

Null controllability for stochastic coupled systems of fourth order parabolic equations*

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Abstract

This paper aims to establish null controllability for systems coupled by two backward fourth order stochastic parabolic equations. The main goal is to control both equations with only one control act on the drift term. To achieve this, we develop a new global Carleman estimate for fourth order stochastic parabolic equations, which allows us to deduce a suitable observability inequality for the adjoint systems.

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1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Denote by $\mathbb{F}$ the progressive σ -field with respect to $\mathbf{F}$. Let H be a Banach space. Write $L^2_{\mathbb{F}_t}(\Omega; H)$ for the space of all $\mathcal{F}_t$ -measurable random variables ξ such that $\mathbb{E}|\xi|_H^2 < \infty$; $L^2_{\mathbb{F}}(0, T; H)$ for the space consisting of all H -valued $\mathbf{F}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty$; $L^\infty_{\mathbb{F}}(0, T; H)$ for the space consisting of all H -valued $\mathbf{F}$ -adapted bounded processes. All these spaces are Banach spaces with the canonical norms (e.g., [17, Section 2.6]).

Let $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded connected domain with a C^6 boundary Γ . Let $G_0 \subset G$ be a nonempty subset. Set $Q = (0, T) \times G$ and $\Sigma = (0, T) \times \Gamma$. Denote by $\nu(x) = (\nu^1(x), \dots, \nu^n(x))$ the unit outward normal vector of Γ at point x . As usual, χ_{G_0} denotes the characteristic function of G_0 .

Consider the following coupled fourth order backward stochastic parabolic equations:

$$\begin{cases} dy_1 - \Delta^2 y_1 dt = (a_1 y_1 + a_2 y_2 + a_3 Y_1) dt + Y_1 dW(t) & \text{in } Q, \\ dy_2 - \Delta^2 y_2 dt = (b_1 y_1 + b_2 y_2 + b_3 Y_2 + \chi_{G_0} u) dt + Y_2 dW(t) & \text{in } Q, \\ y_1 = \frac{\partial y_1}{\partial \nu} = 0, \quad y_2 = \frac{\partial y_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ y_1(T) = y_1^T, \quad y_2(T) = y_2^T & \text{in } G, \end{cases} \quad (1.1)$$

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where $(y_1^T, y_2^T) \in [L_{\mathcal{F}_T}^2(\Omega; L^2(G))]^2$ is a terminal state, $u \in L_{\mathbb{F}}^2(0, T; L^2(G_0))$ is a control and

$$\begin{cases} a_1, a_2 \in L_{\mathbb{F}}^\infty(0, T; L^\infty(G)), & a_3 \in L_{\mathbb{F}}^\infty(0, T; W^{2,\infty}(G)), \\ b_1, b_2 \in L_{\mathbb{F}}^\infty(0, T; L^\infty(G)), & b_3 \in L_{\mathbb{F}}^\infty(0, T; W^{2,\infty}(G)). \end{cases}$$

By the classical wellposedness result for backward stochastic evolution equations (e.g., [17, Theorem 4.11]), we know that (1.1) admits a unique weak solution

$$(y_1, y_2; Y_1, Y_2) \in [L_{\mathbb{F}}^2(\Omega; C([0, T]; L^2(G))) \cap L_{\mathbb{F}}^2(0, T; H^2(G) \cap H_0^1(G))]^2 \times [L_{\mathbb{F}}^2(0, T; L^2(G))]^2.$$

Now we give the definition of the null controllability for (1.1).

Definition 1.1. The system (1.1) is called null controllable at a time $T > 0$ if for any given $(y_1^T, y_2^T) \in [L_{\mathcal{F}_T}^2(\Omega; L^2(G))]^2$, there exists a control $u \in L_{\mathbb{F}}^2(0, T; L^2(G_0))$ such that the solution $(y_1, y_2; Y_1, Y_2)$ to (1.1) satisfies that $(y_1(0), y_2(0)) = (0, 0)$, $\mathbb{P}$ -a.s.

Note that we only act one control in the second equation. To control the entire system, we expect that the action of y_2 to y_1 is sufficiently effective. Hence, it is reasonable to make the following assumption.

Assumption 1.1. *There exists a nonempty open set $G_1 \subset G_0$ and a constant $\sigma > 0$ such that $a_2(x, t) \geq \sigma$ or $a_2(x, t) \leq -\sigma$, a.e. $(x, t) \in G_1 \times (0, T)$, $\mathbb{P}$ -a.s.*

The main result of this paper is the following.

Theorem 1.1. *Under the Assumption 1.1, system (1.1) is null controllable at any time $T > 0$.*

Remark 1.1. From the proof of Theorem 1.2, it is not difficult to see that Theorem 1.1 still holds when two equations of (1.1) contains ∇y_1 , $\nabla^2 y_1$, $\nabla \Delta y_1$ and Y_1 in the drift terms. Indeed, for instance, by adding Y_1 to the drift term of the second equation in (1.1), direct computation reveals that Proposition 4.1 and Proposition 4.3 still hold. Therefore, Theorem 1.1 remains valid. We do not introduce these terms in the system (1.1) for the simplicity of notations.

Remark 1.2. Using the Carleman inequality (1.6) and a proof similar to Theorem 1.2, we can establish the controllability of the backward fourth order stochastic parabolic equation. That is, for any $y_T \in L_{\mathcal{F}_T}^2(\Omega; L^2(G))$, there exists a control $u \in L_{\mathbb{F}}^2(0, T; L^2(G_0))$ such that the solution y of equation

$$\begin{cases} dy - \Delta^2 y dt = (b_2 y + b_3 Y + \chi_{G_0} u) dt + Y dW(t) & \text{in } Q, \\ y = \frac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma, \\ y(T) = y_T & \text{in } G. \end{cases}$$

satisfying $y(0) = 0$, $\mathbb{P}$ -a.s.

