\delta t}^2(L^2(\Omega))} &\leq C \|q_n^2 v^{n-1}\|_{\ell_{\delta t}^2(L^2(\Omega))} = C \|q_{n+1}^2 v^n\|_{\ell_{\delta t}^2(L^2(\Omega))} \\ &\leq C \|q_n^2 v^n\|_{\ell_{\delta t}^2(L^2(\Omega))} \leq CL^{-2m} \|\mu_n v^n\|_{\ell_{\delta t}^2(L^2(\Omega))}. \end{aligned} \quad (3.30)$$

By (3.28)–(3.30), choosing L sufficiently large allows us to absorb $[A, \sigma_n] \sigma_n^{-1} v^n$ and $H_n v^{n-1}$ into the left-hand side of (3.27). The identity

$$\sigma_n D_+ u^n = D_+ v^n - H_n v^{n-1},$$

together with the product rule and (3.8), then yields (3.23).

Existence and uniqueness follow by expansion in the eigenbasis of A : each scalar recurrence has factors $1 + \delta t(\lambda + \mu_n) > 1$ and a unique periodic solution. Estimate (3.27) permits passage from finite spectral sums to arbitrary $g \in \ell_{\delta t}^2(L^2(\Omega))$. $\square$

The continuous weighted graph estimate also holds when the shift is frozen on each time cell. This will be used to compare the discrete and continuous auxiliary problems.

Lemma 3.4. *Let L_0 be as in Lemma 3.1. There exists $C > 0$ such that, for every $L \geq L_0(1)$, one can choose $\delta_{\text{time}}(L) > 0$ for which the following holds whenever $\delta t \leq 1$ and $\delta t q_{\max}^2 \leq \delta_{\text{time}}(L)$. For either sign of ϵ and either cellwise extension $\bar{\mu}$ of μ_n , the periodic problem*

$$(\epsilon \partial_t + A + \bar{\mu})U = G \quad (3.31)$$

has a unique solution in $H^1(0, T; H) \cap L^2(0, T; D(A))$ with $U(0) = U(T)$ for every $G \in L^2(0, T; H)$, and

$$\|\sigma U_t\|_{L^2(\mathcal{Q})} + \mathcal{R}_{\sigma, \mathcal{Q}}(U) \leq C \|\sigma G\|_{L^2(\mathcal{Q})}. \quad (3.32)$$

Proof. Let $\mathcal{L}_\mu U := (\epsilon \partial_t + A + \mu)U$ on the subspace of $H^1(0, T; H) \cap L^2(0, T; D(A))$ with $U(0) = U(T)$. By Lemma 3.1, $\mathcal{L}_\mu$ is invertible. Denote the constant in (3.2) by C_0 . From (2.3) and (3.15),

$$\|\sigma(\bar{\mu} - \mu)U\|_{L^2(\mathcal{Q})} \leq C(L) \delta t q_{\max} \|q^{2m} \sigma U\|_{L^2(\mathcal{Q})} \leq C(L) \delta t q_{\max} \mathcal{R}_{\sigma, \mathcal{Q}}(U). \quad (3.33)$$

Choose $\delta_{\text{time}}(L)$ within the range of Lemma 3.2 so that $C_0 C(L) \delta_{\text{time}}(L) \leq 1/2$, with $C(L)$ from (3.33). Since $q_{\max} \geq 1$, we have $\delta t q_{\max} \leq \delta t q_{\max}^2 \leq \delta_{\text{time}}(L)$. For every $F \in L^2(\mathcal{Q})$, applying (3.2) and (3.33) to $\mathcal{L}_\mu^{-1} F$ gives

$$\|\sigma(\bar{\mu} - \mu) \mathcal{L}_\mu^{-1} F\|_{L^2(\mathcal{Q})} \leq C_0 C(L) \delta t q_{\max} \|\sigma F\|_{L^2(\mathcal{Q})} \leq \frac{1}{2} \|\sigma F\|_{L^2(\mathcal{Q})}. \quad (3.34)$$

By (3.34), the Neumann series in $L^2(\mathcal{Q})$, equipped with the norm $\|\sigma \cdot\|_{L^2(\mathcal{Q})}$, yields a unique solution of

$$F + (\bar{\mu} - \mu) \mathcal{L}_\mu^{-1} F = G.$$

The solution satisfies

$$\|\sigma F\|_{L^2(\mathcal{Q})} \leq 2 \|\sigma G\|_{L^2(\mathcal{Q})}.$$

Then $U = \mathcal{L}_\mu^{-1} F$ is the unique solution of (3.31), and (3.2) gives

$$\|\sigma U_t\|_{L^2(\mathcal{Q})} + \mathcal{R}_{\sigma, \mathcal{Q}}(U) \leq C_0 \|\sigma F\|_{L^2(\mathcal{Q})} \leq 2C_0 \|\sigma G\|_{L^2(\mathcal{Q})}.$$

Thus (3.32) holds with $C = 2C_0$, independent of L . $\square$

The estimates (3.23) and (3.32) yield the temporal L^2 error by duality. Spatial interpolation then gives the energy error.

Theorem 3.5. *Under the hypotheses and temporal restrictions of Lemmas 3.3 and 3.4, there exists $C > 0$ such that, for every $g \in \ell_{\delta t}^2(L^2(\Omega))$ and every allowed weight, the solutions u of (3.22) and U of (3.31) with $G = \bar{g}$ satisfy*

$$\|\sigma(U - \bar{u})\|_{L^2(\mathcal{Q})} \leq C\delta t \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (3.35)$$

Their weighted energy error satisfies

$$\mathcal{N}_{\sigma, \mathcal{Q}}(U - \bar{u}) \leq C\sqrt{\delta t} \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (3.36)$$

Proof. Step 1. In this step, we estimate the L^2 error by duality.

Put $\delta = \hat{u} - \bar{u}$, $E = U - \hat{u}$, and $e_t = U - \bar{u} = E + \delta$. The interpolation formula and (3.14) and (3.23) give

$$\|\sigma\delta\|_{L^2(\mathcal{Q})} \leq C\delta t \|\sigma D_{\pm} u\|_{\ell_{\delta t}^2(L^2(\Omega))} \leq C\delta t \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (3.37)$$

Subtracting (3.22) and (3.31) yields

$$\epsilon \partial_t E + (A + \bar{\mu})e_t = 0 \quad \text{in } L^2(0, T; H). \quad (3.38)$$

Let Y solve the periodic adjoint problem

$$(-\epsilon \partial_t + A + \bar{\mu})Y = \sigma^2 e_t. \quad (3.39)$$

Applying (3.32) with σ^{-1} gives $\|\sigma^{-1}Y_t\|_{L^2(\mathcal{Q})} \leq C\|\sigma e_t\|_{L^2(\mathcal{Q})}$. Both E and Y belong to $H^1(0, T; H) \cap L^2(0, T; D(A))$ and satisfy $E(0) = E(T)$ and $Y(0) = Y(T)$. Using (3.38) and (3.39), periodic integration by parts gives

$$\|\sigma e_t\|_{L^2(\mathcal{Q})}^2 = \int_0^T ((\epsilon \partial_t E, Y)_H + a_{\bar{\mu}}(e_t, Y)) dt - \epsilon \int_0^T (\delta, Y_t)_H dt = -\epsilon \int_0^T (\delta, Y_t)_H dt. \quad (3.40)$$

Cauchy's inequality in (3.40), together with (3.32) and (3.37), gives

$$\|\sigma e_t\|_{L^2(\mathcal{Q})} \leq C\|\sigma\delta\|_{L^2(\mathcal{Q})} \leq C\delta t \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (3.41)$$

Step 2. In this step, we obtain the energy error by spatial interpolation.

The graph estimates (3.23) and (3.32) and the cell bounds (3.14) give

$$\mathcal{R}_{\sigma, \mathcal{Q}}(e_t) \leq \mathcal{R}_{\sigma, \mathcal{Q}}(U) + \mathcal{R}_{\sigma, \mathcal{Q}}(\bar{u}) \leq C\|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (3.42)$$

For $v \in L^2(0, T; H^{2m}(\Omega))$, fix time and put $w = \sigma v$. Sobolev interpolation and Young's inequality give, for $0 \leq j \leq m$,

$$q^{2(m-j)} \|D^j w\|_{L^2(\Omega)}^2 \leq C\|w\|_{L^2(\Omega)} (\|w\|_{H^{2m}(\Omega)} + q^{2m} \|w\|_{L^2(\Omega)}). \quad (3.43)$$

Since $q \geq 1$, the product rule and (2.2) and (3.8) convert the sum of (3.43) over j to

$$\mathcal{N}_{\sigma, \Omega}(v)^2 \leq C\|\sigma v\|_{L^2(\Omega)} \mathcal{R}_{\sigma, \Omega}(v).$$

Integration in time, Cauchy's inequality, and (2.3) yield

$$\mathcal{N}_{\sigma, \mathcal{Q}}(v)^2 \leq C\|\sigma v\|_{L^2(\mathcal{Q})} \mathcal{R}_{\sigma, \mathcal{Q}}(v). \quad (3.44)$$

Applying (3.44) to e_t and using (3.41) and (3.42) gives $\mathcal{N}_{\sigma, \mathcal{Q}}(e_t)^2 \leq C\delta t \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}^2$. $\square$

Remark 3.1. The constants in (3.23), (3.35) and (3.36) are uniform in $\delta t \lambda$, where λ ranges over the eigenvalues of A . Thus no spectral CFL condition $\delta t \lambda \leq C$ is required.

4 Weighted error estimates for spatial discretization

In this section we compare the spatially continuous and fully discrete auxiliary problems on the same Euler grid. The temporal estimate in Theorem 3.5 will then yield the full space–time error.

Throughout this section we assume Assumptions (A1) to (A3), Assumptions (N1), (N3) and (N4), and a lift $\mathcal{L}_h$ satisfying $J_h \mathcal{L}_h = \Gamma_h$ and (2.14). We choose the margin B in (3.4) sufficiently large in terms of the constant in (2.19) and the fixed weight bounds, and then fix L as in Lemmas 3.1 and 3.3.

Under these assumptions, there exist $\eta_*, \delta_x, \delta_{\text{time}}, C > 0$, with $\delta_x \leq \delta_{\text{num}}$, such that the estimates below hold whenever

$$a_E \leq 1, \quad \eta_A \leq \eta_*, \quad \delta t \leq 1, \quad h q_{\max} \leq \delta_x, \quad \delta t q_{\max}^2 \leq \delta_{\text{time}}.$$

The temporal threshold is chosen within the ranges of Lemmas 3.3 and 3.4.

For $g \in \ell_{\delta t}^2(H)$, let u solve (3.22), and let $u_h \in \ell_{\delta t}^2(X_h)$ solve the periodic auxiliary problem

$$D_{\pm} u_h^n + A_h u_h^n + \mu_n G_h u_h^n = J_h g^n. \quad (4.1)$$

Here the time difference is taken with respect to the inner product of X_h , while $G_h = J_h R_h$ represents the reconstruction mass in the shift. It is readily seen that (4.1) has a unique periodic solution.

We now compare these two solutions. The following lemma combines weighted Céa-type energy and Aubin–Nitsche L^2 estimates, allowing nonorthogonal transfers and stiffness defects.

Lemma 4.1. *Under the assumptions of this section, let u and u_h solve (3.22) and (4.1) for $g \in \ell_{\delta t}^2(H)$. For every allowed weight σ , $e_s = u - R_h u_h$ satisfies*

$$\mathcal{N}_{\sigma, \delta t}(e_s) \leq C(a_E + \eta_A) \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (4.2)$$

The weighted L^2 error satisfies

$$\|\sigma e_s\|_{\ell_{\delta t}^2(L^2(\Omega))} \leq C(a_0 + a_E^2 + \eta_A) \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (4.3)$$

Recall that a_E, a_0 are defined in (2.16) and (2.17) and η_A in (2.13).

Proof. Step 1. In this step, we estimate the weighted energy error.

We only prove the estimate for D_+ . Time reflection gives the case D_- . Put $\eta^n = u^n - Q_h u^n$ and $e_h^n = u_h^n - J_h u^n$, so that $e_s = \eta - R_h e_h$. Subtracting (3.22) and (4.1) and using (2.9) and (2.12), we obtain, for every $v \in X_h$,

$$(D_+ e_h^n, v)_{X_h} + a_{\mu_n}(R_h e_h^n, R_h v) + \mathfrak{b}_h(e_h^n, v) = a_{\mu_n}(\eta^n, R_h v) - \mathfrak{b}_h(J_h u^n, v). \quad (4.4)$$

Set $\mathcal{T}_n = \mathcal{T}_{\sigma, h}(t_n)$ and extend this sequence N -periodically, using the endpoint compatibility in Assumption (N4). We test (4.4) with $\mathcal{T}_n e_h^n$ and sum over one period. Self-adjointness and periodic reindexing give

$$\delta t \sum_{n=0}^{N-1} (D_+ e_h^n, \mathcal{T}_n e_h^n)_{X_h} = \frac{1}{2} \sum_{n=0}^{N-1} (e_h^n, (\mathcal{T}_n - \mathcal{T}_{n+1}) e_h^n)_{X_h} + \frac{1}{2} \sum_{n=0}^{N-1} (e_h^n - e_h^{n-1}, \mathcal{T}_n (e_h^n - e_h^{n-1}))_{X_h}. \quad (4.5)$$

The second sum is nonnegative. By (2.19) and (3.14), the absolute value of the first sum is bounded by $C \|q \sigma R_h e_h\|_{\ell_{\delta t}^2(L^2(\Omega))}^2$ and is absorbed by the margin B in (3.4).

The product rule and (2.20) and (3.8) imply

$$\mathcal{N}_{\sigma^{-1}, \Omega}(R_h \mathcal{T}_{\sigma, h} z) \leq C \mathcal{N}_{\sigma, \Omega}(R_h z), \quad z \in X_h.$$

By (2.13), (2.20) and (3.3),

$$\delta t \sum_{n=0}^{N-1} \left(|a_{\mu_n}(R_h e_h^n, R_h \mathcal{T}_n e_h^n - \sigma_n^2 R_h e_h^n)| + |\mathfrak{b}_h(e_h^n, \mathcal{T}_n e_h^n)| \right) \leq (C(L) h q_{\max} + C \eta_A) \mathcal{N}_{\sigma, \delta t}(R_h e_h)^2. \quad (4.6)$$

After fixing L , we choose δ_x and η_* sufficiently small. Absorbing the left-hand side of (4.6) into the coercive term and estimating the right side of (4.4) by (2.13) and (3.3), we obtain

$$\mathcal{N}_{\sigma,\delta t}(R_h e_h) \leq C(\mathcal{N}_{\sigma,\delta t}(\eta) + \eta_A \mathcal{N}_{\sigma,\delta t}(Q_h u)).$$

Now (2.16) gives $\mathcal{N}_{\sigma,\delta t}(\eta) \leq a_E \mathcal{R}_{\sigma,\delta t}(u)$ and $\mathcal{N}_{\sigma,\delta t}(Q_h u) \leq (1 + a_E) \mathcal{R}_{\sigma,\delta t}(u)$. Since $a_E \leq 1$, (3.23) yields

$$\mathcal{N}_{\sigma,\delta t}(R_h e_h) \leq C(a_E + \eta_A) \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (4.7)$$

Using $e_s = \eta - R_h e_h$ proves the energy estimate in (4.2). The same bounds, together with $R_h u_h = Q_h u + R_h e_h$, give the stability estimate

$$\mathcal{N}_{\sigma,\delta t}(R_h u_h) \leq C \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (4.8)$$

Step 2. In this step, we estimate the L^2 error by duality.