Remark 1.3. It would be quite interesting to study whether the system (1.1) remains controllable if they are coupled through diffusion terms, i.e., by adding Y_2 in the drift term of the first equation and assuming $a_2 \equiv 0$. We currently do not know how to prove it.

To prove the null controllability of (1.1), we introduce the adjoint equation of (1.1):

$$\begin{cases} dz_1 + \Delta^2 z_1 dt = -(a_1 z_1 + b_1 z_2) dt - a_3 z_1 dW(t) & \text{in } Q, \\ dz_2 + \Delta^2 z_2 dt = -(a_2 z_1 + b_2 z_2) dt - b_3 z_2 dW(t) & \text{in } Q, \\ z_1 = \frac{\partial z_1}{\partial \nu} = 0, \quad z_2 = \frac{\partial z_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ z_1(0) = z_1^0, \quad z_2(0) = z_2^0 & \text{in } G. \end{cases} \quad (1.2)$$

By the classical wellposedness result for stochastic evolution equations (e.g., [17, Theorem 3.24]), we know that (1.2) admits a unique weak solution

$$(z_1, z_2) \in [L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G))]^2.$$

By a standard duality argument, Theorem 1.1 is implied by the following observability estimate (e.g., [17, Theorem 7.17]).

Theorem 1.2. *There exists a constant $C > 0$ such that the solution (z_1, z_2) to system (1.2) satisfies*

$$|(z_1(T), z_2(T))|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G) \times L^2(G))} \leq C |z_2|_{L^2_{\mathbb{F}}(0, T; L^2(G_0))}.$$

Here and in what follows, we denote by C a generic positive constant depending on G , G_0 , T and a_i, b_i , $i = 1, 2, 3$, whose value may vary from line to line.

A plenty of works on controllability and observability estimates for coupled deterministic parabolic systems exist in the literature, including works such as [2, 3, 6, 12, 18, 19] and the references therein. The main issue, compared to the single-equation cases, is that one wants to control the system with fewer controls than equations by indirectly affecting the equations where no control term appears, thanks to the coupling terms in the system. Hence, it is more difficult to obtain the controllability for coupled systems than for the single equations. It is a fundamental problem since, in some complex systems, only a few quantities can be effectively controlled. Additional details regarding this specific subject can be found in the survey article [1].

As far as we know, there are only a few works studying controllability problems for coupled stochastic systems with fewer controls than equations, see, [8, 10, 13, 14]. In particular, in [8], the authors obtain null controllability for one dimensional linear backward stochastic systems coupling fourth- and second-order parabolic equations.

In order to prove Theorem 1.2, we will use Carleman estimates, which is derived taking inspiration from the literature [5, 7, 15, 20]. To state the Carleman estimate, we need to introduce the weight functions:

$$\alpha(x, t) = \frac{e^{\lambda(|2\eta|_{C(\overline{G})} + \eta(x))} - e^{4\lambda|\eta|_{C(\overline{G})}}}{t^{1/2}(T-t)^{1/2}}, \quad \xi(x, t) = \frac{e^{\lambda(|2\eta|_{C(\overline{G})} + \eta(x))}}{t^{1/2}(T-t)^{1/2}}. \quad (1.3)$$

Let G_i be three nonempty open sets such that $\overline{G_i} \subset G_{i-1}$ and denote by $Q_i = (0, T) \times G_i$ for $i = 1, 2, 3$. Let the function $\eta(x)$ satisfy

$$\eta \in C^6(\overline{G}), \quad \eta > 0 \text{ in } G, \quad \eta|_{\Gamma} = 0, \quad |\nabla \eta| \geq C > 0 \text{ in } \overline{G \setminus G_3} \quad (1.4)$$

For the existence of η , see [4, Lemma 1.1]. We remark that our weight functions are the same as that in [16].

We have the following Carleman estimate.

Theorem 1.3. *For any $T > 0$, there are positive constants C, λ_0 and $s_0 = s_0(T)$, such that for all $\lambda \geq \lambda_0$, $s \geq s_0$ and $\varphi \in L^2_{\mathbb{F}}(0, T; H^2(G))$, $f \in L^2_{\mathbb{F}}(0, T; L^2(G))$, $g \in L^2_{\mathbb{F}}(0, T; H^2_0(G))$ satisfying*

$$\begin{cases} d\varphi + \Delta^2 \varphi dt = f dt + g dW(t) & \text{in } Q, \\ \varphi = \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \Sigma, \end{cases} \quad (1.5)$$

it holds that

$$\begin{aligned} & \mathbb{E} \int_Q e^{2s\alpha} (s^6 \lambda^8 \xi^6 |\varphi|^2 + s^4 \lambda^6 \xi^4 |\nabla \varphi|^2 + s^3 \lambda^4 \xi^3 |\Delta \varphi|^2 + s^2 \lambda^4 \xi^2 |\nabla^2 \varphi|^2 + \lambda |\nabla \Delta \varphi|^2) dx dt \\ & \leq C \left(\mathbb{E} \int_{Q_2} s^7 \lambda^8 \xi^7 e^{2s\alpha} |\varphi|^2 dx dt + \mathbb{E} \int_Q e^{2s\alpha} (|f|^2 + s^5 \lambda^5 \xi^5 |g|^2 + s^3 \lambda^3 \xi^3 |\nabla g|^2 + s \lambda \xi |\nabla^2 g|^2) dx dt \right). \end{aligned} \quad (1.6)$$

Remark 1.4. The orders of some terms in (1.6) seem inconsistent. For instance, it is an interesting question whether $\lambda^2 |\nabla \Delta \varphi|^2$ can be included on the left-hand side of (1.6). To do this, one possible approach is to replace Φ_2 in Theorem 2.1 with $\Phi_2 = -2s\lambda^2 \xi |\nabla \eta|^2 - 2s\lambda \xi \Delta \eta - \lambda^2$. This will result in the term $-2s^2 \lambda^4 \xi^2 |\nabla \eta|^2 |\Delta \psi|^2$ appearing in M (see (3.14)), and we currently lack a method to handle this term. Hence, we currently do not know how to achieve $\lambda^2 |\nabla \Delta \varphi|^2$.