Let $Y \in \ell_{\delta t}^2(D(A))$ solve the periodic problem

$$D_- Y^n + (A + \mu_n) Y^n = \sigma_n^2 e_s^n. \quad (4.9)$$

Applying (3.23) with σ^{-1} gives

$$\mathcal{R}_{\sigma^{-1},\delta t}(Y) \leq C \|\sigma e_s\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (4.10)$$

Testing (4.1) with $J_h Y^n$ and using (2.9) and (2.12), we obtain

$$(R_h D_+ u_h^n, Y^n)_H + a_{\mu_n} (R_h u_h^n, Q_h Y^n) + \mathfrak{b}_h(u_h^n, J_h Y^n) = (g^n, Q_h Y^n)_H.$$

Since $D_+^* = D_-$ and R_h is time independent, (3.22) and (4.9) now give

$$\begin{aligned} \|\sigma e_s\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 &= \delta t \sum_{n=0}^{N-1} ((D_+ e_s^n, Y^n)_H + a_{\mu_n} (e_s^n, Y^n)) \\ &= \delta t \sum_{n=0}^{N-1} ((g^n, Y^n - Q_h Y^n)_H - a_{\mu_n} (R_h u_h^n, Y^n - Q_h Y^n)) + \delta t \sum_{n=0}^{N-1} \mathfrak{b}_h(u_h^n, J_h Y^n) \\ &= \delta t \sum_{n=0}^{N-1} ((D_+ u^n, Y^n - Q_h Y^n)_H + a_{\mu_n} (e_s^n, Y^n - Q_h Y^n)) + \delta t \sum_{n=0}^{N-1} \mathfrak{b}_h(u_h^n, J_h Y^n). \end{aligned} \quad (4.11)$$

We estimate the three terms in the final expression separately. By (2.17) and (3.23),

$$\left| \delta t \sum_{n=0}^{N-1} (D_+ u^n, Y^n - Q_h Y^n)_H \right| \leq C a_0 \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))} \mathcal{R}_{\sigma^{-1},\delta t}(Y).$$

Estimate (4.2), together with (2.16) and (3.3), gives

$$\left| \delta t \sum_{n=0}^{N-1} a_{\mu_n} (e_s^n, Y^n - Q_h Y^n) \right| \leq C a_E (a_E + \eta_A) \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))} \mathcal{R}_{\sigma^{-1},\delta t}(Y).$$

Finally, (2.13), (2.16) and (4.8) and $\mathcal{N}_{\sigma^{-1},\delta t}(Q_h Y) \leq (1 + a_E) \mathcal{R}_{\sigma^{-1},\delta t}(Y)$ imply

$$\left| \delta t \sum_{n=0}^{N-1} \mathfrak{b}_h(u_h^n, J_h Y^n) \right| \leq C \eta_A \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))} \mathcal{R}_{\sigma^{-1},\delta t}(Y).$$

Combining these estimates with (4.10) and (4.11) and $a_E \leq 1$ proves the L^2 estimate in (4.3). $\square$

Combining the spatial and temporal estimates, we obtain the full space-time error.

Theorem 4.2. *Under the assumptions of this section, let U solve (3.31) with $G = \bar{g}$, and let u_h solve (4.1), for $g \in \ell_{\delta t}^2(H)$. For either orientation and every allowed weight, with $\varepsilon_E, \varepsilon_0$ defined in (2.25),*

$$\mathcal{N}_{\sigma, \mathcal{Q}}(U - R_h \bar{u}_h) \leq C \varepsilon_E \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (4.12)$$

The weighted L^2 error satisfies

$$\|\sigma(U - R_h \bar{u}_h)\|_{L^2(\mathcal{Q})} \leq C \varepsilon_0 \|\sigma g\|_{\ell_{\delta t}^2(L^2(\Omega))}. \quad (4.13)$$

Proof. Decompose $U - R_h \bar{u}_h = (U - \bar{u}) + \overline{u - R_h u_h}$ and apply (3.14), (3.35), (3.36), (4.2) and (4.3). $\square$

5 Proof of the transfer theorem

We choose the source term of the shifted continuous problem so that its fully discrete solution coincides with the prescribed state. We then apply the continuous Carleman estimate and use Theorem 4.2 to recover the original discrete energy, residual, and observation.

Proof of Theorem 2.1. Step 1. In this step, we choose the auxiliary load.

Given the pair (z, f) in (2.28), we set

$$g^n = \mathcal{L}_h f^n + \mu_n R_h z^n, \quad (5.1)$$

and let U solve (3.31) with $G = \bar{g}$. Recall the extension defined in (3.13). By Assumption (N2) and (2.10) and (2.28),

$$J_h g^n = \Gamma_h f^n + \mu_n G_h z^n = D_{\pm} z^n + A_h z^n + \mu_n G_h z^n.$$

Uniqueness in (4.1) gives $u_h = z$. Put $e = U - R_h \bar{z}$. Using (3.31) with $G = \bar{g}$, then (3.13) and (5.1) and $U = R_h \bar{z} + e$, we obtain

$$(\epsilon \partial_t + A)U = \bar{g} - \bar{\mu}U = \overline{\mathcal{L}_h f} + \bar{\mu}R_h \bar{z} - \bar{\mu}(R_h \bar{z} + e) = \overline{\mathcal{L}_h f} - \bar{\mu}e. \quad (5.2)$$

This identity is the parabolic counterpart of (1.14).

Step 2. In this step, we compare the continuous and discrete quantities.

We now apply the continuous Carleman estimate (2.8) to $U = R_h \bar{z} + e$. By (2.11), (2.14), (2.21), (3.14) and (5.2),

$$\begin{aligned} \mathcal{E}_{q,h,\delta t}(z) &\leq C(\|f\|_{\ell_{\delta t}^2(\mathfrak{F}_h),\theta}^2 + \|q^{o/2} B_h^* z\|_{\ell_{\delta t}^2(U_h),\theta}^2) + C \sum_{j=1}^m \|q^{c_j/2} \theta \mathcal{D}_j e\|_{L^2(\mathcal{Q})}^2 + C \|q^{c_0/2} \theta e\|_{L^2(\mathcal{Q})}^2 \\ &\quad + C \|q^{o/2} \theta e\|_{L^2(\mathcal{Q}_{\omega_0})}^2 + C \|\bar{\mu} \theta e\|_{L^2(\mathcal{Q})}^2. \end{aligned} \quad (5.3)$$

Step 3. In this step, we estimate the error terms on the right-hand side of (5.3).

By the order of $\mathcal{D}_j$ in Assumption (A4) and (2.3), $\|q^{m-j} \sigma \mathcal{D}_j e\|_{L^2(\mathcal{Q})} \leq C \mathcal{N}_{\sigma, \mathcal{Q}}(e)$. For each j , we apply the energy estimate in (4.12) with $\sigma = q^{(c_j - 2(m-j))/2} \theta$ when the exponent is positive, and with $\sigma = \theta$ otherwise. These weights are covered by (2.24). Using (2.23) and (5.1), we obtain

$$\sum_{j=1}^m \|q^{c_j/2} \theta \mathcal{D}_j e\|_{L^2(\mathcal{Q})}^2 \leq C \varepsilon_E^2 (\|q^{d_*/2} \theta \mathcal{L}_h f\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 + \|q^{2m+d_*/2} \theta R_h z\|_{\ell_{\delta t}^2(L^2(\Omega))}^2). \quad (5.4)$$

For the remaining errors, we use the L^2 estimate in (4.13) with the weights $q^{c_0/2} \theta$, $q^{o/2} \theta$, and $q^{2m} \theta$.

By (3.13) and (3.14), $\bar{\mu}(t) = \mu_n = (Lq_n)^{2m} \asymp (Lq(t))^{2m}$ on each cell. Together with $b_* = \max\{c_0, o, 4m\}$ in (2.23), this gives

$$\|q^{c_0/2} \theta e\|_{L^2(\mathcal{Q})}^2 + \|q^{o/2} \theta e\|_{L^2(\mathcal{Q}_{\omega_0})}^2 + \|\bar{\mu} \theta e\|_{L^2(\mathcal{Q})}^2 \leq C \varepsilon_0^2 (\|q^{b_*/2} \theta \mathcal{L}_h f\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 + \|q^{2m+b_*/2} \theta R_h z\|_{\ell_{\delta t}^2(L^2(\Omega))}^2). \quad (5.5)$$

Step 4. In this step, we absorb the state error.

For $\nu \in \{d_*, b_*\}$, (2.11), (2.14) and (2.22) give

$$\|q^{\nu/2}\theta\mathcal{L}_hf\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 \leq Cq_{\max}^\nu\|f\|_{\ell_{\delta t}^2(\mathfrak{F}_h),\theta}^2.$$

The state term is bounded by

$$\|q^{2m+\nu/2}\theta R_h z\|_{\ell_{\delta t}^2(L^2(\Omega))}^2 \leq C\max\{1, q_{\max}^{4m+\nu-c_0}\}\mathcal{E}_{q,h,\delta t}(z).$$

Combining (5.3)–(5.5) with (2.26) and (2.27), we obtain

$$\mathcal{E}_{q,h,\delta t}(z) \leq C(1 + \Lambda_f)\|f\|_{\ell_{\delta t}^2(\mathfrak{F}_h),\theta}^2 + C\|q^{o/2}B_h^*z\|_{\ell_{\delta t}^2(U_h),\theta}^2 + C\Lambda_z\mathcal{E}_{q,h,\delta t}(z). \quad (5.6)$$

With the constant C in (5.6) fixed, choose c_* such that $Cc_* \leq 1/2$. Since $\Lambda_f \leq 1$ and $\Lambda_z \leq c_*$, we get (2.29). Finally, taking $f = \mathcal{I}_h(D_\pm z + A_h z)$ and using (2.15) gives (2.30). $\square$

Proof of Corollary 2.2. By (2.25) and the approximation bounds in Corollary 2.2, for $h \leq 1$, we have

$$\varepsilon_E^2 \leq C(h^{2m} + \delta t), \quad \varepsilon_0^2 \leq C(h^{4m} + |\delta t|^2).$$

Since $q_{\max} \geq 1$, (2.26), (2.27) and (2.31) then give

$$\Lambda_f + \Lambda_z \leq C[(h^{2m} + \delta t)q_{\max}^{\nu_E} + (h^{4m} + |\delta t|^2)q_{\max}^{\nu_0}].$$

For each positive exponent, (2.32) bounds the spatial terms by $C\kappa^{\nu_E}$ or $C\kappa^{\nu_0}$, and likewise for the temporal terms. All remaining powers of h and δt are nonnegative. For zero exponents, $\lim_{h,\delta t \rightarrow 0}(h^{2m} + \delta t) = 0$ and $\lim_{h,\delta t \rightarrow 0}(h^{4m} + |\delta t|^2) = 0$.

We choose κ so that the positive-exponent terms are below the fixed thresholds for Λ_f and Λ_z , and also $\kappa \leq \delta_x$, $\kappa^2 \leq \delta_{\text{time}}$. Then we decrease $h_0, \Delta_0 \leq 1$ to control any zero-exponent terms and to ensure $a_E \leq 1$, $\eta_A \leq \eta_*$.

Since $\alpha_x \leq 1$ and $\alpha_t \leq 1/2$, the same choice gives $hq_{\max} \leq \delta_x$ and $\delta tq_{\max}^2 \leq \delta_{\text{time}}$. Thus Theorem 2.1 applies. $\square$

Proof of Corollary 2.3. For $m = 1$ and $(c_0, c_1; o) = (3, 1; 3)$, (2.23) and (2.31) give $(d_*, b_*, \nu_E, \nu_0) = (1, 4, 2, 5)$. Substituting these values into (2.25)–(2.27), we obtain

$$\Lambda_f \leq C[(h^2 + \delta t)q_{\max} + (h^4 + |\delta t|^2)q_{\max}^4].$$

The state defect satisfies

$$\Lambda_z \leq C[(h^2 + \delta t)q_{\max}^2 + (h^4 + |\delta t|^2)q_{\max}^5].$$

Finally, (2.32) gives $(\alpha_x, \alpha_t) = (4/5, 2/5)$, which proves (2.33). $\square$

6 Finite-element and finite-difference discretizations

We now verify Assumptions (N1) to (N5) for two spatial discretizations of the heat equation. The first is the consistent-mass P_1 finite-element method on locally graded simplicial meshes in dimensions two and three, and the second is the standard Cartesian finite-difference scheme in any fixed dimension. In both cases, the continuous analytic assumptions are supplied by Corollary A.2 and Proposition A.3, with $\tau_* = 4$, so that the transfer theorem and its corollaries apply once the numerical hypotheses are checked. Throughout this section, weighted estimates are evaluated at a fixed time. For each realization, we fix an open control region ω with $\omega_0 \Subset \omega \Subset \Omega$. Constants may also depend on ω . The finite-element case is treated in Subsection 6.1, and the finite-difference case in Subsection 6.2.

6.1 Consistent-mass P_1 elements in two and three dimensions

We now verify the numerical assumptions for the consistent-mass P_1 finite-element discretization of the heat equation in two and three dimensions.

Let $d \in \{2, 3\}$ and let $\Omega \subset \mathbb{R}^d$ be a fixed bounded convex polytope. Take $V = H_0^1(\Omega)$, $a(v, w) = (\nabla v, \nabla w)_{L^2(\Omega)}$, and $A = -\Delta_D$. Let $\mathcal{T}_h = \{K\}$ be a conforming, uniformly shape-regular simplicial mesh of Ω , with $h_K = \text{diam } K$ and $h = \max_{K \in \mathcal{T}_h} h_K$.

Set $\Gamma_d = (\sqrt{d+2} + 1)/(\sqrt{d+2} - 1)$, so that $\Gamma_2 = 3$ and $\Gamma_3 = (3 + \sqrt{5})/2$. Assume there are positive numbers ℓ_K satisfying

$$ch_K \leq \ell_K \leq Ch_K, \quad \gamma^{-1} \leq \ell_K/\ell_{K'} \leq \gamma < \Gamma_d \quad \text{whenever } K, K' \text{ share a vertex}, \quad (6.1)$$

where $c, C > 0$ and $1 \leq \gamma < \Gamma_d$ are fixed. Write $\ell|_K = \ell_K$ for the associated elementwise mesh-size function.

Set $X_h = V_h = \{v \in H_0^1(\Omega) : v|_K \in P_1\}$, equipped with the $L^2(\Omega)$ inner product, and define $(A_h z, w)_{X_h} = a(z, w)$. The reconstruction R_h is inclusion and $J_h = \Pi_h$, where Π_h is the $L^2(\Omega)$ orthogonal projection onto V_h . Thus $Q_h = R_h \Pi_h$ is identified with Π_h , $G_h = \text{Id}_{X_h}$, and $\eta_A = 0$.

Take $\mathfrak{F}_h = H$, $\Gamma_h = J_h$, $\mathcal{L}_h = \text{Id}_H$, and $\mathcal{I}_h = R_h$. For $U_h = L^2(\omega)$, define $(B_h u, z)_{X_h} = \int_\omega u z \, dx$, so that $B_h^* z = z|_\omega$. The weighted norms are

$$\|z\|_{X_h, \sigma} := \|\sigma z\|_{L^2(\Omega)}, \quad \|f\|_{\mathfrak{F}_h, \sigma} := \|\sigma f\|_{L^2(\Omega)}, \quad \|w\|_{U_h, \sigma} := \|\sigma w\|_{L^2(\omega)}.$$

The following proposition verifies Assumptions (N1) to (N5) for P_1 elements under the mesh grading condition.

Proposition 6.1. *Under (6.1), there exist $\delta_{\text{num}}, C > 0$, depending additionally on the grading bounds in (6.1) and the shape-regularity bound, such that Assumptions (N1) to (N5) hold whenever $hq_{\max} \leq \delta_{\text{num}}$, with $m = 1$, $a_E = Ch$, $a_0 = Ch^2$, $\eta_A = 0$, and $\mathcal{T}_{\sigma, h} z = \Pi_h(\sigma^2 z)$.*

Proof. We verify the assumptions in three steps: weighted stability of the L^2 projection, weighted approximation estimates, and the weighted test estimates.

Step 1. In this step, we establish the weighted stability of the L^2 projection.

We use the weighted projection stability in [10, Theorems 4.4 and 4.14(i), Remark 4.5] for degree one, $p = 2$, and zero boundary values on $\partial\Omega$. The admissible weight-grading threshold is Γ_d . Consider the four weights

$$\sigma, \quad \sigma^{-1}, \quad \ell^{-1}\sigma, \quad \ell^{-1}\sigma^{-1}. \quad (6.2)$$

For each such weight ζ , (3.8) and (6.1) give $\zeta(x)/\zeta(y) \leq \gamma e^{Chq}$ for a.e. $x \in K$, $y \in K'$ whenever K, K' share a vertex. Choose $\delta_{\text{num}} \leq 1$ so that

$$\gamma e^{Chq_{\max}} \leq (\gamma + \Gamma_d)/2 < \Gamma_d, \quad \text{for } hq_{\max} \leq \delta_{\text{num}}.$$

The cited stability result then gives, uniformly for every ζ in (6.2),

$$\|\zeta \Pi_h v\|_{L^2(\Omega)} \leq C \|\zeta v\|_{L^2(\Omega)}. \quad (6.3)$$

Step 2. In this step, we derive the weighted approximation estimates.

For each $K \in \mathcal{T}_h$, choose $x_K \in K$ and set $\sigma_K = \sigma(x_K)$. By (3.8),

$$e^{-Chq} \leq \frac{\sigma(x)}{\sigma_K} \leq e^{Chq}, \quad x \in K. \quad (6.4)$$

Thus $hq \leq \delta_{\text{num}}$ makes σ uniformly comparable to σ_K on K . The same holds for σ^{-1} and, by convexity of Ω , on patches of diameter $O(h)$. Let $v \in D(A)$ and let I_h be the nodal interpolant, which is well defined since $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ for $d = 2, 3$. The standard local interpolation estimate in [8, Theorem 4.4.4], together with (6.4) and $\ell_K \asymp h_K$, gives

$$\|\sigma(v - I_h v)\|_{L^2(\Omega)} \leq C \|\ell^2 \sigma D^2 v\|_{L^2(\Omega)}, \quad (6.5)$$

and

$$\|\sigma \nabla(v - I_h v)\|_{L^2(\Omega)} + \|\ell^{-1} \sigma(v - I_h v)\|_{L^2(\Omega)} \leq C \|\ell \sigma D^2 v\|_{L^2(\Omega)}. \quad (6.6)$$

The local inverse estimate [8, Lemma 4.5.3], with the same weight comparison, yields

$$\|\sigma \nabla z_h\|_{L^2(\Omega)} \leq C \|\ell^{-1} \sigma z_h\|_{L^2(\Omega)}, \quad z_h \in V_h. \quad (6.7)$$

Using $v - \Pi_h v = (v - I_h v) - \Pi_h(v - I_h v)$, (6.3) and (6.5)–(6.7), and $hq \leq 1$, we obtain

$$\|\sigma(v - \Pi_h v)\|_{L^2(\Omega)} \leq Ch^2 \mathcal{R}_{\sigma, \Omega}(v).$$

The corresponding energy estimate is

$$\mathcal{N}_{\sigma, \Omega}(v - \Pi_h v) \leq Ch \mathcal{R}_{\sigma, \Omega}(v).$$

Step 3. In this step, we construct the test operator and verify the superapproximation estimate.

For $z_h \in V_h$, set $w = \sigma^2 z_h - I_h(\sigma^2 z_h)$. We use the elementwise superapproximation argument in the proof of [9, Theorem 2.1], with the nodal interpolant. By (3.8) and (6.4), the normalized weight σ/σ_K is uniformly comparable to one and satisfies $|D^j(\sigma/\sigma_K)| \leq Cq^j$ on K for $j = 1, 2$. The cited argument therefore applies with length scale $q^{-1} \geq h_K$. For the L^2 bounds, since $D^2 z_h = 0$ on each simplex, we also use (6.5) elementwise and

$$|D^2(\sigma^2 z_h)| \leq C\sigma^2(q|\nabla z_h| + q^2|z_h|).$$

Together these give

$$\|\sigma^{-1} \nabla w\|_{L^2(\Omega)} + \|\ell^{-1} \sigma^{-1} w\|_{L^2(\Omega)} \leq Ch q_{\max} \mathcal{N}_{\sigma, \Omega}(z_h). \quad (6.8)$$

The zeroth-order term also satisfies

$$q \|\sigma^{-1} w\|_{L^2(\Omega)} \leq Ch^2 q_{\max}^2 \mathcal{N}_{\sigma, \Omega}(z_h). \quad (6.9)$$

Since $I_h(\sigma^2 z_h) \in V_h$, we have

$$\Pi_h(\sigma^2 z_h) - \sigma^2 z_h = \Pi_h w - w. \quad (6.10)$$

Applying (6.3) with $\zeta = \sigma^{-1}, \ell^{-1} \sigma^{-1}$ and (6.7) with σ^{-1} , combining (6.8)–(6.10), we obtain (2.20).

The test family is T -periodic because σ is. Orthogonality gives the norm identity

$$(z_h, \mathcal{T}_{\sigma, h} z_h)_{X_h} = \|\sigma z_h\|_{L^2(\Omega)}^2 \quad (6.11)$$

and the temporal derivative identity

$$(z_h, \mathcal{T}'_{\sigma, h} z_h)_{X_h} = \int_{\Omega} (\sigma^2)' z_h^2 dx. \quad (6.12)$$

From (3.8), (6.11) and (6.12), we get (2.18) and (2.19).

Finally, the definitions give the exact stiffness identity

$$(A_h z, w)_{X_h} = a(R_h z, R_h w), \quad z, w \in X_h.$$

The load maps satisfy $J_h \mathcal{L}_h = \Gamma_h$ and $\Gamma_h \mathcal{I}_h = \text{Id}_{X_h}$. Since R_h is inclusion and A_h is self-adjoint and nonnegative, these verify Assumptions (N1) and (N2) with $\eta_A = 0$. All norm comparisons in (2.11), (2.14) and (2.15) are equalities. Moreover, $B_h^* z = z|_{\omega}$ and $\omega_0 \subset \omega$ give

$$\|\sigma R_h z\|_{L^2(\omega_0)} \leq \|\sigma z\|_{L^2(\omega)} = \|B_h^* z\|_{U_h, \sigma},$$

which proves Assumption (N5). $\square$

From Corollary A.2 and Proposition A.3, the preceding verification allows us to apply Theorem 2.1 and Corollary 2.3, giving the heat range in (2.33).

6.2 Standard Cartesian finite-difference discretization

We next verify the numerical assumptions for the standard Cartesian finite-difference discretization of the heat equation in any fixed dimension.

Fix $d \geq 1$ and let $\Omega = \prod_{r=1}^d (0, L_r)$, $V = H_0^1(\Omega)$, and $a(v, w) = (\nabla v, \nabla w)_{L^2(\Omega)}$. For integers $N_r \geq 2$, define the uniform Cartesian grid by

$$h_r = L_r/N_r, \quad x_{r,j} = jh_r, \quad 0 \leq j \leq N_r, \quad h = \max_r h_r.$$

Assume the steps h_r are mutually comparable. The largest reconstruction-element diameter is then comparable to h .

For interior multi-indices $i \in \prod_{r=1}^d \{1, \dots, N_r - 1\}$, impose zero boundary values and set $\omega_i = \prod_{r=1}^d h_r$. The diagonal physical inner product is

$$(z, w)_{X_h} = \sum_i \omega_i z_i w_i.$$

The standard $(2d+1)$ -point Laplacian is

$$(A_h z)_i = \sum_{r=1}^d \frac{2z_i - z_{i+e_r} - z_{i-e_r}}{h_r^2}, \quad (6.13)$$

where e_r is the r th coordinate unit vector.

To connect this scheme with the continuous problem, we subdivide each box into $d!$ simplices by the Kuhn triangulation, using the same orientation in every box. For the interior nodal basis functions ϕ_i , set $R_h z = \sum_i z_i \phi_i$ and $(J_h v)_i = \omega_i^{-1} (v, \phi_i)_H$. Thus $Q_h = R_h J_h$ and $(G_h z)_i = \omega_i^{-1} \sum_j (\phi_j, \phi_i)_H z_j$.

Take $\mathfrak{F}_h = X_h$ and $\Gamma_h = \mathcal{I}_h = \text{Id}_{X_h}$. The control space consists of nodal values at nodes in ω , with the restricted X_h inner product. B_h is zero extension and B_h^* is restriction. The weighted norms are

$$\|z\|_{X_h, \sigma}^2 := \sum_i \omega_i \sigma(x_i)^2 z_i^2, \quad \|f\|_{\mathfrak{F}_h, \sigma} := \|f\|_{X_h, \sigma}, \quad \|w\|_{U_h, \sigma}^2 := \sum_{i: x_i \in \omega} \omega_i \sigma(x_i)^2 w_i^2.$$

The following proposition shows that the Cartesian difference scheme satisfies the same numerical assumptions as the finite-element scheme, through its piecewise affine reconstruction.

Proposition 6.2. *There exist $\delta_{\text{num}}, C > 0$, depending additionally on the comparability bound for h_r , such that (6.13) satisfies Assumptions (N1) to (N5) whenever $h q_{\max} \leq \delta_{\text{num}}$, with $m = 1$, $a_E = Ch$, $a_0 = Ch^2$, $\eta_A = 0$, and the test operator*

$$(\mathcal{T}_{\sigma, h} z)_i = \sigma(x_i)^2 z_i.$$

Proof. We first prove the exact stiffness identity

$$(A_h z, w)_{X_h} = a(R_h z, R_h w). \quad (6.14)$$

A Kuhn simplex associated with a permutation π has vertices $v_0, \dots, v_d$ with $v_j - v_{j-1} = h_{\pi_j} e_{\pi_j}$. Since $R_h z$ is affine on the simplex,

$$\partial_{\pi_j} R_h z = \frac{z(v_j) - z(v_{j-1})}{h_{\pi_j}}.$$

Each coordinate edge $[x_i, x_{i+e_r}]$ with interior transverse indices belongs to $d!$ such simplices, one for each permutation. Each simplex has volume $\prod_s h_s / d!$, so the total contribution of this edge to $a(R_h z, R_h w)$ is

$$\left(\prod_{s=1}^d h_s \right) \frac{z_{i+e_r} - z_i}{h_r} \frac{w_{i+e_r} - w_i}{h_r}.$$

Edges contained in the boundary contribute zero. Summation by parts in (6.13) gives the same contributions to $(A_h z, w)_{X_h}$, proving (6.14) and hence $\mathfrak{b}_h = 0$. The mass, load, approximation, test, and observation estimates are proved in Appendix B. $\square$

By Corollaries A.2 and 2.3 and Proposition A.3, the estimate (2.30) holds for $h \leq h_0$, $\delta t \leq \Delta_0$, and $q_{\max} \leq S_{h,\delta t}$, with the constants in Corollary 2.3.

Remark 6.1. The result extends to the standard conservative $(2d+1)$ -point scheme on smoothly stretched tensor grids $x_{r,j} = F_r(j/N_r)$, where the $F_r : [0, 1] \rightarrow [0, L_r]$ are fixed C^2 diffeomorphisms with $F'_r \geq c > 0$ and the N_r^{-1} are mutually comparable. We use the same Kuhn reconstruction and diagonal mass weights given by products of averaged adjacent steps.

With h denoting the largest step, adjacent steps differ by $O(h^2)$. The local stiffness and nodal-moment estimates give the weighted bounds $a_E = O(h)$ and $a_0 + \eta_A = O(h^2)$. Thus Corollary 2.3 gives the same heat parameter range, with constants depending additionally on the fixed C^2 bounds, the lower bound for F'_r , and the step comparability bounds.

7 Observability and control for the fully discrete heat system

We now turn from the Carleman estimate to its consequences in controllability theory. Fix one of the heat discretizations in section 6 and a control time $T > 0$. Throughout this section, $A = -\Delta_D$ and $V = H_0^1(\Omega)$. Let $N \geq 2$, $\delta t = T/N$, and $t_n = n\delta t$. The control space and operator are those of the chosen realization. In each case, $\|B_h\|_{\mathcal{L}(U_h, X_h)} \leq 1$. Here $C(T, M)$ denotes a generic positive constant that may also depend on the fixed domain, control region, and mesh-regularity and grading bounds, but not on h , δt , or the initial datum.

When the finite element method is used to construct numerical solutions, the coefficient b^n is represented at the discrete level by a bounded potential. To fix the setting, let $b^n \in L^\infty(\Omega; \mathbb{R})$ with $\|b^n\|_{L^\infty(\Omega)} \leq M$, and define the potential operator V_h^n by

$$(V_h^n y, z)_{X_h} = \int_{\Omega} b^n(R_h y)(R_h z) dx.$$

For the finite difference discretization, the continuous coefficient is replaced by nodal values of the potential. Specifically, V_h^n is defined as multiplication by real nodal values b_i^n with $|b_i^n| \leq M$. The averaging rules used for a fixed continuous coefficient are specified in Subsection 7.4.

For $y_{0,h} \in X_h$, consider the discrete control system

$$\begin{cases} \frac{y^n - y^{n-1}}{\delta t} + (A_h + V_h^n)y^n = B_h u^n, & 1 \leq n \leq N, \\ y^0 = y_{0,h}. \end{cases} \quad (7.1)$$

For $p_T \in X_h$, the corresponding adjoint system is

$$\begin{cases} \frac{p^{n-1} - p^n}{\delta t} + (A_h + V_h^n)^* p^{n-1} = 0, & 1 \leq n \leq N, \\ p^N = p_T. \end{cases} \quad (7.2)$$

All adjoints and terminal norms use the inner product of X_h . For controls, $\ell_{\delta t}^2(U_h)$ uses the index set $\mathcal{J} = \{1, \dots, N\}$. The time mass is the consistent L^2 mass for finite elements and the original diagonal mass for differences.

We first record the energy bounds and the exact duality identity for the forward and adjoint schemes.

Lemma 7.1. *Assume that $\delta t M \leq 1/2$. Then the systems (7.1) and (7.2) are well posed, and there exists a constant $C(T, M) > 0$ with the following properties:*

1. *Every solution of (7.2) satisfies the backward stability bound*

$$\|p^j\|_{X_h} \leq e^{2M(t_\ell - t_j)} \|p^\ell\|_{X_h}, \quad 0 \leq j \leq \ell \leq N. \quad (7.3)$$

2. *The solution to (7.1) satisfies the a priori bound*

$$\max_{0 \leq n \leq N} \|y^n\|_{X_h} \leq C(T, M) (\|y_{0,h}\|_{X_h} + \|u\|_{\ell_{\delta t}^2(U_h)}). \quad (7.4)$$

3. For solutions to (7.1) and (7.2), we have the duality identity

$$(y^N, p^N)_{X_h} - (y^0, p^0)_{X_h} = \delta t \sum_{n=1}^N (u^n, B_h^* p^{n-1})_{U_h}. \quad (7.5)$$

Proof. For completeness, we recall the standard backward-Euler stability and discrete duality argument.

Since V_h^n is self-adjoint with norm at most M , the operator $\text{Id}_{X_h} + \delta t(A_h + V_h^n)$ is bounded below by $1 - \delta tM > 0$. Hence

$$\|(\text{Id}_{X_h} + \delta t(A_h + V_h^n))^{-1}\| \leq (1 - \delta tM)^{-1} \leq e^{2\delta tM}.$$

Iterating this one-step bound from j to ℓ yields (7.3).

Next, applying the same bound to (7.1) and using

$$\delta t \sum_{n=1}^N \|u^n\|_{U_h} \leq \sqrt{T} \|u\|_{\ell_{\delta t}^2(U_h)},$$

gives (7.4).

It remains to prove the duality identity. For each time step, (7.1) and (7.2) yield

$$\begin{aligned} (y^n, p^n)_{X_h} &= (y^n, p^{n-1})_{X_h} + \delta t((A_h + V_h^n)y^n, p^{n-1})_{X_h} \\ &= (y^{n-1}, p^{n-1})_{X_h} + \delta t(B_h u^n, p^{n-1})_{X_h}. \end{aligned}$$

Summing over $n = 1, \dots, N$ gives (7.5). $\square$

7.1 Relaxed observability with exponentially small terminal remainder

We now derive the relaxed observability estimate from the discrete Carleman estimate. Fix the constants κ, h_0, Δ_0 supplied by Corollary 2.3. Recall that $S_{h,\delta t}$ is defined in (2.33). For $h \leq h_0$ and $\delta t \leq \Delta_0$, the discrete Carleman estimate is valid whenever $q_{\max} \leq S_{h,\delta t}$.

The discrete Carleman estimate yields observability with an exponentially small terminal remainder.