Moreover, the term $s^7 \lambda^8 \xi^7 e^{2s\alpha} |\varphi|^2$ appears on the right-hand side instead of $s^6 \lambda^8 \xi^6 e^{2s\alpha} |\varphi|^2$. It is because we need to estimate local term $\mathbb{E} \int_{(0,T) \times G_3} s^3 \lambda^4 \xi^3 \theta^2 |\Delta \varphi|^2 dx dt$ (see (3.38)). Note that the term $s^6 \lambda^8 \xi^6 e^{2s\alpha} |\varphi|^2$ on the left-hand side has a lower order of s than that on the right-hand side. This is because we utilize Lemma 2.3 to obtain this term (see (3.31)), hence, we cannot obtain results of a higher order in s . Indeed, as mentioned in [9, Remark 1.7], for a fourth-order equation, it seems difficult to let $s^7 \lambda^8 \xi^7 e^{2s\alpha} |\varphi|^2$ appear in the left hand side of (1.6).

Remark 1.5. From the proof of Theorem 1.3 and that of [16, Theorem 1.3], it is not difficult to see that Carleman inequality (1.6) is still valid when φ satisfies the boundary condition $\varphi = \Delta \varphi = 0$ on Σ . Hence, Theorem 1.1 holds true when y_i satisfies $y_i = \Delta y_i = 0$ on Σ for $i = 1, 2$.

Remark 1.6. Using the pointwise weighted identity (2.1) with $\kappa = -1$, we can deduce the Carleman estimate for the backward fourth order stochastic parabolic equations. In comparison to existing results (see [16, Theorem 1.3]), we further estimate $\nabla \Delta \varphi$, enabling us to contain $\nabla \Delta y$ in the drift term of the control system in [16].

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to prove Theorem 1.3. By means of this Carleman estimate, we prove Theorem 1.2 in Section 4. Then, Section 5 is addressed to the proof of Theorem 1.1. Finally, we finished our article with the appendix in which we prove lemmas that are useful in deriving the Carleman estimate.

2 Preliminary results

In this section, we present some preliminary results for the proof of [Theorem 1.3](#).

We have a pointwise weighted identity for the stochastic partial differential operator $\kappa d\varphi + \Delta^2 \varphi dt$, which plays a fundamental role in the proof of the Carleman estimates.

In what follows, we will use the notation $y_{x_i} \equiv y_{x_i}(x) = \partial y(x)/\partial x_i$, where x_i is the i -th coordinate of a generic point $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$.

Theorem 2.1. *Let φ be an $H^4(G)$ -valued Itô process. Set $\theta = e^\ell$, $\ell = s\alpha$, $\psi = \theta\varphi$. κ is a nonzero constant. The function α and ξ are as in [\(1.3\)](#). Let $\eta \in C^6(\mathbb{R}^n)$. Then, for any $t \in [0, T]$ and a.e. $(x, \omega) \in G \times \Omega$,*

$$\begin{aligned}
& 2\theta K_1(\kappa d\varphi + \Delta^2 \varphi dt) - 2\kappa \sum_{i,j=1}^n (\psi_{x_i x_i x_j} d\psi - \psi_{x_i x_j} d\psi_{x_i} + \Psi_2 \psi_{x_i} \delta_{ij} d\psi + \Psi_3^{ij} \psi_{x_i} d\psi)_{x_j} - 2 \operatorname{div}(V_1 + V_2) dt \\
&= 2K_1^2 dt + 2K_1 K_3 + 2M dt + 2\mathcal{R} dt + 2 \sum_{i,j,k,l=1}^n \Lambda_1^{ijkl} \psi_{x_i x_j} \psi_{x_k x_l} dt + 2 \sum_{i,j=1}^n \Lambda_2^{ij} \psi_{x_i} \psi_{x_j} dt + 2\Lambda_3 \psi^2 dt \\
&+ 2\kappa d \left(\frac{1}{2} |\nabla^2 \psi|^2 - \frac{1}{2} \Psi_2 |\nabla \psi|^2 - 2s^2 \lambda^2 \xi^2 |\nabla \eta \nabla \psi|^2 + \frac{1}{2} \Psi_5 \psi^2 \right) + \kappa \theta^2 \Psi_2 |d\nabla \varphi + \nabla \ell d\varphi|^2 - \kappa \Psi_5 \theta^2 (d\varphi)^2 \\
&+ 4\kappa s^2 \lambda^2 \xi^2 \theta^2 |\nabla \eta d\nabla \varphi + \nabla \eta \nabla \ell d\varphi|^2 - \kappa \theta^2 \sum_{i,j=1}^n [d\varphi_{x_i x_j} + \ell_{x_i} d\varphi_{x_j} + \ell_{x_j} d\varphi_{x_i} + (\ell_{x_i x_j} + \ell_{x_i} \ell_{x_j}) d\varphi]^2
\end{aligned} \tag{2.1}$$

The additional remainder is

$$\mathcal{R} = \frac{1}{2} \sum_{i,j,k=1}^n \partial_{x_k} (\Psi_4^j \Phi_1^i - \Psi_4^i \Phi_1^j) \psi_{x_i} \psi_{x_j x_k}.$$