Theorem 7.2. *There exist constants $C_b, d_0 > 0$, depending additionally on ω and on the mesh bounds in Propositions 6.1 and 6.2 for the chosen realization, such that the following holds:*

Given $T > 0$ and $M \geq 0$, set $q_b = C_b(q_c + T^{-1} + M^{2/3})$. Then there exists a constant $C(T, M) > 0$ such that, for every spatial mesh size h and time step $\delta t = T/N$ satisfying

$$h \leq h_0, \quad \delta t \leq \min\{\Delta_0, T/12\}, \quad \delta tM \leq \frac{1}{2}, \quad 16q_b \leq S_{h,\delta t} \leq \frac{q_b T}{4\delta t}, \quad (7.6)$$

every solution of (7.2) satisfies

$$\|p^0\|_{X_h}^2 \leq C(T, M) \left(\delta t \sum_{n=1}^N \|B_h^* p^{n-1}\|_{U_h}^2 + e^{-2d_0 S_{h,\delta t}} \|p^N\|_{X_h}^2 \right). \quad (7.7)$$

Proof. We first fix the constants, then argue in three steps: we introduce a periodic Carleman weight and cut off the adjoint, we absorb the potential and estimate the endpoint contribution, and finally we recover the initial adjoint norm.

With the Carleman and load constants fixed, we choose C_b sufficiently large, independently of $T, M, h, \delta t$, so that the weight bounds in Step 1 and the potential absorption condition in Step 2 hold. This is possible because

$$q_b \geq C_b q_c, \quad q_b T \geq C_b, \quad q_b^3 \geq C_b^3 M^2.$$

We fix this choice before taking $h, \delta t$ satisfying (7.6).

Step 1: Choice of the periodic weight and cutoff of the adjoint. Put $\delta = q_b/S_{h,\delta t} \leq 1/16$ and define

$$q(t) = \frac{q_b}{\sqrt{\sin^2(\pi t/T) + \delta^2}}. \quad (7.8)$$

Then

$$\min_{[0,T]} q = \frac{q_b}{\sqrt{1 + \delta^2}}, \quad \max_{[0,T]} q = S_{h,\delta t},$$

and differentiation gives

$$\frac{|q'|}{q^2} \leq \frac{C}{q_b T}, \quad \frac{|q''|}{q^3} \leq \frac{C}{q_b^2 T^2}.$$

The choice of C_b ensures (2.1). By (2.32) and (2.33), the mesh conditions also give

$$hS_{h,\delta t} \leq \delta_x, \quad \delta t S_{h,\delta t}^2 \leq \delta_{\text{time}}.$$

Let χ be a smooth function with values in $[0, 1]$ that vanishes at distance at most δT from $\{0, T\}$, equals one at distance at least $2\delta T$, and satisfies $|\chi'| \leq C/(\delta T)$. For the cutoff sequence and its loads, the discrete weighted norms below use the periodic index set $\{0, \dots, N-1\}$. Define the periodic sequence

$$z^j = \chi(t_j)p^j, \quad 0 \leq j \leq N-1, \quad z^N = z^0 = 0,$$

and set

$$r^j = \frac{\chi(t_{j+1}) - \chi(t_j)}{\delta t} p^{j+1}, \quad 0 \leq j \leq N-1.$$

Since $\chi(t_N)p^N = 0$, (7.2) gives

$$D_- z^j + A_h z^j = -\mathbf{V}_h^{j+1} z^j - r^j. \quad (7.9)$$

Define the potential load by

$$f_{\text{pot}}^j = \begin{cases} -b^{j+1} R_h z^j, & \text{for finite elements,} \\ (-b_i^{j+1} z_i^j)_i, & \text{for finite differences.} \end{cases}$$

In both cases, $\Gamma_h f_{\text{pot}}^j = -\mathbf{V}_h^{j+1} z^j$, and the load satisfies

$$\|f_{\text{pot}}^j\|_{\mathfrak{F}_h, \theta_j} \leq CM \|z^j\|_{X_h, \theta_j}.$$

Set $f^j = f_{\text{pot}}^j - \mathcal{I}_h r^j$. By Assumption (N2) and (7.9),

$$D_- z^j + A_h z^j = \Gamma_h f^j, \quad 0 \leq j \leq N-1.$$

Applying (2.29) to the periodic pair (z, f) , with $o = 3$, and using (2.15), we obtain

$$\begin{aligned} \mathcal{E}_{q,h,\delta t}(z) &\leq C(\|f\|_{\ell_{\delta t}^2(\mathfrak{F}_h), \theta}^2 + \|q^{3/2} B_h^* z\|_{\ell_{\delta t}^2(U_h), \theta}^2) \\ &\leq CM^2 \|z\|_{\ell_{\delta t}^2(X_h), \theta}^2 + C\|r\|_{\ell_{\delta t}^2(X_h), \theta}^2 + C\|q^{3/2} B_h^* z\|_{\ell_{\delta t}^2(U_h), \theta}^2. \end{aligned} \quad (7.10)$$

Step 2: Absorption of the potential and estimate of the endpoint contribution. By (2.22) and $q \geq q_b/\sqrt{1 + \delta^2}$, our choice of C_b gives

$$CM^2 \|z\|_{\ell_{\delta t}^2(X_h), \theta}^2 \leq \frac{CM^2(1 + \delta^2)^{3/2}}{q_b^3} \|q^{3/2} z\|_{\ell_{\delta t}^2(X_h), \theta}^2 \leq \frac{1}{2} \|q^{3/2} z\|_{\ell_{\delta t}^2(X_h), \theta}^2 \leq \frac{1}{2} \mathcal{E}_{q,h,\delta t}(z).$$

Since $z^j = \chi(t_j)p^j$, $0 \leq \chi \leq 1$, and $\rho \leq -\rho_*$,

$$\|q^{3/2} B_h^* z\|_{\ell_{\delta t}^2(U_h), \theta}^2 = \delta t \sum_{j=0}^{N-1} q_j^3 \chi(t_j)^2 \|B_h^* p^j\|_{U_h, \theta_j}^2 \leq \left(\sup_{s \geq q_c} s^3 e^{-2\rho_* s} \right) \delta t \sum_{j=0}^{N-1} \|B_h^* p^j\|_{U_h}^2 \leq C \delta t \sum_{j=0}^{N-1} \|B_h^* p^j\|_{U_h}^2.$$

Let $\mathcal{J}_\chi$ denote the set of indices where the cutoff difference does not vanish:

$$\mathcal{J}_\chi := \{j \in \{0, \dots, N-1\} : \chi(t_{j+1}) \neq \chi(t_j)\}.$$

Each corresponding cell meets $[\delta T, 2\delta T]$ or $[T - 2\delta T, T - \delta T]$. Counting these cells and using $\delta t \leq \delta T/4$, we obtain

$$\delta t \sum_{j \in \mathcal{J}_\chi} 1 \leq 2\delta T + 4\delta t \leq 3\delta T.$$

Moreover, $\min\{t_j, T - t_j\} \leq 3\delta T$ for every $j \in \mathcal{J}_\chi$. Define $c_{\text{end}} := 2\rho_*/\sqrt{1+9\pi^2}$, a positive constant depending only on ρ_* . Since $|\sin(\pi t_j/T)| \leq 3\pi\delta$ at these indices, (7.8) gives

$$q_j \geq \frac{S_{h,\delta t}}{\sqrt{1+9\pi^2}}, \quad \theta_j(x)^2 \leq e^{-c_{\text{end}} S_{h,\delta t}}, \quad x \in \Omega, \quad j \in \mathcal{J}_\chi.$$

Using the derivative bound on χ , (7.3), and $\delta = q_b/S_{h,\delta t}$, we conclude

$$\begin{aligned} \|r\|_{\ell_{\delta t}^2(X_h), \theta}^2 &= \delta t \sum_{j \in \mathcal{J}_\chi} \left| \frac{\chi(t_{j+1}) - \chi(t_j)}{\delta t} \right|^2 \|p^{j+1}\|_{X_h, \theta_j}^2 \\ &\leq \frac{C\delta t \sum_{j \in \mathcal{J}_\chi} 1}{(\delta T)^2} e^{-c_{\text{end}} S_{h,\delta t}} e^{4MT} \|p^N\|_{X_h}^2 \\ &\leq \frac{C e^{4MT}}{q_b T} S_{h,\delta t} e^{-c_{\text{end}} S_{h,\delta t}} \|p^N\|_{X_h}^2 \\ &\leq C(T, M) S_{h,\delta t} e^{-c_{\text{end}} S_{h,\delta t}} \|p^N\|_{X_h}^2. \end{aligned} \tag{7.11}$$

After absorption, (7.10) and (7.11) therefore give

$$\mathcal{E}_{q,h,\delta t}(z) \leq C\delta t \sum_{j=0}^{N-1} \|B_h^* p^j\|_{U_h}^2 + C(T, M) S_{h,\delta t} e^{-c_{\text{end}} S_{h,\delta t}} \|p^N\|_{X_h}^2. \tag{7.12}$$

Step 3: Recovery of the initial adjoint norm. On $[T/3, 2T/3]$, we have $\chi = 1$, $q \asymp q_b$, and $\theta \geq e^{-Cq_b}$. Since $\delta t \leq T/12$, the nodes in this interval have total discrete measure at least $T/6$. By (2.22) and (7.3),

$$\mathcal{E}_{q,h,\delta t}(z) \geq c q_b^3 e^{-Cq_b} \delta t \sum_{t_j \in [T/3, 2T/3]} \|p^j\|_{X_h}^2 \geq c(T, M) \|p^0\|_{X_h}^2. \tag{7.13}$$

Choose $d_0 > 0$ so that $2d_0 < c_{\text{end}}$. Then

$$S_{h,\delta t} e^{-c_{\text{end}} S_{h,\delta t}} \leq C e^{-2d_0 S_{h,\delta t}},$$

and (7.12) and (7.13) imply (7.7). $\square$

7.2 Uniform approximate null controllability with exponentially small terminal error

By duality, Theorem 7.2 yields approximate null controls with a cost uniform in h and δt .

Theorem 7.3. *Under the hypotheses of Theorem 7.2, there exists $C(T, M) > 0$ such that, for every $y_{0,h} \in X_h$, there is a control $u \in \ell_{\delta t}^2(U_h)$ satisfying*

$$\|u\|_{\ell_{\delta t}^2(U_h)} \leq C(T, M) \|y_{0,h}\|_{X_h}. \tag{7.14}$$

The corresponding solution of (7.1) satisfies

$$\|y^N\|_{X_h} \leq C(T, M) e^{-d_0 S_{h,\delta t}} \|y_{0,h}\|_{X_h}, \tag{7.15}$$

with d_0 as in Theorem 7.2.

Proof. We use the standard penalized HUM argument; see [1, Propositions 2.2 and 2.4]. For completeness, we present the details in four steps: we introduce a strictly convex functional, take its unique minimizer, use the optimality condition to identify the resulting control and terminal state, and finally estimate them.

For each $p_T \in X_h$, let p solve (7.2), and define

$$J(p_T) = \frac{\delta t}{2} \sum_{n=1}^N \|B_h^* p^{n-1}\|_{U_h}^2 + \frac{e^{-2d_0 S_{h,\delta t}}}{2} \|p_T\|_{X_h}^2 - (y_{0,h}, p^0)_{X_h}.$$

For fixed $h, \delta t$, the positive penalty coefficient makes J strictly convex and coercive. Hence J admits a unique minimizer $\hat{p}_T \in X_h$. Let $\hat{p}$ denote the corresponding solution of (7.2).

Since $\hat{p}_T$ minimizes J , for every $p_T \in X_h$ and its associated adjoint solution p , we have

$$\delta t \sum_{n=1}^N (B_h^* \hat{p}^{n-1}, B_h^* p^{n-1})_{U_h} + e^{-2d_0 S_{h,\delta t}} (\hat{p}_T, p_T)_{X_h} = (y_{0,h}, p^0)_{X_h}. \quad (7.16)$$

Choose $u^n = -B_h^* \hat{p}^{n-1}$, $1 \leq n \leq N$, and let y solve (7.1). Combining the duality identity (7.5) with the optimality condition (7.16) yields

$$(y^N, p_T)_{X_h} = e^{-2d_0 S_{h,\delta t}} (\hat{p}_T, p_T)_{X_h}, \quad p_T \in X_h.$$

Since $p_T \in X_h$ is arbitrary, it follows that $y^N = e^{-2d_0 S_{h,\delta t}} \hat{p}_T$.

Taking $p_T = \hat{p}_T$ in (7.16) and applying (7.7) to $\hat{p}$, we obtain

$$\|u\|_{\ell_{\delta t}^2(U_h)}^2 + e^{-2d_0 S_{h,\delta t}} \|\hat{p}_T\|_{X_h}^2 = (y_{0,h}, \hat{p}^0)_{X_h} \leq \sqrt{C(T, M)} \|y_{0,h}\|_{X_h} (\|u\|_{\ell_{\delta t}^2(U_h)}^2 + e^{-2d_0 S_{h,\delta t}} \|\hat{p}_T\|_{X_h}^2)^{1/2}.$$

Dividing by the square-root factor gives

$$\|u\|_{\ell_{\delta t}^2(U_h)}^2 + e^{-2d_0 S_{h,\delta t}} \|\hat{p}_T\|_{X_h}^2 \leq C(T, M) \|y_{0,h}\|_{X_h}^2.$$

In particular, this yields (7.14). Finally, combining the identity $y^N = e^{-2d_0 S_{h,\delta t}} \hat{p}_T$ with the preceding bound gives

$$\|y^N\|_{X_h} \leq C(T, M) e^{-d_0 S_{h,\delta t}} \|y_{0,h}\|_{X_h},$$

which is (7.15). $\square$

Remark 7.1. Fix T, M and consider the parabolic scaling $\delta t \asymp h^2$ with $\delta t \leq T^{-2} h^2$. Under this scaling, (2.33) and (7.15) yield the terminal-error decay factor $e^{-ch^{-4/5}}$. For the same scaling, [13, Proposition 3.1 and Theorem 1.10] gives the factor $e^{-ch^{-1/2}}$ for one-dimensional finite differences with the backward Euler scheme. Thus, in this common setting, Theorem 7.3 improves the decay exponent and, in addition, applies to the higher-dimensional realizations in Propositions 6.1 and 6.2. Since both bounds decay faster than any fixed algebraic power of h , they are consistent with the practical accuracy criterion discussed in [7, Remark 4.4].

Remark 7.2. Theorem 7.3 yields ϕ -null controllability in the sense of [1, Definition 2.1 and Theorem 3.2]. Indeed, taking $\phi(h, \delta t) = e^{-2d_0 S_{h,\delta t}}$, we obtain a control cost that is uniform in $h, \delta t$, together with a terminal error bounded by $C(T, M) \sqrt{\phi(h, \delta t)} \|y_{0,h}\|_{X_h}$.

Remark 7.3. For fixed T, M , the conditions in (7.6) hold for all sufficiently small $h, \delta t$ when $h, \delta t \rightarrow 0$ independently, with $\delta t = T/N$. Indeed, (2.33) gives

$$\lim_{h, \delta t \rightarrow 0} S_{h,\delta t} = +\infty.$$

Moreover, $0 \leq \delta t S_{h,\delta t} \leq \kappa(\delta t)^{3/5}$ implies

$$\lim_{h, \delta t \rightarrow 0} \delta t S_{h,\delta t} = 0.$$

Thus the two-sided bound on $S_{h,\delta t}$ is eventually satisfied. If, in addition, one imposes the optional scaling $\delta t \leq ch^2$, then $S_{h,\delta t} \asymp h^{-4/5}$ is retained.

7.3 Uniform observability under coarse-space filtering

Suitable filtering makes it possible to control the retained component of the terminal state exactly to zero, with a cost that is bounded independently of $h, \delta t$ and a full terminal state that is exponentially small [21]. In our setting, the terminal remainder in (7.7) can be absorbed in an analogous way, provided that the terminal adjoint data have a sufficiently small discrete Rayleigh quotient.

Throughout this subsection, we assume (7.6), fix $0 < \vartheta < 1$, write $S = S_{h, \delta t}$, and set

$$C_0 = \max\{1, C(T, M)\}, \quad (7.17)$$

with $C(T, M)$ as in Theorem 7.2. For a terminal subspace $F \subset X_h$, define the discrete Rayleigh quotient

$$\Lambda_F = \sup_{0 \neq v \in F} \frac{(A_h v, v)_{X_h}}{\|v\|_{X_h}^2}, \quad (7.18)$$

with $\Lambda_{\{0\}} = 0$. We then set

$$D_{T, M} = MT + \frac{1}{2}M^2T^2, \quad K_F = e^{T\Lambda_F + D_{T, M}}.$$

The restriction is imposed only on p^N , while the fine-grid equations remain unchanged. In particular, neither A_h nor the potential is required to preserve F .

We first bound the terminal norm by the initial adjoint norm.

Lemma 7.4. *Every solution of (7.2) with $p^N \in F$ satisfies*

$$\|p^N\|_{X_h} \leq K_F \|p^0\|_{X_h}. \quad (7.19)$$

The proof of Lemma 7.4 is given in subsection D.1. It shows that, at each backward-Euler step, the Rayleigh quotient can increase by at most $\delta t M^2$. Importantly, the resulting constants are independent of both $\dim X_h$ and $\|A_h\|$.

Theorem 7.5. *Recall C_0 is given by (7.17). If*

$$T\Lambda_F \leq \vartheta d_0 S, \quad (1 - \vartheta)d_0 S \geq D_{T, M} + \frac{1}{2} \log(4C_0), \quad (7.20)$$

then every solution of (7.2) with $p^N \in F$ satisfies

$$\|p^0\|_{X_h}^2 \leq 2C_0 \delta t \sum_{n=1}^N \|B_h^* p^{n-1}\|_{U_h}^2. \quad (7.21)$$

Proof. By (7.20), $4C_0 e^{-2d_0 S} K_F^2 \leq 1$. Inserting (7.19) into (7.7) bounds the remainder by $\frac{1}{4} \|p^0\|_{X_h}^2$. Absorption proves (7.21). $\square$

A penalty on the complementary terminal space gives one control that annihilates the retained component and leaves an exponentially small full terminal state. Let Π_F denote the X_h -orthogonal projection onto F .

Corollary 7.6. *Assume (7.20). Then, for every $y_{0, h} \in X_h$, there exists a control $u \in \ell_{\delta t}^2(U_h)$ whose corresponding solution of (7.1) satisfies $\Pi_F y^N = 0$. Moreover, the control cost satisfies*

$$\|u\|_{\ell_{\delta t}^2(U_h)} \leq \sqrt{2C_0} \|y_{0, h}\|_{X_h}, \quad (7.22)$$

while the full terminal state fulfills

$$\|y^N\|_{X_h} \leq C(T, M) e^{-(1-\vartheta)d_0 S} \|y_{0, h}\|_{X_h}, \quad (7.23)$$

where the constant $C(T, M)$ is as in Theorem 7.2, which is independent of $h, \delta t$ and F .

The proof of Corollary 7.6 is given in subsection D.2.

Coarse-space realizations are motivated by the two-grid strategies studied for numerical wave controllability in [15, 20], where interpolation from a coarser grid yields adjoint data with mesh-uniform observability. In the present parabolic setting, the admissible coarse scale follows from (7.20).

For consistent-mass finite elements, a nested coarse space gives the required Rayleigh bound.

Proposition 7.7. *Let V_h be the space in Proposition 6.1. Suppose a conforming, shape-regular, quasi-uniform coarse triangulation of maximum diameter H_c has a P_1 space $V_{H_c} \subset V_h$. Then $F = V_{H_c}$ satisfies*

$$\Lambda_F \leq CH_c^{-2}, \quad (7.24)$$

where C depends only on the coarse mesh regularity bounds and the dimension.

Proof. The coarse-space inverse estimate gives $\|\nabla v\|_{L^2(\Omega)}^2 \leq CH_c^{-2} \|v\|_{L^2(\Omega)}^2$ for $v \in V_{H_c}$. Since the physical inner product is the consistent L^2 product and $(A_h v, v)_{X_h} = \|\nabla v\|_{L^2(\Omega)}^2$, this proves (7.24). The fine mesh need only satisfy Proposition 6.1. $\square$

For Cartesian differences, tensor-product linear interpolation gives the corresponding bound in the original diagonal mass.

Proposition 7.8. *Consider (6.13) and a nested Cartesian coarse grid with steps $H_r = m_r h_r$, where $m_r \in \mathbb{N}$ divides N_r , $1 \leq r \leq d$. Let $H_c = \max_r H_r$. Let $I_{H_c}^h$ prolong zero-boundary coarse nodal values by tensor-product piecewise linear interpolation, and set $F = \text{Ran } I_{H_c}^h$. Then the discrete Rayleigh quotient of F satisfies*

$$\Lambda_F \leq 12 \sum_{r=1}^d H_r^{-2}. \quad (7.25)$$

Proof. We first consider the one-dimensional case on $(0, L)$, and write H for the coarse step. Let v_H be the continuous coarse piecewise linear function, and let z_h denote its vector of values on the fine grid. Since the endpoint values vanish, summation over the fine intervals gives

$$(A_h z_h, z_h)_{X_h} = \|v_H'\|_{L^2(0,L)}^2.$$

The induced mass satisfies

$$\|z_h\|_{X_h}^2 = \|v_H\|_{L^2(0,L)}^2 + \frac{1}{6} h^2 \|v_H'\|_{L^2(0,L)}^2.$$

By the linear-element inverse estimate

$$\|v_H'\|_{L^2(0,L)}^2 \leq 12H^{-2} \|v_H\|_{L^2(0,L)}^2,$$

we therefore obtain

$$(A_h z_h, z_h)_{X_h} \leq 12H^{-2} \|z_h\|_{X_h}^2.$$

This proves (7.25) in one dimension.

In dimension d , the induced mass is the tensor product of the one-dimensional induced masses. Tensorizing the preceding quadratic-form bound with these masses in the other directions, and summing the resulting directional stiffness terms, yields (7.25). $\square$

When the Cartesian coarse steps are comparable, both realizations have $\Lambda_F \leq CH_c^{-2}$. Thus, on mesh hierarchies with compatible coarse levels, (2.33) gives the sufficient choice

$$H_c \geq C(T, \vartheta) \max\{h^{2/5}, (\delta t)^{1/5}\}, \quad (7.26)$$

with the constant sufficiently large and $S_{h,\delta t}$ satisfying the second condition in (7.20). This sufficient scale is generally larger than a fixed multiple of the fine scale.

Remark 7.4. The spectral space spanned by eigenvectors of A_h with eigenvalues at most $\vartheta d_0 S/T$ also satisfies the first condition in (7.20). The coarse-space constructions above avoid computing these eigenvectors.

7.4 Weak convergence of reconstructed controls to continuous null controls

We finally show that the discrete controls constructed above converge, along weakly convergent subsequences, to continuous null controls.

Fix $y_0 \in H$ and a real potential $b \in L^\infty(\mathcal{Q})$ with $\|b\|_{L^\infty(\mathcal{Q})} \leq M$. We discretize these data by projection and averaging. For P_1 finite elements, take

$$y_{0,h} = \Pi_h y_0, \quad b^n(x) = \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} b(t, x) dt. \quad (7.27)$$

For differences, define the interior dual cells

$$D_i = \prod_{r=1}^d (x_{r,i_r} - h_r/2, x_{r,i_r} + h_r/2), \quad |D_i| = \omega_i, \quad (7.28)$$

and use

$$(y_{0,h})_i = \frac{1}{\omega_i} \int_{D_i} y_0(x) dx, \quad b_i^n = \frac{1}{\delta t \omega_i} \int_{t_{n-1}}^{t_n} \int_{D_i} b(t, x) dx dt. \quad (7.29)$$

Define $E_h : X_h \rightarrow H$ by $E_h = R_h$ for finite elements. For differences, let $E_h z = z_i$ on D_i and zero on the uncovered boundary strip. Define $K_h : U_h \rightarrow H$ as zero extension from ω for finite elements and as $E_h B_h$ for differences. These reconstructions preserve the physical norms:

$$(E_h z, E_h w)_H = (z, w)_{X_h}, \quad \|K_h v\|_{L^2(\Omega)} = \|v\|_{U_h}. \quad (7.30)$$

They also preserve the control pairing $(K_h v, E_h z)_H = (v, B_h^* z)_{U_h}$. For $t \in (t_{n-1}, t_n]$, set

$$U_{h,\delta t}(t) = K_h u^n, \quad Y_{h,\delta t}(t) = R_h y^n, \quad \tilde{Y}_{h,\delta t}(t) = E_h y^n.$$

For differences, fix an open set $\omega_\sharp$ with $\omega_0 \Subset \omega_\sharp \Subset \omega$ and restrict the active nodes to $\omega_\sharp$, extending the controls by zero to U_h . Their cellwise reconstructions are then supported in ω for sufficiently small h .

Denote by $V' = H^{-1}(\Omega)$ the dual of $V = H_0^1(\Omega)$. Let $A : V \rightarrow V'$ be the linear operator defined by

$$\langle Av, w \rangle_{V',V} = a(v, w) = \int_{\Omega} \nabla v \cdot \nabla w dx, \quad v, w \in V,$$

where $\langle \cdot, \cdot \rangle_{V',V}$ denotes the duality pairing. Since H is embedded in V' through the $L^2(\Omega)$ inner product, this operator extends the Dirichlet Laplacian $A = -\Delta_D$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Consequently, the energy solution belongs to $L^2(0, T; V) \cap H^1(0, T; V') \subset C([0, T]; H)$.

With this functional setting in place, we may pass from the discrete controls to the continuous equation. The reconstructed discrete controls have weakly convergent subsequences whose limits drive the continuous state to zero at time T . This is made precise in the following corollary.

Corollary 7.9. *Fix a realization in Section 6, with the data in (7.27) or (7.29). Then there exists $C(T, M) > 0$ such that the following holds. For every sequence of admissible mesh sizes h_j and time steps $\delta t_j = T/N_j$ satisfying $\lim_{j \rightarrow \infty} h_j = \lim_{j \rightarrow \infty} \delta t_j = 0$, one can choose, for all sufficiently large j , controls satisfying (7.14) and (7.15) whose reconstructions $U_{h_j, \delta t_j}$ vanish outside ω and form a relatively weakly compact family in $L^2(0, T; H)$. Moreover, every weak accumulation point of this family is the zero extension of a control $u \in L^2(\mathcal{Q}_\omega)$ for which the energy solution of*

$$\partial_t y + Ay + b(t, \cdot)y = \mathbf{1}_\omega u \quad \text{in } V', \quad y(0) = y_0, \quad (7.31)$$

satisfies $y(T) = 0$. The limiting control satisfies

$$\|u\|_{L^2(\mathcal{Q}_\omega)} \leq C(T, M) \|y_0\|_{L^2(\Omega)}. \quad (7.32)$$

Proof. We use the standard compactness argument for control approximation; see [1, Theorem 2.5] for the semidiscrete setting. The proof proceeds in four steps: uniform bounds for the controls, weak compactness of the states, identification of the limit equation, and identification of the endpoint values together with the control bound.

Since the conditions in (7.6) hold eventually along the given sequence, Theorem 7.3 applies. In the finite difference case, restricting the controls to $\omega_\sharp$ preserves Assumption (N5) by the local norm comparison in Appendix B. Consequently, (C.3), (7.14) and (7.30) show that the controls $U_{h,\delta t}$ are uniformly bounded in $L^2(0, T; H)$.

Next, we test (7.1) with $\delta t y^n$, sum over n , and use (C.2) and (7.4). This yields a uniform bound for $Y_{h,\delta t}$ in $L^2(0, T; V)$. Therefore, starting from any weakly convergent control subsequence and passing to a further subsequence if necessary, relabeled by j , we obtain

$$\lim_{j \rightarrow \infty} U_{h_j, \delta t_j} = \mathbf{1}_\omega u \quad \text{in the weak topology of } L^2(0, T; H).$$

Along the same subsequence, the states satisfy

$$\lim_{j \rightarrow \infty} Y_{h_j, \delta t_j} = y \quad \text{in the weak topology of } L^2(0, T; V).$$

By (C.1), the modified states satisfy

$$\lim_{j \rightarrow \infty} \tilde{Y}_{h_j, \delta t_j} = y \quad \text{in the weak topology of } L^2(0, T; H).$$

It remains to identify the limit equation and the endpoint values. For the smooth tests ϕ and their interpolants ϕ_h^n from Appendix C, we test (7.1) with $\delta t \phi_h^{n-1}$ and sum by parts. Since $\lim_{h, \delta t \rightarrow 0} S_{h, \delta t} = +\infty$, (7.15) and the uniform bounds for $y_{0, h}$ and ϕ_h^N give

$$\lim_{h, \delta t \rightarrow 0} (y^N, \phi_h^N)_{X_h} = 0, \quad N = T/\delta t.$$

This limit holds in particular along the chosen subsequence. Using the consistency properties from Appendix C, we therefore obtain

$$-\int_0^T (y, \partial_t \phi)_H dt + \int_0^T (a(y, \phi) + (by, \phi)_H) dt = (y_0, \phi(0))_H + \int_0^T \int_\omega u \phi dx dt. \quad (7.33)$$

Interior time tests give

$$\partial_t y = \mathbf{1}_\omega u - Ay - by \in L^2(0, T; V'). \quad (7.34)$$

To see this, let $\psi \in C_c^\infty(0, T)$ and $w \in V$, and take $\phi(t, x) = \psi(t)w(x)$ in (7.33). Since ψ vanishes near 0 and T , the endpoint term $(y_0, \phi(0))_H$ drops out, and we obtain

$$-\int_0^T \psi'(t) (y(t), w)_H dt + \int_0^T \psi(t) (a(y(t), w) + (b(t)y(t), w)_H) dt = \int_0^T \psi(t) \int_\omega u(t, x)w(x) dx dt.$$

This is the distributional formulation of (7.34). Indeed, each term on the right-hand side of (7.34) lies in $L^2(0, T; V')$: the control term $\mathbf{1}_\omega u$ lies in $L^2(0, T; H) \subset L^2(0, T; V')$, the term Ay lies in $L^2(0, T; V')$ since $y \in L^2(0, T; V)$ and $A : V \rightarrow V'$ is bounded, and the potential term by lies in $L^2(0, T; V')$ by the boundedness of b and the embedding $H \hookrightarrow V'$. Hence (7.34) holds.

Consequently, $y \in C([0, T]; H)$, so the unrestricted endpoint values of ϕ in (7.33) identify $y(0) = y_0$ and $y(T) = 0$. Finally, the control bound in (7.32) follows from weak lower semicontinuity. $\square$

Remark 7.5. The independent refinement in Corollary 7.9 follows from the transferred observability estimate for bounded real space-time potentials. To relate this to the spectral framework in [6], suppose that the uniform partial spectral inequality holds up to frequencies of order $h^{-\beta_{\text{sp}}}$, with $\beta_{\text{sp}} > 0$, and set $\eta = \max\{h, (\delta t)^{1/\beta_{\text{sp}}}\}$. With this choice, the same discrete system satisfies the spectral hypothesis on the smaller range of order $\eta^{-\beta_{\text{sp}}}$, and moreover $\delta t \leq \eta^{\beta_{\text{sp}}}$. Thus, in the autonomous backward-Euler case, Theorems 3.5 and 3.8 of [6] apply with the effective cutoff parameter η , upon choosing $\gamma = \beta_{\text{sp}}$ and $C_T = 1$. This yields an independent-refinement consequence of their results. We emphasize that η only changes the cutoff used in the argument; it does not alter the underlying spatial discretization.

Remark 7.6. For fixed $0 < \vartheta < 1$, the compactness argument in Corollary 7.9 applies equally well to the controls constructed in Corollary 7.6, under the same data approximations and consistency assumptions. Indeed, the bounds in (7.22) and (7.23) provide uniform control norms together with

$$\lim_{h, \delta t \rightarrow 0} \|y^N\|_{X_h} = 0, \quad N = T/\delta t.$$

Consequently, every weak accumulation point of their reconstructions is a continuous null control.

8 Numerical experiments for heat control

We now illustrate the control construction of Theorem 7.3 on the consistent-mass P_1 /backward Euler scheme. The experiments examine three aspects of the construction: control effectiveness on nonuniform meshes, independent spatial and temporal refinement, and the balance between terminal accuracy and iterative cost. In each case, the computed control is also applied, without re-optimization, in an independently refined approximation of the continuous heat equation.

8.1 Model and numerical method

We take $\Omega = (0, 1)^2$, $T = 0.1$, and $\omega = (0.4, 0.6)^2$, and impose homogeneous Dirichlet boundary conditions on

$$\partial_t y - \Delta y + b(t, x)y = \mathbf{1}_\omega U.$$

The main experiments use $b = 0$ and two initial data: the sine datum (S), $y_{0,S}(x) = 2 \sin(\pi x_1) \sin(\pi x_2)$, and the localized datum (L),

$$y_{0,L}(x) = c_L \begin{cases} \exp(-1/(1 - |x - x_c|^2/r^2)), & |x - x_c| < r, \\ 0, & |x - x_c| \geq r. \end{cases}$$

Here $x_c = (0.25, 0.70)$, $r = 0.1$, and c_L normalizes the $L^2(\Omega)$ norm to one. Both initial data therefore have unit norm, and we use their L^2 projections onto V_h .