It arises when integrating $\sum_{i,j} \Psi_4^j \Phi_1^i \psi_{x_i} (\Delta \psi)_{x_j}$ twice: the matrix $\Psi_4^i \Phi_1^j$ need not be symmetric. Here

$$K_1 = \Delta^2 \psi + \Psi_2 \Delta \psi + \sum_{i,j=1}^n \Psi_3^{ij} \psi_{x_i x_j} + \sum_{i=1}^n \Psi_4^i \psi_{x_i} + \Psi_5 \psi,$$

with

$$\begin{cases} \Psi_2 = 2s^2 \lambda^2 \xi^2 |\nabla \eta|^2, & \Psi_3^{ij} = 4s^2 \lambda^2 \xi^2 \eta_{x_i} \eta_{x_j}, \\ \Psi_4^i = 12s^2 \lambda^3 \xi^2 |\nabla \eta|^2 \eta_{x_i} + 4s^2 \lambda^2 \xi^2 \Delta \eta \eta_{x_i} + \sum_{j=1}^n 8s^2 \lambda^2 \xi^2 \eta_{x_i x_j} \eta_{x_j}, & i, j = 1, \dots, n. \\ \Psi_5 = s^4 \lambda^4 \xi^4 |\nabla \eta|^4, \end{cases}$$

$$K_2 = \sum_{i=1}^n \Phi_1^i \Delta \psi_{x_i} dt + \Phi_2 \Delta \psi dt + \sum_{i,j=1}^n \Phi_3^{ij} \psi_{x_i x_j} dt + \sum_{i=1}^n \Phi_4^i \psi_{x_i} dt + \Phi_5 \psi dt + \kappa d\psi,$$

with

$$\begin{cases} \Phi_1^i = -4s\lambda\xi\eta_{x_i}, & \Phi_2 = -2s\lambda^2\xi|\nabla\eta|^2 - 2s\lambda\xi\Delta\eta - \lambda, \\ \Phi_3^{ij} = 4s\lambda\xi\eta_{x_i x_j} - 4s\lambda^2\xi\eta_{x_i}\eta_{x_j}, & \Phi_4^i = -4s^3\lambda^3\xi^3|\nabla\eta|^2\eta_{x_i}, \\ \Phi_5 = -6s^3\lambda^4\xi^3|\nabla\eta|^4 - 12s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta) - 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta - s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4, \end{cases} \quad i, j = 1, \dots, n.$$

$$K_3 = -8s\lambda\xi \sum_{i,j=1}^n \eta_{x_i x_j} \psi_{x_i x_j} dt - 4\nabla\Delta\ell \cdot \nabla\psi dt + 4(\nabla\ell \cdot \nabla\Delta\ell)\psi dt + 2|\nabla^2\ell|^2\psi dt - \Delta^2\ell\psi dt \\ + |\Delta\ell|^2\psi dt + 8s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)\psi dt - \kappa s\alpha_t\psi dt + s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4\psi dt + \lambda\Delta\psi dt,$$

$$V_1 = (V_1^1, V_1^2, \dots, V_1^n), \quad V_2 = (V_2^1, V_2^2, \dots, V_2^n), \\ V_1^j = \sum_{i,k=1}^n \left[\sum_{l=1}^n \Phi_1^l \psi_{x_k x_k x_l} \psi_{x_i x_i x_j} - \frac{1}{2} \sum_{l=1}^n \Phi_1^j \psi_{x_k x_k x_l} \psi_{x_i x_i x_l} + \frac{1}{2} \Psi_2 \Phi_1^j \psi_{x_i x_i} \psi_{x_k x_k} \right. \\ \left. + \sum_{l=1}^n \Psi_3^{ik} \Phi_1^l \psi_{x_i x_k} \psi_{x_l x_j} - \frac{1}{2} \sum_{l=1}^n \Psi_3^{ij} \Phi_1^l \psi_{x_i x_k} \psi_{x_k x_l} + \Phi_4^k \psi_{x_i x_i x_j} \psi_{x_k} \right. \\ \left. - \Phi_4^k \psi_{x_i x_j} \psi_{x_i x_k} + \frac{1}{2} \Phi_4^j \psi_{x_i x_k}^2 + \left(\Psi_2 \Phi_4^k \delta_{ij} - \frac{1}{2} \Psi_2 \Phi_4^j \delta_{ik} - \frac{1}{2} \Psi_5 \Phi_1^j \delta_{ik} + \frac{1}{2} \Psi_3^{ij} \Phi_4^k \right) \psi_{x_i} \psi_{x_k} \right], \\ V_2^j = \sum_{i,l,r,m=1}^n \Theta_1^{ijlrm} \psi_{x_i x_i x_l} \psi_{x_r x_m} + \sum_{i,k,l,r=1}^n \Theta_2^{ijklr} \psi_{x_i x_k} \psi_{x_l x_r} + \sum_{i,k,l=1}^n \Theta_3^{ijkl} \psi_{x_i x_k} \psi_{x_l} \\ + \sum_{i,k=1}^n \Theta_4^{ijk} \psi_{x_i} \psi_{x_k} + \sum_{i=1}^n \Theta_5 \psi_{x_i x_i x_j} \psi + \sum_{i,k=1}^n \Theta_6^{ijk} \psi_{x_i x_k} \psi + \sum_{i=1}^n \Theta_7^{ij} \psi_{x_i} \psi + \Theta_8^j \psi^2,$$

$$M = - \sum_{i,j,k,l=1}^n (\Phi_{1x_j}^l + \Phi_3^{jl}) \psi_{x_k x_k x_l} \psi_{x_i x_i x_j} + \left(\frac{1}{2} \sum_{i=1}^n \Phi_{1x_i}^i - \Phi_2 \right) |\nabla\Delta\psi|^2 \\ + \sum_{i,j,k,l=1}^n [(-\Psi_3^{ij} \Phi_1^l)_{x_k} + \Psi_3^{il} \Phi_3^{kj}] \psi_{x_i x_j} \psi_{x_k x_l} \\ + \sum_{i,j,k=1}^n \left[2\Phi_{4x_j}^k + \Psi_2 \Phi_3^{jk} + \frac{1}{2} \sum_{l=1}^n (\Psi_3^{lj} \Phi_1^k)_{x_l} + \Psi_3^{jk} \Phi_2 - \Psi_4^j \Phi_1^k \right] \psi_{x_i x_j} \psi_{x_i x_k} \\ - \frac{1}{2} \sum_{i=1}^n \Phi_{4x_i}^i |\nabla^2\psi|^2 + \left[\Phi_5 - \frac{1}{2} \sum_{i=1}^n (\Psi_2 \Phi_1^i)_{x_i} + \Phi_2 \Psi_2 \right] |\Delta\psi|^2 \\ + \sum_{i,j=1}^n \left[-(\Psi_2 \Phi_4^i)_{x_j} - \frac{1}{2} \sum_{k=1}^n (\Psi_3^{ik} \Phi_4^j)_{x_k} - \Psi_3^{ij} \Phi_5 + \Psi_4^i \Phi_4^j + (\Psi_5 \Phi_1^i)_{x_j} - \Psi_5 \Phi_3^{ij} \right] \psi_{x_i} \psi_{x_j} \\ + \left[-\frac{1}{2} \sum_{i=1}^n (\Psi_5 \Phi_4^i)_{x_i} + \Psi_5 \Phi_5 \right] |\psi|^2 + \left[\frac{1}{2} \sum_{i=1}^n (\Psi_2 \Phi_4^i)_{x_i} - \Psi_2 \Phi_5 + \frac{1}{2} \sum_{i=1}^n (\Psi_5 \Phi_1^i)_{x_i} - \Psi_5 \Phi_2 \right] |\nabla\psi|^2,$$

$$\Lambda_1^{ijkl} = \sum_{r=1}^n \left(\frac{1}{2} \Phi_{2x_r x_r} \delta_{ij} \delta_{kl} - \Phi_{3x_j x_r}^{kr} \delta_{il} - \Phi_{3x_i x_r}^{kr} \delta_{lj} + \frac{1}{2} \sum_{m=1}^n \Phi_{3x_r x_m}^{rm} \delta_{ik} \delta_{lj} \right) + \Phi_{3x_i x_j}^{kl} + \Phi_{3x_i x_l}^{kj}, \\ \Lambda_2^{ij} = \sum_{k=1}^n \left[-\Phi_{4x_i x_k x_k}^j + \frac{1}{2} \Phi_{4x_i x_j x_k}^k + (\Psi_2 \Phi_3^{jk})_{x_i x_k} - \frac{1}{2} (\Psi_2 \Phi_3^{ij})_{x_k x_k} - \sum_{l=1}^n \frac{1}{2} (\Psi_2 \Phi_3^{kl} \delta_{ij})_{x_k x_l} \right. \\ \left. + (\Psi_3^{ik} \Phi_2)_{x_k x_j} - \sum_{l=1}^n \frac{1}{2} (\Psi_3^{kl} \Phi_2 \delta_{ij})_{x_k x_l} - \frac{1}{2} (\Psi_3^{ij} \Phi_2)_{x_k x_k} + \sum_{l=1}^n (\Psi_3^{ik} \Phi_3^{jl})_{x_k x_l} \right]$$

$$\begin{aligned}
& - \sum_{l=1}^n \frac{1}{2} (\Psi_3^{ij} \Phi_3^{kl})_{x_k x_l} - \sum_{l=1}^n \frac{1}{2} (\Psi_3^{kl} \Phi_3^{ij})_{x_k x_l} + \frac{1}{2} (\Psi_4^i \Phi_1^j)_{x_k x_k} + \frac{1}{2} (\Psi_4^k \Phi_2 \delta_{ij})_{x_k} \\
& - (\Psi_4^i \Phi_3^{jk})_{x_k} + \frac{1}{2} (\Psi_4^k \Phi_3^{ij})_{x_k} \Big] - 2\Phi_{5x_i x_j} - (\Psi_4^i \Phi_2)_{x_j} + \frac{\kappa}{2} (\Psi_{2t} \delta_{ij} + \Psi_{3t}^{ij}), \\
\Lambda_3 &= \frac{1}{2} \Delta^2 \Phi_5 + \frac{1}{2} \Delta (\Psi_2 \Phi_5) + \frac{1}{2} \sum_{i,j=1}^n (\Psi_3^{ij} \Phi_5)_{x_i x_j} - \frac{1}{2} \sum_{i=1}^n (\Psi_4^i \Phi_5)_{x_i} \\
& - \frac{1}{2} \sum_{i,j=1}^n (\Psi_5 \Phi_1^i)_{x_i x_j x_j} + \frac{1}{2} \Delta (\Psi_5 \Phi_2) + \frac{1}{2} \sum_{i,j=1}^n (\Psi_5 \Phi_3^{ij})_{x_i x_j} - \frac{\kappa}{2} \Psi_{5t}.
\end{aligned}$$