For the nodal basis $\{\phi_i\}_{i=1}^{N_h}$, $N_h = \dim V_h$, define

$$M_{ij} = \int_{\Omega} \phi_i \phi_j dx, \quad K_{ij} = \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j dx, \quad W_{ij} = \int_{\omega} \phi_i \phi_j dx.$$

The observation matrix is assembled on the exact intersections of the triangles with ω . We represent the state by $y_h^n = \sum_{i=1}^{N_h} \mathbf{y}_i^n \phi_i$ and the control by $u_h^n = (\sum_{i=1}^{N_h} \mathbf{u}_i^n \phi_i)|_{\omega}$, and write $\|\mathbf{v}\|_M^2 = \mathbf{v}^T M \mathbf{v}$. For $\varepsilon > 0$, the penalized problem minimizes

$$\frac{\delta t}{2} \sum_{n=1}^N (\mathbf{u}^n)^T W \mathbf{u}^n + \frac{1}{2\varepsilon} \|\mathbf{y}^N\|_M^2,$$

subject to (1.3). We solve the dual problem by mass-preconditioned conjugate gradients, with sparse direct solves for the time steps. With the sign convention of Theorem 7.3, $u_h^n = -p_h^{n-1}|_{\omega}$, where $p_h^n = \sum_{i=1}^{N_h} \mathbf{p}_i^n \phi_i$ is the computed adjoint. From the forward state driven by the computed control, we evaluate the primal-dual gap

$$\text{gap} = \frac{\|\mathbf{y}^N - \varepsilon \mathbf{p}_T\|_M^2}{2\varepsilon},$$

where $\mathbf{p}_T = \mathbf{p}^N$. We stop the iterations when

$$\sqrt{T} e^{2MT} \sqrt{2 \text{gap}} \leq 10^{-9}, \quad (8.1)$$

where M is the potential bound. Work is measured by the number of forward and adjoint time-step linear solves, including stopping checks.

Let $U_{h,\delta t}$ be the zero extension of u_h^n on $(t_{n-1}, t_n]$. We compute a reference terminal state $y_{\text{ref}}(T; U_{h,\delta t}, y_0)$ by sine-Galerkin propagation of the same control. The discrete terminal error is

$$E_d = \frac{\|y_h^N\|_{L^2(\Omega)}}{\|y_0\|_{L^2(\Omega)}},$$

and the reference terminal error is

$$E_r = \frac{\|y_{\text{ref}}(T; U_{h,\delta t}, y_0)\|_{L^2(\Omega)}}{\|y_0\|_{L^2(\Omega)}}.$$

We also report the control cost

$$C_u = \frac{\|U_{h,\delta t}\|_{L^2((0,T) \times \omega)}}{\|y_0\|_{L^2(\Omega)}}.$$

The control is projected onto tensor-product sine modes by integration over the intersections with ω and is kept constant on its original time intervals. For $b = 0$, modal time propagation is exact. We compare 12,

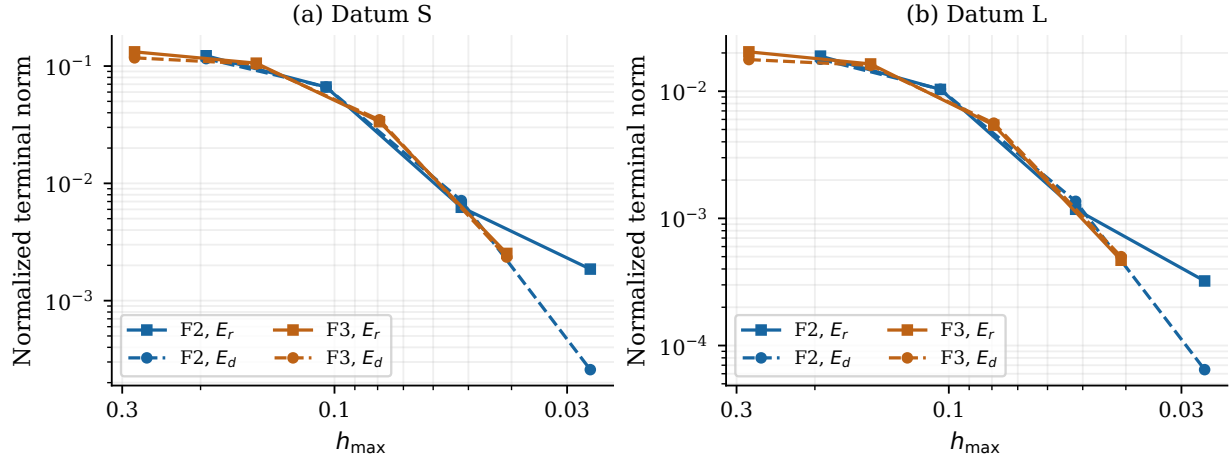


Figure 1: Control effectiveness on F2 and F3 meshes for (a) the sine datum S and (b) the localized datum L, with $T = 0.1$, $N = 128$, $b = 0$, and (8.2). Solid squares show the reference terminal norm E_r ; dashed circles show the discrete terminal norm E_d . The horizontal coordinate is the measured maximum element diameter.

Table 1: Terminal accuracy and control cost at the finest F2/F3 levels with $N = 128$, and at the finest slow-time level with $N = 53$. All rows use (8.2); slow denotes F2 with (8.3).

Mesh/data	$h_{\max}$	N	E_r	C_u	E_r/E_{free}
F2/L	0.0266	128	3.22×10^{-4}	1.024	1.50×10^{-2}
F2/S	0.0266	128	1.86×10^{-3}	5.314	1.34×10^{-2}
F3/L	0.0410	128	4.69×10^{-4}	0.708	2.19×10^{-2}
F3/S	0.0410	128	2.53×10^{-3}	4.148	1.82×10^{-2}
slow/L	0.0266	53	8.65×10^{-4}	0.718	4.04×10^{-2}
slow/S	0.0266	53	5.11×10^{-3}	4.186	3.68×10^{-2}

24, 48, and, when needed, 96 modes per coordinate, together with increased quadrature orders. In the three main figures, the largest empirical refinement indicator, measured by the change in the reference terminal state relative to E_r , is 2.27%.

With $h = \max_{K \in \mathcal{T}_h} \text{diam } K$, we choose, unless another value is specified,

$$\varepsilon_{\text{exp}}(h, \delta t) = \exp[-\min\{h^{-4/5}, (\delta t)^{-2/5}\}]. \quad (8.2)$$

This choice follows the scale of the theoretical penalty $e^{-2d_0 S_{h, \delta t}}$, with the experimental coefficient fixed at one after preliminary runs.

8.2 Control effectiveness on nonuniform meshes

We first examine whether the control remains effective on nonuniform meshes. To this end, we use two families of Delaunay meshes: perturbed Cartesian sites (F2) and a fixed smooth coordinate grading with density $1 + 2 \exp(-((x - 1/2)/0.18)^2)$, followed by the same perturbation (F3). The perturbations are at most 0.18 of the coordinate spacing. The four nominal spatial resolutions are $n_{\text{sp}} \in \{8, 16, 32, 64\}$, with $N = 128$ time steps. On these meshes, $\ell_K = h_K$ satisfies (6.1), with vertex-neighbor ratios below $2.04 < 3$. Both terminal norms decrease along the reported refinements in Figure 1. To compare the controlled state with natural heat decay, let $E_{\text{free}} = \|y_{\text{ref}}(T; 0, y_0)\|_{L^2(\Omega)} / \|y_0\|_{L^2(\Omega)}$. Its value is $e^{-2\pi^2 T} = 0.1389111$ for S and 0.0214074 for L. At the finest reported F2/F3 levels, E_r/E_{free} lies between 0.0134 and 0.0219, while the control norms range from 0.708 to 5.314; see Table 1. At a common nominal resolution, F2 and F3 have different measured diameters and hence different penalties, so this comparison concerns refinement rather than equal-accuracy efficiency.

A complementary experiment with local refinement in dyadic rings keeps $h_{\max} = 0.0625$, $N = 256$, and $\varepsilon = 1.02097 \times 10^{-4}$ fixed, while increasing the global diameter contrast from 1.41 to 90.51, with vertex

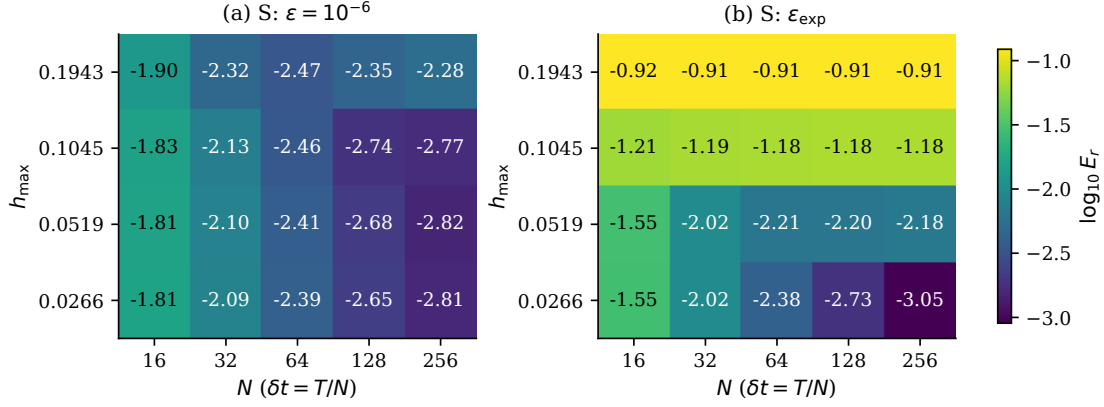


Figure 2: Independent spatial and temporal refinement for S on F2, with $T = 0.1$ and $b = 0$: (a) fixed $\varepsilon = 10^{-6}$; (b) (8.2). The entries are $\log_{10} E_r$ on a common color scale. Rows correspond to measured $h_{\max}$ and columns to N , with $\delta t = 0.1/N$.

grading ratio two. The measured control norms vary only from 2.9233 to 2.9254, and the reference terminal norms from 0.01478 to 0.01483. Thus, when the mesh is refined locally at a fixed outer scale, both the measured control cost and the terminal norm remain nearly constant.

8.3 Independent spatial and temporal refinement

We next test whether the control remains accurate under independent spatial and temporal refinement. To this end, we combine the four F2 spatial levels with $N = 16, 32, 64, 128, 256$. Figure 2 compares E_r for S with a fixed penalty $\varepsilon = 10^{-6}$ and with (8.2). For the fixed penalty on the finest spatial mesh, increasing N from 16 to 256 reduces E_r from 1.55×10^{-2} to 1.56×10^{-3} , whereas E_d changes from 2.17×10^{-3} to 1.46×10^{-3} . Thus, a small discrete terminal state can coexist with a substantially larger terminal state in the reference PDE. On the coarsest mesh, E_r is not monotone in N . For the exponential penalty, refinement also changes the optimization problem, so the observed trends combine discretization and regularization effects.

We also consider the slower temporal refinement

$$N(h) = \max\{12, \lceil 4[\log(1/h)]^2 \rceil\}, \quad (8.3)$$

using the natural logarithm and the measured diameter. This gives $N = 12, 21, 36, 53$. At the finest level, the S reference terminal norm is 5.11×10^{-3} , compared with 1.86×10^{-3} for $N = 128$. This illustrates independent refinement along a path distinct from the parabolic balance, with a different accuracy at the tested resolutions.

8.4 Penalty selection and computational cost

Finally, we examine how the penalty parameter governs the trade-off between terminal accuracy and computational cost. We take $(n_{\text{sp}}, N) = (16, 32)$ and $(32, 64)$, and vary $\varepsilon \in \{10^{-2}, 10^{-4}, \dots, 10^{-12}\}$. Figure 3 compares the discrete and reference terminal norms and the number of time-step solves for S. Across the S/L scans, six configurations at the smallest penalties hit the iteration limit before satisfying (8.1); the unconverged S iterates are marked by crosses.

On the 225-degree-of-freedom mesh, reducing ε from 10^{-6} to 10^{-10} decreases E_d by a factor of 91.3, but decreases E_r only from 7.392×10^{-3} to 6.151×10^{-3} , an improvement of 16.8%. In contrast, the number of time-step solves rises from 4512 to 63392. On the 961-degree-of-freedom mesh, the change from 10^{-6} to 10^{-8} decreases E_d by a factor of 9.18 and E_r by 10.7%, while the solve count rises from 13248 to 73408. We note that these counts measure iterative effort rather than equal-cost work across different meshes. The penalty therefore controls the discrete target more directly than the accuracy attained by the resulting control in the continuous equation. Once the latter changes only slightly, further reductions of the penalty mainly increase the iterative cost. The experiments distinguish this effect from the improvements obtained by spatial and temporal refinement.

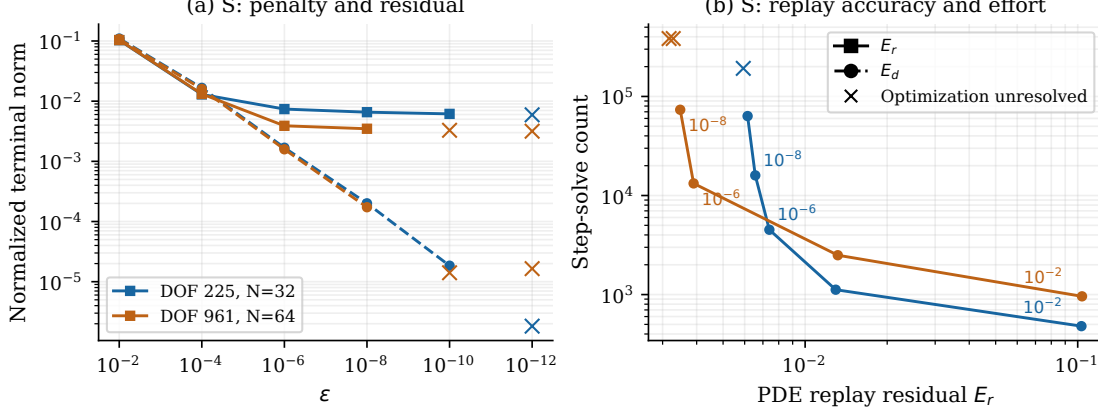


Figure 3: Penalty scans for S on F2: 225 state degrees of freedom with $N = 32$, and 961 with $N = 64$. (a) Discrete terminal norm E_d (dashed circles) and reference terminal norm E_r (solid squares); the penalty decreases to the right. (b) Number of forward and adjoint time-step solves versus E_r . Crosses mark iterates that did not satisfy (8.1) and are excluded from the connecting curves.

A Verification of the continuous analytic assumptions

We first give a sufficient class of spatial operators for arbitrary $m \geq 1$, and then verify all continuous analytic assumptions for the Dirichlet Laplacian used in Section 6.

A.1 Symmetric coercive spatial operators

Throughout this subsection, weighted estimates are understood at each fixed time.

We begin with a sufficient class of spatial operators. The following proposition shows that symmetric coercive differential forms with full elliptic regularity satisfy the spatial analytic assumptions.

Proposition A.1. *Let $m \geq 1$ be a fixed integer, set $V = H_0^m(\Omega)$, and let*

$$a(u, v) = \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} a_{\alpha\beta}(x) D^{\beta} u D^{\alpha} v dx, \quad (\text{A.1})$$

where the coefficients are real, satisfy $a_{\alpha\beta} = a_{\beta\alpha}$, and belong to $W^{2m, \infty}(\Omega)$. Suppose that a is bounded and coercive on $H_0^m(\Omega)$ with uniform positive constants, and that its associated operator has domain $D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega)$ and satisfies

$$\|v\|_{H^{2m}(\Omega)} \leq C \|Av\|_{L^2(\Omega)}.$$

Then Assumptions (A1) to (A3) hold for every fixed smooth spatial weight ρ and every fixed finite τ_* . The constants depend additionally on the $W^{2m, \infty}(\Omega)$ norms of $a_{\alpha\beta}$.