and

$$\begin{aligned}
\Theta_1^{ijlrm} &= \Phi_2 \delta_{lj} \delta_{rm} + \Phi_3^{rm} \delta_{lj} - \Phi_3^{mj} \delta_{lr} + \Phi_3^{lr} \delta_{mj}, \\
\Theta_2^{ijklr} &= -\frac{1}{2} \Phi_{2x_j} \delta_{ik} \delta_{lr} - \Phi_{3x_i}^{lr} \delta_{kj} + \Phi_{3x_k}^{rj} \delta_{il} + \sum_{m=1}^n \Phi_{3x_m}^{rm} \delta_{kj} \delta_{il} - \frac{1}{2} \sum_{m=1}^n \Phi_{3x_m}^{jm} \delta_{il} \delta_{kr} - \Phi_{3x_i}^{kl} \delta_{rj}, \\
\Theta_3^{ijkl} &= -\Phi_{4x_i}^l \delta_{kj} - \Phi_5 \delta_{ik} \delta_{lj} + \Psi_2 \Phi_3^{ik} \delta_{lj} - \Psi_2 \Phi_3^{ij} \delta_{kl} + \Psi_3^{lj} \Phi_2 \delta_{ik} - \Psi_3^{kl} \Phi_2 \delta_{ij} + \Psi_3^{ik} \Phi_3^{lj} \\
& - \Psi_3^{ij} \Phi_3^{kl} + \Psi_4^l \Phi_1^k \delta_{ij}, \\
\Theta_4^{ijk} &= \Phi_{4x_i x_j}^k - \frac{1}{2} \Phi_{4x_i x_k}^j + 2\Phi_{5x_i} \delta_{kj} - \Phi_{5x_j} \delta_{ik} - (\Psi_2 \Phi_3^{jk})_{x_i} + \frac{1}{2} (\Psi_2 \Phi_3^{ik})_{x_j} + \sum_{l=1}^n \frac{1}{2} (\Psi_2 \Phi_3^{jl})_{x_l} \delta_{ik} \\
& - \sum_{l=1}^n (\Psi_3^{il} \Phi_2)_{x_l} \delta_{kj} + \frac{1}{2} \sum_{l=1}^n (\Psi_3^{lj} \Phi_2)_{x_l} \delta_{ik} + \frac{1}{2} (\Psi_3^{ik} \Phi_2)_{x_j} - \sum_{l=1}^n (\Psi_3^{ij} \Phi_3^{kl})_{x_l} + \frac{1}{2} \sum_{l=1}^n (\Psi_3^{ik} \Phi_3^{jl})_{x_l} \\
& + \frac{1}{2} \sum_{l=1}^n (\Psi_3^{lj} \Phi_3^{ik})_{x_l} - \frac{1}{2} (\Psi_4^i \Phi_1^k)_{x_j} + \Psi_4^i \Phi_2 \delta_{kj} - \frac{1}{2} \Psi_4^j \Phi_2 \delta_{ik} + \Psi_4^i \Phi_3^{kj} - \frac{1}{2} \Psi_4^j \Phi_3^{ik}, \\
\Theta_5 &= \Phi_5, \\
\Theta_6^{ijk} &= -\Phi_{5x_j} \delta_{ik} + \Psi_5 \Phi_1^k \delta_{ij}, \\
\Theta_7^{ij} &= \sum_{k=1}^n \Phi_{5x_k x_k} \delta_{ij} + \Psi_2 \Phi_5 \delta_{ij} + \Psi_3^{ij} \Phi_5 - (\Psi_5 \Phi_1^i)_{x_j} + \Psi_5 \Phi_2 \delta_{ij} + \Psi_5 \Phi_3^{ij}, \\
\Theta_8^j &= -\frac{1}{2} \sum_{i=1}^n \Phi_{5x_i x_i x_j} - \frac{1}{2} (\Psi_2 \Phi_5)_{x_j} - \frac{1}{2} \sum_{i=1}^n (\Psi_3^{ij} \Phi_5)_{x_i} + \frac{1}{2} \Psi_4^j \Phi_5 + \frac{1}{2} \sum_{i=1}^n (\Psi_5 \Phi_1^j)_{x_i x_i} - \frac{1}{2} (\Psi_5 \Phi_2)_{x_j} \\
& - \frac{1}{2} \sum_{i=1}^n (\Psi_5 \Phi_3^{ij})_{x_i} + \frac{1}{2} \Psi_5 \Phi_4^j.
\end{aligned}$$

Theorem 2.1 can be proved by following the steps of the proof of the fundamental identity presented in [16, Theorem 2.2]. So we omit the proof here.

It is easy to see that for all $t \in (0, T)$ and $x \in G$,

$$|\xi_t| \leq \frac{T}{2} \xi^3, \quad |\alpha_t| \leq \frac{T}{2} \xi^3, \quad \xi^{-1} \leq \frac{T}{2}. \quad (2.2)$$

In what follows, for a nonnegative integer m , we denote by $O(\lambda^m)$ a function of order λ^m for large λ . Similarly to [16, Proposition 2.2], we have the following estimate.

Proposition 2.2. *When s and λ are large enough, it holds that*

$$\left\{ \begin{array}{l} \sum_{i,j,k,l=1}^n \Lambda_1^{ijkl} \zeta_{ij} \zeta_{kl} \geq s \xi O(\lambda^4) |\zeta|^2, \\ \sum_{i,j=1}^n \Lambda_2^{ij} \zeta_i \zeta_j \geq [s^3 \xi^3 O(\lambda^6) + T s^2 \xi^4 O(\lambda^2)] |\zeta|^2, \\ \Lambda_3 \geq s^5 \xi^5 O(\lambda^8) + T s^4 \xi^6 O(\lambda^4). \end{array} \right. \quad \forall \zeta = (\zeta_{ij}) \in \mathbb{R}^{n \times n}, \quad \forall \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n, \quad (2.3)$$

for any $t \in (0, T)$ and $x \in G$.

From the Carleman inequality for the Laplace operator with homogeneous Dirichlet boundary conditions (see [7, Lemma 2.4]), we obtain the following result.

Lemma 2.3. [16, Lemma 2.3] *Let $q \in H^2(G) \cap H_0^1(G)$ and η be given by (1.4). Then there exists a constant $C > 0$ independent of s and λ , and parameters $\hat{\lambda} > 1$ and $\hat{s} > 1$ such that for all $\lambda \geq \hat{\lambda}$ and for all $s \geq \hat{s}$,*

$$\begin{aligned} & \int_Q s^6 \lambda^8 \xi^6 \theta^2 |q|^2 dx dt + \int_Q s^4 \lambda^6 \xi^4 \theta^2 |\nabla q|^2 dx dt \\ & \leq C \left(\int_{(0,T) \times G_2} s^6 \lambda^8 \xi^6 \theta^2 |q|^2 dx dt + \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\Delta q|^2 dx dt \right). \end{aligned} \quad (2.4)$$

Let

$$\alpha_\star(t) = \frac{e^{\lambda|2\eta|_{C(\overline{G})}} - e^{4\lambda|\eta|_{C(\overline{G})}}}{t^{1/2}(T-t)^{1/2}}, \quad \xi_\star(t) = \frac{e^{\lambda|2\eta|_{C(\overline{G})}}}{t^{1/2}(T-t)^{1/2}}. \quad (2.5)$$