Proof. The representation theorem for closed symmetric forms, together with the assumed graph estimate, gives Assumption (A1). Smooth multiplication preserves V and $D(A)$. By Leibniz' rule, every term of $[A, \sigma] \sigma^{-1} v$ contains a derivative of the weight; the coefficient of $D^j v$, $j \leq 2m - 1$, is bounded by $C q^{2m-j}$ by (3.8). This proves Assumption (A2).

The Cauchy–Schwarz inequality in (A.1) gives (2.6). For the lower bound, set $w = \sigma v$. Leibniz' rule and Sobolev interpolation give, for every $\eta > 0$,

$$\begin{aligned} |a(v, \sigma^2 v) - a(w, w)| &\leq C \sum_{\substack{0 \leq r, s \leq m \\ r+s < 2m}} q^{2m-r-s} \|D^r w\|_{L^2(\Omega)} \|D^s w\|_{L^2(\Omega)} \\ &\leq \eta \|w\|_{H^m(\Omega)}^2 + C(\eta) q^{2m} \|w\|_{L^2(\Omega)}^2. \end{aligned}$$

Choose $\eta > 0$ smaller than the coercivity constant of a . Coercivity, together with the product rule and interpolation bound

$$\mathcal{N}_{\sigma,\Omega}(v)^2 \leq C(\|w\|_{H^m(\Omega)}^2 + q^{2m}\|w\|_{L^2(\Omega)}^2),$$

then yields (2.7). $\square$

For variable-coefficient forms, the spatial analytic assumptions are supplied by Proposition A.1, whereas the Carleman estimate and numerical stiffness compatibility must be verified separately. Weight preservation also depends on the choice of boundary condition. In particular, a general smooth weight does not preserve a Neumann domain, since $\partial_\nu(\sigma v) = \sigma \partial_\nu v + v \partial_\nu \sigma$.

The Dirichlet Laplacian satisfies the spatial analytic assumptions on convex polytopes and boxes.

Corollary A.2. *Assumptions (A1) to (A3) hold for the Dirichlet Laplacian on a fixed bounded convex polytope in $\mathbb{R}^d$, $d = 2, 3$, or on a box in any fixed dimension, with $m = 1$, $V = H_0^1(\Omega)$, and $a(v, w) = (\nabla v, \nabla w)_{L^2(\Omega)}$.*

Proof. Grisvard's convex-domain regularity theorem [14, Theorem 3.2.1.2] gives $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and (2.4). Poincaré's inequality and Proposition A.1 complete the verification. $\square$

A.2 Continuous Carleman estimate with matching time endpoints

We now verify the continuous Carleman assumption for the heat operator. Since the auxiliary estimates are applied to periodic-in-time functions with matching endpoints, we state the estimate in that setting. In this subsection, integrals are over $\mathcal{Q}$ unless another domain is indicated.

Proposition A.3. *Consider the Dirichlet Laplacian on any domain in Corollary A.2, and fix a nonempty $\omega_0 \Subset \Omega$. Then there exist a negative smooth weight ρ and constants $0 < \beta \leq 1$, $q_c \geq 1$, and $C > 0$, determined only by d, Ω, ω_0 , such that (2.8) holds for every $T > 0$, every q satisfying (2.1), both signs of ϵ , and every $v \in H^1(0, T; H) \cap L^2(0, T; D(A))$ with $v(0) = v(T)$.*

Here $m = 1$, $\mathcal{D}_1 v = \nabla v$, and the Carleman exponents are $(c_0, c_1; o) = (3, 1; 3)$.

Proof. We prove the estimate by choosing a suitable weight, specializing the pointwise Carleman inequality of [11, Corollary 3.1], and then absorbing the lower-order terms.

Choose a ball $B_* \Subset \omega_0$ centred at x_* , and set

$$\psi = -|x - x_*|^2, \quad \phi = e^{\lambda \psi}, \quad \rho = \phi - 2.$$

We prove the estimate for $\epsilon = 1$ by specializing the pointwise inequality of [11, Corollary 3.1] to the periodic weight $\ell = q\rho$.

First take v smooth and periodic in time with values in $D(A)$, and set $z = \theta v$. Apply the cited inequality to v , with weighted unknown z , choosing

$$p^{jk} = \delta_{jk}, \quad \Psi = -\Delta \ell - 8q\lambda\phi, \quad \Phi = 0.$$

The total time derivative integrates to zero by periodicity. The spatial flux on each face is $2q\partial_\nu \phi |\partial_\nu z|^2 \leq 0$ by convexity, and the face integrals are justified by the Green identities on $H^2(\Omega) \cap H_0^1(\Omega)$.

The matrix (c^{jk}) in (3.5) of [11] equals

$$4q\lambda\phi \text{Id}_{\mathbb{R}^d} + 2q\lambda^2\phi \nabla \psi \otimes \nabla \psi.$$

Substituting this into (3.3)–(3.5) of [11], and then applying Young's inequality together with (2.1), yields, with $C > 0$ independent of λ ,

$$\int q\lambda\phi |\nabla z|^2 + \int q^3 \lambda^3 \phi^3 |\nabla \psi|^2 (\lambda |\nabla \psi|^2 - C) z^2 \leq C \|\theta P_1 v\|_{L^2(\mathcal{Q})}^2 + C(\lambda) \int (q^2 + (\beta + \beta^2)q^3) z^2. \quad (\text{A.2})$$

The constants in this estimate are independent of β, q_c, T , and the dependence on λ is displayed explicitly.

Choose λ so that $\lambda |\nabla \psi|^2 \geq 2C$ outside B_* . Then choose β sufficiently small and q_c sufficiently large to absorb the right-hand remainder outside B_* . The remaining zeroth-order terms on B_* are bounded by $C \int_{\mathcal{Q}_{\omega_0}} q^3 z^2$. Returning from z to v proves (2.8) with the exponents $(c_0, c_1; o) = (3, 1; 3)$.

Density extends the estimate to the stated space, and time reflection gives the case $\epsilon = -1$. $\square$

Remark A.1. Let $b \in C^\infty(\bar{\Omega})$ be nonnegative and set $A_b = -\Delta_D + b$ on a convex polytope as above. Bounded perturbation, coercivity, and the Dirichlet regularity estimate give $D(A_b) = H^2(\Omega) \cap H_0^1(\Omega)$ and (2.4), so Proposition A.1 applies. In Proposition A.3, the additional residual bv is absorbed by the q^3 term after increasing q_c , with the same Carleman exponents. Taking the exact stiffness form $a_b(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)} + (bu, v)_{L^2(\Omega)}$ for consistent P_1 elements then gives the same transfer result.

B Cartesian difference schemes and numerical assumptions

We complete the proof of Proposition 6.2. All norms in this appendix are spatial.

Step 1. In this step, we verify the moments and load lift.

The basis on the infinite uniform Kuhn grid is translation invariant and forms a partition of unity. Each interior nodal star is complete and centrally symmetric, including those adjacent to the boundary. Hence

$$\int_{\Omega} \phi_i dx = \omega_i, \quad \int_{\Omega} (x - x_i) \phi_i dx = 0. \quad (\text{B.1})$$

In particular, $(J_h p)_i = p(x_i)$ for affine p .

On each simplex K , with barycentric coordinates λ_i , define

$$(\mathcal{L}_h f)|_K = \sum_{i: x_i \in K \cap \Omega} f_i((d+2)\lambda_i - 1).$$

For each interior vertex x_j , the identities

$$\int_K ((d+2)\lambda_i - 1) \lambda_j dx = \frac{\delta_{ij}|K|}{d+1}, \quad \sum_{K \ni x_j} \frac{|K|}{d+1} = \omega_j$$

give $J_h \mathcal{L}_h = \text{Id}_{X_h}$. Local polynomial norm equivalence and (6.4) give the load stability bound

$$\|\sigma \mathcal{L}_h f\|_{L^2(\Omega)} \leq C \|f\|_{X_h, \sigma}.$$

The reconstruction satisfies the norm equivalence

$$\|\sigma R_h z\|_{L^2(\Omega)} \asymp \|z\|_{X_h, \sigma}. \quad (\text{B.2})$$

The same argument gives the local stability of Q_h :

$$\|Q_h v\|_{L^2(K)} \leq C \|v\|_{L^2(B_K)}, \quad (\text{B.3})$$

where B_K is the smallest axis-aligned rectangular block containing K and the supports of its active basis functions. The blocks have diameter $O(h)$, bounded aspect ratios, and bounded overlap. Together with (6.14), these estimates prove Assumptions (N1) and (N2).

Step 2. In this step, we establish the weighted approximation estimates.

The first lemma gives a local affine approximation that respects the physical boundary, including at corners. Let $\mathcal{P}_1(B_K)$ be the space of affine polynomials on B_K , let $\mathcal{F}_K$ be the set of faces of B_K contained in $\partial\Omega$, and define

$$\mathcal{P}_{1, \mathcal{F}_K}(B_K) := \{p \in \mathcal{P}_1(B_K) : p|_F = 0 \text{ for every } F \in \mathcal{F}_K\}.$$

Lemma B.1. *For every $v \in H^2(B_K)$ with zero trace on each $F \in \mathcal{F}_K$, there exists $p_K \in \mathcal{P}_{1, \mathcal{F}_K}(B_K)$ such that*

$$\|v - p_K\|_{L^2(B_K)} + h \|\nabla(v - p_K)\|_{L^2(B_K)} \leq Ch^2 \|D^2 v\|_{L^2(B_K)}, \quad (\text{B.4})$$

where the constant C depends only on the dimension and the grid-step comparability bound.

Proof. Map B_K to the unit cube $\widehat{B}$, and denote the transformed function and physical faces by $\widehat{v}$ and $\widehat{\mathcal{F}}_K$. By the Bramble-Hilbert lemma [8, Lemma 4.3.8], there exists $\widetilde{p} \in \mathcal{P}_1(\widehat{B})$ such that

$$\|\widehat{v} - \widetilde{p}\|_{H^1(\widehat{B})} \leq C \|D^2 \widehat{v}\|_{L^2(\widehat{B})}.$$

In the finite-dimensional space $\mathcal{P}_1(\widehat{B})$, the kernel of the face-trace map is $\mathcal{P}_{1,\widehat{\mathcal{F}}_K}(\widehat{B})$. Norm equivalence on its orthogonal complement therefore gives

$$\inf_{\widehat{p} \in \mathcal{P}_{1,\widehat{\mathcal{F}}_K}(\widehat{B})} \|\widetilde{p} - \widehat{p}\|_{H^1(\widehat{B})} \leq C \left(\sum_{\widehat{F} \in \widehat{\mathcal{F}}_K} \|\widetilde{p}\|_{L^2(\widehat{F})}^2 \right)^{1/2} \leq C \|\widehat{v} - \widetilde{p}\|_{H^1(\widehat{B})},$$

where the last inequality uses the zero traces of $\widehat{v}$ together with the trace theorem. For an interior block, we may take $\widehat{p} = \widetilde{p}$. For a boundary block, we choose a minimizer $\widehat{p}$ in the preceding estimate. Applying the triangle inequality and scaling back to B_K proves (B.4). The constants are uniform because the reference cube has only finitely many subsets of faces and the block side lengths are comparable to h .

Finally, if B_K meets two nonparallel physical boundary faces, then the only affine function vanishing on both is zero. Hence $p_K = 0$, and the same argument shows that the zero traces eliminate the affine kernel of the Hessian seminorm. $\square$

The moment reconstruction then gives a global weighted estimate without using point values of v .

Lemma B.2. *For every $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and every allowed weight σ with $hq \leq 1$ at the fixed time,*

$$\|\sigma(v - Q_h v)\|_{L^2(\Omega)} + h \|\sigma \nabla(v - Q_h v)\|_{L^2(\Omega)} \leq Ch^2 \|\sigma D^2 v\|_{L^2(\Omega)}, \quad (\text{B.5})$$

where the constant is independent of h, q, σ and v .

Proof. Fix K . Choose p_K as in Lemma B.1. The moment identities (B.1) reproduce p_K at every interior vertex of K , while the boundary vertices have value zero by the choice of p_K . Hence $Q_h p_K = p_K$ on K , and therefore

$$(v - Q_h v)|_K = (v - p_K)|_K - Q_h(v - p_K)|_K.$$

Since only values on B_K enter this local action of Q_h , (B.3), the finite-element inverse estimate, and (B.4) give

$$\begin{aligned} \|v - Q_h v\|_{L^2(K)} + h \|\nabla(v - Q_h v)\|_{L^2(K)} &\leq C(\|v - p_K\|_{L^2(B_K)} + h \|\nabla(v - p_K)\|_{L^2(B_K)}) \\ &\leq Ch^2 \|D^2 v\|_{L^2(B_K)}. \end{aligned}$$

It remains to pass from this local estimate to the global one. Since $hq \leq 1$, (3.8) makes σ uniformly comparable to any of its values on each B_K . Multiplying the local estimate by such a value, squaring, and summing over K therefore yields (B.5), by the bounded overlap of the blocks. No point values of v are used, so the argument applies in every fixed dimension. $\square$

Since $hq \leq 1$, Lemma B.2 proves Assumption (N3) with $a_E = Ch$ and $a_0 = Ch^2$.

Step 3. In this step, we verify the weighted test and observation estimates.

The diagonal map $\mathcal{T}_{\sigma,h}$ is T -periodic, positive and self-adjoint, and (B.2) and (3.8) give (2.18) and (2.19). Moreover, $R_h \mathcal{T}_{\sigma,h} z = I_h(\sigma^2 R_h z)$, where I_h is nodal interpolation. On each element, normalize the weight by σ_K from (6.4). For $hq \leq \delta_{\text{num}}$, this normalized weight is uniformly comparable to one and its derivatives of order $j \leq 2$ are $O(q^j)$. The local superapproximation argument in [9, Theorem 2.1], rescaled to a unit-size element with weight scale $(h_K q)^{-1}$, yields

$$\|\sigma^{-1}(R_h \mathcal{T}_{\sigma,h} z - \sigma^2 R_h z)\|_{L^2(\Omega)} + h \|\sigma^{-1} \nabla(R_h \mathcal{T}_{\sigma,h} z - \sigma^2 R_h z)\|_{L^2(\Omega)} \leq Ch^2 (q \|\sigma \nabla R_h z\|_{L^2(\Omega)} + q^2 \|\sigma R_h z\|_{L^2(\Omega)}).$$

Here the interpolated product is smooth on each simplex, so the estimate holds in every fixed dimension. Since $hq \leq 1$, it implies (2.20). Finally, for small h , every vertex of an element meeting ω_0 lies in ω ; local norm equivalence gives (2.21).

C Consistency for the control limit

We record the consistency properties used in Corollary 7.9. Throughout this appendix, T , y_0 , and b are fixed; spatial limits are taken over the admissible meshes, and temporal limits over $\delta t = T/N$ with $N \in \mathbb{N}$. Joint limits impose no coupling between h and δt . The definitions give (7.30), and the reconstructions satisfy

$$\|E_h z - R_h z\|_{L^2(\Omega)} \leq Ch \|R_h z\|_{H^1(\Omega)}. \quad (\text{C.1})$$

For finite differences, this estimate follows by scaling on each dual-cell patch, since $z_i - R_h z$ vanishes for constant nodal data; the same bound on the uncovered boundary strip of width $O(h)$ follows from the zero-trace Poincaré estimate. For finite elements, we simply have $E_h = R_h$. Finally, in both realizations the exact stiffness identity

$$(A_h z, w)_{X_h} = a(R_h z, R_h w) \quad (\text{C.2})$$

holds, either by definition or by (6.14).