Lemma 2.4. *There exists a constant $C > 0$, such that for $\varphi \in L_{\mathbb{F}}^2(0, T; H^4(G))$, $f \in L_{\mathbb{F}}^2(0, T; L^2(G))$ and $g \in L_{\mathbb{F}}^2(0, T; H_0^2(G))$ satisfying*

$$\left\{ \begin{array}{l} d\varphi + \Delta^2 \varphi dt = f dt + g dW(t) \quad \text{in } Q, \\ \varphi = \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \Sigma, \end{array} \right. \quad (2.6)$$

for $s \geq CT^{1/2}$, it holds that

$$\begin{aligned} & \mathbb{E} \int_{\Sigma} \left[s^{\frac{9}{4}} \lambda^3 \xi^{\frac{9}{4}} \theta^2 |\nabla^2 \varphi|^2 + s^{\frac{3}{4}} \lambda \xi^{\frac{3}{4}} \theta^2 |\nabla^3 \varphi|^2 \right] d\Gamma dt + \mathbb{E} \int_0^T e^{2s\alpha_\star} |\varphi|_{H^4(G)}^2 dt \\ & \leq C \mathbb{E} \int_Q s^6 \lambda^8 \xi^6 \theta^2 |\varphi|^2 dx dt + C \mathbb{E} \int_Q \theta^2 (|f|^2 + |\Delta g|^2 + s^2 \lambda^2 \xi^2 |\nabla g|^2 + s^4 \lambda^4 \xi^4 g^2) dx dt. \end{aligned}$$

The proof of Lemma 2.4 is similar to that of [16, Lemma 2.4], which is put in section A for the sake of completeness.

3 Proof of Carleman estimate

Proof of Theorem 1.3. By a density argument, we assume $\varphi \in L^2_{\mathbb{F}}(0, T; H^4(G))$.

In order to shorten the formulas used in the sequel, we define the lower order terms

$$\begin{aligned}\mathcal{A}_1 &\triangleq s^5 \mathbb{E} \int_Q \xi^5 [O(\lambda^8) + s\xi O(\lambda^7)] |\psi|^2 dx dt, & \mathcal{A}_2 &\triangleq s^3 \mathbb{E} \int_Q \xi^3 [O(\lambda^6) + s\xi O(\lambda^5)] |\nabla \psi|^2 dx dt, \\ \mathcal{A}_3 &\triangleq s \mathbb{E} \int_Q \xi [O(\lambda^4) + s\xi O(\lambda^3)] |\nabla^2 \psi|^2 dx dt, & \mathcal{A}_4 &\triangleq \mathbb{E} \int_Q O(1) |\nabla \Delta \psi|^2 dx dt, \\ \mathcal{B}_1 &\triangleq s^2 \mathbb{E} \int_{\Sigma} \xi^2 O(\lambda^3) \theta^2 |\nabla^2 \varphi|^2 d\Gamma dt, & \mathcal{B}_2 &\triangleq s^{1/2} \mathbb{E} \int_{\Sigma} \xi^{1/2} O(\lambda) \theta^2 |\nabla^3 \varphi|^2 d\Gamma dt,\end{aligned}\tag{3.1}$$

and

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4, \quad \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2.$$

Writing $h = (\nabla^2 \eta) \nabla \eta$, we have $\Psi_4^j \Phi_1^i - \Psi_4^i \Phi_1^j = -32s^3 \lambda^3 \xi^3 (h_j \eta_{x_i} - h_i \eta_{x_j})$. Consequently, Young's inequality yields

$$\left| \mathbb{E} \int_Q \mathcal{R} dx dt \right| \leq C \mathbb{E} \int_Q (s^2 \lambda^3 \xi^2 |\nabla^2 \psi|^2 + s^4 \lambda^5 \xi^4 |\nabla \psi|^2) dx dt.$$

These are already lower order terms of the types $\mathcal{A}_3$ and $\mathcal{A}_2$, respectively, and are absorbed with them below. Integrating the equality (2.1) with $\kappa = 1$ on Q , taking mathematical expectation in both sides, and noting (2.2), (2.3), we conclude that, there exists a $s_1 \geq CT^{1/2}$, such that for $s \geq s_1$, it holds that

$$\begin{aligned}& 2\mathbb{E} \int_Q \theta K_1 (d\varphi + \Delta^2 \varphi dt) dx - 2\mathbb{E} \int_Q \operatorname{div}(V_1 + V_2) dx dt \\ & - 2\mathbb{E} \int_Q \sum_{i,j=1}^n (\psi_{x_i x_i x_j} d\psi - \psi_{x_i x_j} d\psi_{x_i} + \Psi_2 \psi_{x_i} \delta_{ij} d\psi + \Psi_3^{ij} \psi_{x_i} d\psi)_{x_j} dx \\ & \geq 2\mathbb{E} \int_Q K_1^2 dx dt + 2\mathbb{E} \int_Q K_1 K_3 dx + 2\mathbb{E} \int_Q M dx dt + 2\mathbb{E} \int_Q \mathcal{R} dx dt \\ & - \mathbb{E} \int_Q \theta^2 \sum_{i,j=1}^n [d\varphi_{x_i x_j} + \ell_{x_i} d\varphi_{x_j} + \ell_{x_j} d\varphi_{x_i} + (\ell_{x_i x_j} + \ell_{x_i} \ell_{x_j}) d\varphi]^2 dx + \mathbb{E} \int_Q \theta^2 \Psi_2 |d\nabla \varphi + \nabla \ell d\varphi|^2 dx \\ & + 4\mathbb{E} \int_Q s^2 \lambda^2 \xi^2 \theta^2 |\nabla \eta d\nabla \varphi + \nabla \eta \nabla \ell d\varphi|^2 dx - \mathbb{E} \int_Q \Psi_5 \theta^2 (d\varphi)^2 dx + \mathcal{A}.\end{aligned}\tag{3.2}$$

We will estimate the divergence terms, the interior terms and the stochastic terms in the following Steps 1–3.