We now record the consistency properties of the discretized data. For finite elements, let $b_{h,\delta t} = b^n$ on each time cell; for finite differences, let $b_{h,\delta t} = b_i^n$ on $(t_{n-1}, t_n] \times D_i$, extended by zero. Standard projection and cell-average approximation then yield

$$\|y_{0,h}\|_{X_h} \leq \|y_0\|_H$$

and

$$\lim_{h \rightarrow 0} \|E_h y_{0,h} - y_0\|_H = 0. \quad (\text{C.3})$$

The averaged potential satisfies $\|b_{h,\delta t}\|_{L^\infty(\mathcal{Q})} \leq M$ and

$$\lim_{h, \delta t \rightarrow 0} \|b_{h,\delta t} - b\|_{L^2(\mathcal{Q})} = 0, \quad (\text{C.4})$$

and, on the n th time cell, we have the exact pairing

$$(\mathbf{V}_h^n z, w)_{X_h} = (b_{h,\delta t} E_h z, E_h w)_H.$$

We conclude by recording the consistency of the test functions. For a fixed finite sum $\phi(t, x) = \sum_\ell \eta_\ell(t) \psi_\ell(x)$ with $\eta_\ell \in C^\infty([0, T])$ and $\psi_\ell \in C_c^\infty(\Omega)$, set $\phi_h^n = I_h \phi(t_n)$ and $\Phi_{h,\delta t}^-(t) = \phi_h^{n-1}$ on $(t_{n-1}, t_n]$. Here I_h denotes nodal interpolation for finite elements and nodal sampling for finite differences. Spatial interpolation and time sampling yield

$$\lim_{h, \delta t \rightarrow 0} \|R_h \Phi_{h,\delta t}^- - \phi\|_{L^2(0,T;V)} = 0. \quad (\text{C.5})$$

The reconstruction using E_h satisfies

$$\lim_{h, \delta t \rightarrow 0} \|E_h \Phi_{h,\delta t}^- - \phi\|_{L^\infty(\mathcal{Q})} = 0.$$

The time difference quotient, defined piecewise on $(t_{n-1}, t_n]$ by the expression below, satisfies

$$\lim_{h, \delta t \rightarrow 0} \left\| E_h \left(\frac{\phi_h^n - \phi_h^{n-1}}{\delta t} \right) - \partial_t \phi \right\|_{L^2(0,T;H)} = 0.$$

At the endpoints, spatial interpolation gives

$$\lim_{h \rightarrow 0} \|E_h I_h \phi(\tau) - \phi(\tau)\|_H = 0, \quad \tau \in \{0, T\}.$$

Finally, the potential and test-function limits give

$$\lim_{h, \delta t \rightarrow 0} \|b_{h,\delta t} E_h \Phi_{h,\delta t}^- - b\phi\|_{L^2(\mathcal{Q})} = 0.$$

D Proofs of the filtering estimates

We use the notation and hypotheses of Subsection 7.3. The proofs are given in two parts: the reverse norm estimate in Subsection D.1, and the complementary-penalty construction in Subsection D.2.

D.1 Reverse norm estimate

We first prove that terminal adjoint data in the filtered subspace remain controlled by the initial adjoint norm.

Proof of Lemma 7.4. All norms and inner products in this proof are those of X_h . For $z \neq 0$, put

$$\nu(z) = \frac{(A_h z, z)}{\|z\|^2}.$$

One resolvent step. Let $V = V^*$ with $\|V\| \leq M$, and let

$$x = [\text{Id} + \delta t(A_h + V)]z.$$

Normalize $\|z\| = 1$, set $r = \nu(z)$ and $b = (Vz, z)$, and define the components orthogonal to z by

$$v = (A_h - r\text{Id})z, \quad w = (V - b\text{Id})z.$$

Put $\xi = v + w$ and $\gamma = 1 + \delta t(r + b)$. Then $\xi \perp z$, $\|w\| \leq M$, $x = \gamma z + \delta t \xi$, and

$$\|x\|^2 = \gamma^2 + (\delta t)^2 \|\xi\|^2.$$

Since $A_h \geq 0$, completion of the square gives

$$\begin{aligned} (A_h x, x) - r\|x\|^2 &= 2\gamma\delta t(v, \xi) + (\delta t)^2((A_h - r\text{Id})\xi, \xi) \\ &\geq \delta t(2\gamma - \delta t r)\|\xi\|^2 - 2\gamma\delta t(w, \xi) \\ &\geq -\frac{\delta t \gamma^2 M^2}{2\gamma - \delta t r}, \end{aligned} \tag{D.1}$$

where $2\gamma - \delta t r \geq 2(1 - \delta t M) > 0$. Dividing (D.1) by $\|x\|^2$ and using $\|x\|^2 \geq \gamma^2$, we obtain

$$\nu(z) \leq \nu(x) + \frac{\delta t M^2}{2(1 - \delta t M)} \leq \nu(x) + \delta t M^2. \tag{D.2}$$

Iteration along the backward Euler recursion. Put $z_j = p^{N-j}$ and $V_j = V_h^{N-j+1}$, so that

$$[\text{Id} + \delta t(A_h + V_j)]z_j = z_{j-1}. \tag{D.3}$$

If $p^N \neq 0$, all z_j are nonzero, since each resolvent is invertible. Writing $\nu_j = \nu(z_j)$, (D.2) gives

$$\nu_j \leq \nu_0 + j\delta t M^2.$$

Applying $S_0 = (\text{Id} + \delta t A_h)^{-1}$ to both sides of (D.3) gives

$$z_j = S_0 z_{j-1} - \delta t S_0 V_j z_j.$$

Taking norms and using the reverse triangle inequality $\|a - b\| \geq \|a\| - \|b\|$, we obtain

$$\|z_j\| = \|S_0 z_{j-1} - \delta t S_0 V_j z_j\| \geq \|S_0 z_{j-1}\| - \delta t \|S_0 V_j z_j\|. \tag{D.4}$$

Since $\|S_0\| \leq 1$ and $\|V_j\| \leq M$, we have

$$\|S_0 V_j z_j\| \leq \|S_0\| \|V_j z_j\| \leq M \|z_j\|.$$

This, together with (D.4), implies

$$\|z_j\| \geq \|S_0 z_{j-1}\| - \delta t M \|z_j\|. \tag{D.5}$$

Adding $\delta t M \|z_j\|$ to both sides yields

$$(1 + \delta t M) \|z_j\| \geq \|S_0 z_{j-1}\|.$$

Thus

$$\|z_j\| \geq \frac{\|S_0 z_{j-1}\|}{1 + \delta t M}. \quad (\text{D.6})$$

It remains to estimate $\|S_0 z_{j-1}\|$ from below. By the spectral lower bound for S_0 ,

$$\|S_0 x\| \geq \frac{\|x\|}{1 + \delta t \nu(x)}, \quad x \neq 0.$$

Taking $x = z_{j-1}$ gives

$$\|S_0 z_{j-1}\| \geq \frac{\|z_{j-1}\|}{1 + \delta t \nu(z_{j-1})} = \frac{\|z_{j-1}\|}{1 + \delta t \nu_{j-1}}. \quad (\text{D.7})$$

Combining (D.6) and (D.7), we obtain

$$\|z_j\| \geq \frac{\|z_{j-1}\|}{(1 + \delta t M)(1 + \delta t \nu_{j-1})}. \quad (\text{D.8})$$

Bounding the norm ratio. Taking logarithms of (D.8) and summing, we obtain

$$\log \frac{\|z_N\|}{\|z_0\|} \geq -MT - \delta t \sum_{j=1}^N \nu_{j-1} \geq -MT - T\nu_0 - \frac{1}{2}M^2 T^2.$$

Since $z_0 = p^N \in F$ gives $\nu_0 \leq \Lambda_F$, this proves (7.19). The zero terminal datum is immediate. $\square$

D.2 Complementary terminal penalty and filtered control

We next prove the filtered-control result by combining the reverse estimate with the relaxed observability inequality.

Proof of Corollary 7.6. Write $P = \Pi_F$, $Q = \text{Id}_{X_h} - P$, and $\varepsilon = e^{-d_0 S}$. The proof is divided into four steps: we first derive a complementary observability inequality, then introduce a penalized functional and identify its minimizer, next construct the control and estimate the terminal state, and finally estimate the penalization parameter η_F .

Step 1: A complementary observability inequality. For a terminal datum $z \in X_h$, let $p[z]$ solve (7.2), and define

$$\mathcal{R}z = p[z]^0, \quad \mathcal{O}z = (B_h^* p[z]^{n-1})_{n=1}^N.$$

By (7.3), the operator $\mathcal{R}$ satisfies

$$\|\mathcal{R}\| \leq L_{T,M} := e^{2MT}.$$

Moreover, (7.19) applied to $Pz \in F$ gives

$$\|Pz\|_{X_h} \leq K_F \|\mathcal{R}Pz\|_{X_h} \leq K_F (\|\mathcal{R}z\|_{X_h} + L_{T,M} \|Qz\|_{X_h}). \quad (\text{D.9})$$

This, together with the orthogonality of P and Q , implies

$$\|z\|_{X_h}^2 \leq 2K_F^2 \|\mathcal{R}z\|_{X_h}^2 + (1 + 2L_{T,M}^2 K_F^2) \|Qz\|_{X_h}^2. \quad (\text{D.10})$$

Set $\eta_F^2 = \varepsilon^2(1 + 2L_{T,M}^2 K_F^2)$. Substituting (D.10) into (7.7) and absorbing the term $2C_0 \varepsilon^2 K_F^2 \leq 1/2$ yields the complementary observability inequality

$$\|\mathcal{R}z\|_{X_h}^2 \leq 2C_0 (\|\mathcal{O}z\|_{\ell_{\delta t}^2(U_h)}^2 + \eta_F^2 \|Qz\|_{X_h}^2). \quad (\text{D.11})$$

Step 2: The penalized functional and its minimizer. Consider on X_h the functional

$$J_F(z) = \frac{1}{2} \|\mathcal{O}z\|_{\ell_{\delta t}^2(U_h)}^2 + \frac{1}{2} \eta_F^2 \|Qz\|_{X_h}^2 - (y_{0,h}, \mathcal{R}z)_{X_h}. \quad (\text{D.12})$$

Its quadratic part is positive definite. Indeed, if both quadratic terms vanish, then (D.11) gives $\mathcal{R}z = 0$, and the invertibility of the resolvents implies $z = 0$. Since X_h is finite-dimensional, J_F therefore has a unique minimizer $\widehat{z} \in X_h$. Its optimality condition reads

$$(\mathcal{O}\widehat{z}, \mathcal{O}z)_{\ell_{\delta t}^2(U_h)} + \eta_F^2 (Q\widehat{z}, Qz)_{X_h} = (y_{0,h}, \mathcal{R}z)_{X_h}, \quad z \in X_h.$$

Step 3: Construction of the control and estimate of the terminal state. Choose the control $u^n = -B_h^* p[\hat{z}]^{n-1}$, $1 \leq n \leq N$, and let y solve (7.1). Combining the optimality condition with the duality identity (7.5) gives $y^N = \eta_F^2 Q\hat{z}$, and hence $Py^N = 0$. It remains to bound the control and the terminal state. Taking $z = \hat{z}$ in the optimality condition yields

$$\|\mathcal{O}\hat{z}\|_{\ell_{\delta t}^2(U_h)}^2 + \eta_F^2 \|Q\hat{z}\|_{X_h}^2 = (y_{0,h}, \mathcal{R}\hat{z})_{X_h}. \quad (\text{D.13})$$

By (D.11), the right-hand side of (D.13) is bounded by $\|y_{0,h}\|_{X_h} (2C_0(\|\mathcal{O}\hat{z}\|_{\ell_{\delta t}^2(U_h)}^2 + \eta_F^2 \|Q\hat{z}\|_{X_h}^2))^{1/2}$. Therefore,

$$\|\mathcal{O}\hat{z}\|_{\ell_{\delta t}^2(U_h)}^2 + \eta_F^2 \|Q\hat{z}\|_{X_h}^2 \leq 2C_0 \|y_{0,h}\|_{X_h}^2. \quad (\text{D.14})$$

Since $u^n = -B_h^* p[\hat{z}]^{n-1}$, it follows from (D.14) that

$$\|u\|_{\ell_{\delta t}^2(U_h)}^2 + \eta_F^2 \|Q\hat{z}\|_{X_h}^2 \leq 2C_0 \|y_{0,h}\|_{X_h}^2.$$

In particular,

$$\|y^N\|_{X_h} = \eta_F^2 \|Q\hat{z}\|_{X_h} \leq \sqrt{2C_0} \eta_F \|y_{0,h}\|_{X_h}.$$

Step 4: Estimate of η_F . Finally, since $K_F \geq 1$ and (7.20) holds, we have

$$\eta_F \leq \sqrt{1 + 2L_{T,M}^2} e^{D_{T,M}} e^{-(1-\vartheta)d_0 S}.$$

Combining this with the bounds obtained in Step 3 proves (7.22) and (7.23). $\square$

References

- [1] F. BOYER, *On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems*, ESAIM Proc., 41 (2013), pp. 15–58.
- [2] F. BOYER AND V. HERNÁNDEZ-SANTAMARÍA, *Carleman estimates for time-discrete parabolic equations and applications to controllability*, ESAIM Control Optim. Calc. Var., 26 (2020), Paper No. 12.
- [3] F. BOYER AND V. HERNÁNDEZ-SANTAMARÍA, *Boundary controllability of time-discrete parabolic systems: A moments method approach*, Numer. Math., 158 (2026), pp. 1–63.
- [4] F. BOYER, F. HUBERT, AND J. LE ROUSSEAU, *Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations*, J. Math. Pures Appl., 93 (2010), pp. 240–276.
- [5] F. BOYER, F. HUBERT, AND J. LE ROUSSEAU, *Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications*, SIAM J. Control Optim., 48 (2010), pp. 5357–5397.
- [6] F. BOYER, F. HUBERT, AND J. LE ROUSSEAU, *Uniform controllability properties for space/time-discretized parabolic equations*, Numer. Math., 118 (2011), pp. 601–661.
- [7] F. BOYER AND J. LE ROUSSEAU, *Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 31 (2014), pp. 1035–1078.
- [8] S. C. BRENNER AND L. R. SCOTT, *The Mathematical Theory of Finite Element Methods*, Springer, New York, third ed., 2008.
- [9] A. DEMLOW, J. GUZMÁN, AND A. H. SCHATZ, *Local energy estimates for the finite element method on sharply varying grids*, Math. Comp., 80 (2011), pp. 1–9.
- [10] L. DIENING, J. STORN, AND T. TSCHERPEL, *On the Sobolev and L^p -stability of the L^2 -projection*, SIAM J. Numer. Anal., 59 (2021), pp. 2571–2607.

- [11] X. FU, Q. LÜ, AND X. ZHANG, *Carleman Estimates for Second Order Partial Differential Operators and Applications: A Unified Approach*, SpringerBriefs in Mathematics, Springer, Cham, 2019.
- [12] A. V. FURSIKOV AND O. Y. IMANUVILOV, *Controllability of Evolution Equations*, no. 34 in Lecture Notes Series, Seoul National University, Seoul, 1996.
- [13] P. GONZÁLEZ CASANOVA AND V. HERNÁNDEZ-SANTAMARÍA, *Carleman estimates and controllability results for fully discrete approximations of 1D parabolic equations*, Adv. Comput. Math., 47 (2021), Paper No. 72.
- [14] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Society for Industrial and Applied Mathematics, 2011.
- [15] L. I. IGNAT AND E. ZUAZUA, *Convergence of a two-grid algorithm for the control of the wave equation*, J. Eur. Math. Soc., 11 (2009), pp. 351–391.
- [16] B. KOVÁCS, B. LI, AND C. LUBICH, *A-stable time discretizations preserve maximal parabolic regularity*, SIAM J. Numer. Anal., 54 (2016), pp. 3600–3624.
- [17] R. LECAROS, J. LÓPEZ-RÍOS, AND A. A. PÉREZ, *Lipschitz stability and reconstruction in inverse problems for semi-discrete parabolic operators*, Inverse Problems, 41 (2025), Paper No. 115012.
- [18] J.-L. LIONS, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Masson, Paris, 1988.
- [19] N. G. MEYERS, *An L^p -estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3), 17 (1963), pp. 189–206.
- [20] M. NEGREANU AND E. ZUAZUA, *Convergence of a multigrid method for the controllability of a 1-d wave equation*, C. R. Math. Acad. Sci. Paris, 338 (2004), pp. 413–418.
- [21] E. ZUAZUA, *Propagation, observation, and control of waves approximated by finite difference methods*, SIAM Rev., 47 (2005), pp. 197–243.
- [22] E. ZUAZUA, *Control and numerical approximation of the wave and heat equations*, in Proceedings of the International Congress of Mathematicians, Madrid, 2006, M. Sanz-Solé, J. Soria, J. L. Varona, and J. Verdera, eds., vol. III, European Mathematical Society, Zürich, 2006, pp. 1389–1417.
- [23] E. ZUAZUA, *Controllability and observability of partial differential equations: some results and open problems*, in Handbook of Differential Equations: Evolutionary Equations. Vol. III, Elsevier/North-Holland, Amsterdam, 2007, pp. 527–621.