Step 1. In this step, we deal with the divergence terms.

Firstly, we estimate V_1 . Since $\varphi = \frac{\partial \varphi}{\partial \nu} = 0$ and $\psi = \theta \varphi$, we have $\psi = 0$ and $\nabla \psi = 0$ on the boundary Σ . Hence, on Σ , we obtain

$$V_1 \cdot \nu = \sum_{i,j,k=1}^n \left[\sum_{l=1}^n \Phi_1^l \psi_{x_k x_k x_l} \psi_{x_i x_i x_j} - \frac{1}{2} \sum_{l=1}^n \Phi_1^j \psi_{x_k x_k x_l} \psi_{x_i x_i x_l} + \frac{1}{2} \Psi_2 \Phi_1^j \psi_{x_i x_i} \psi_{x_k x_k} \right]$$

$$\begin{aligned}
& + \sum_{l=1}^n \Psi_3^{ik} \Phi_1^l \psi_{x_i x_k} \psi_{x_l x_j} - \frac{1}{2} \sum_{l=1}^n \Psi_3^{ij} \Phi_1^l \psi_{x_i x_k} \psi_{x_k x_l} - \Phi_4^k \psi_{x_i x_j} \psi_{x_i x_k} + \frac{1}{2} \Phi_4^j \psi_{x_i x_k}^2 \Big] \nu^j, \\
& = -4 \sum_{i,j,k,p=1}^n s \lambda \xi \eta_{x_p} \psi_{x_k x_k x_p} \psi_{x_i x_i x_j} \nu^j + 2 \sum_{i,k,p=1}^n s \lambda \xi \frac{\partial \eta}{\partial \nu} \psi_{x_k x_k x_p} \psi_{x_i x_i x_p} \\
& \quad - 4 \sum_{i,k=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \psi_{x_i x_i} \psi_{x_k x_k} - 16 \sum_{i,j,k,p=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \psi_{x_i x_k} \nu^i \nu^k \psi_{x_p x_j} \nu^p \nu^j \\
& \quad + 12 \sum_{i,j,k=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \psi_{x_i x_j} \nu^j \psi_{x_i x_k} \nu^k - 2 \sum_{i,k=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \psi_{x_i x_k}^2. \tag{3.3}
\end{aligned}$$

Since $\nabla \psi = 0$ on Σ , it holds that

$$\sum_{i,j=1}^n \psi_{x_i x_j} \nu^i \nu^j = \sum_{i,j=1}^n (\psi_{x_i} \nu^i \nu^j)_{x_j} - \sum_{i,j=1}^n \psi_{x_i} (\nu^i \nu^j)_{x_j} = \sum_{j=1}^n \left(\frac{\partial \psi}{\partial \nu} \nu^j \right)_{x_j} = \sum_{j=1}^n \psi_{x_j x_j}, \tag{3.4}$$

and

$$\begin{aligned}
\sum_{i,j=1}^n \psi_{x_i x_j}^2 & = \sum_{i=1}^n \left(\sum_{j=1}^n \psi_{x_i x_j}^2 \right) = \sum_{i=1}^n (|\nabla \psi_{x_i}|^2) = \sum_{i=1}^n (|\nabla \psi_{x_i} \cdot \nu|^2) \\
& = \sum_{i=1}^n \left(\sum_{j=1}^n \psi_{x_i x_j} \nu^j \right)^2 = \sum_{i,j,k=1}^n \psi_{x_i x_j} \nu^j \psi_{x_i x_k} \nu^k. \tag{3.5}
\end{aligned}$$

Combining (3.3)–(3.5), on the boundary Σ , we deduce that

$$\begin{aligned}
V_1 \cdot \nu & = -4 \sum_{i,j,k,p=1}^n s \lambda \xi \eta_{x_p} \psi_{x_k x_k x_p} \psi_{x_i x_i x_j} \nu^j + 2 \sum_{i,k,p=1}^n s \lambda \xi \frac{\partial \eta}{\partial \nu} \psi_{x_k x_k x_p} \psi_{x_i x_i x_p} \\
& \quad - 20 \sum_{i,k=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \psi_{x_i x_i} \psi_{x_k x_k} + 10 \sum_{i,j=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \psi_{x_i x_j}^2. \tag{3.6}
\end{aligned}$$

We claim that, on the boundary Σ ,

$$\sum_{i,j=1}^n \psi_{x_i x_i} \psi_{x_j x_j} = \sum_{i,j=1}^n \psi_{x_i x_j}^2. \tag{3.7}$$

In fact, for $i = 1, \dots, n$, since $\psi_{x_i} = 0$ on Σ , we can write $\nabla \psi_{x_i} = \mathcal{K}_i \nu$, where $\mathcal{K}_i = \frac{\partial \psi_i}{\partial \nu}$, which means $\psi_{x_i x_j} = \mathcal{K}_i \nu^j$ for all $j = 1, \dots, n$. Notice that, for $i, j = 1, \dots, n$, $\psi_{x_i x_j} = \psi_{x_j x_i}$. We obtain $\mathcal{K}_i \nu^j = \mathcal{K}_j \nu^i$. Hence, it holds that

$$\sum_{i,j=1}^n \psi_{x_i x_i} \psi_{x_j x_j} = \sum_{i,j=1}^n \mathcal{K}_i \nu^i \mathcal{K}_j \nu^j = \sum_{i,j=1}^n \mathcal{K}_i \nu^j \mathcal{K}_j \nu^i = \sum_{i,j=1}^n (\mathcal{K}_i \nu^j)^2 = \sum_{i,j=1}^n \psi_{x_i x_j}^2.$$

Combining (3.6) and (3.7), we obtain

$$\begin{aligned}
V_1 \cdot \nu &= -4 \sum_{i,j,k,p=1}^n s \lambda \xi \eta_{x_p} \psi_{x_k x_k x_p} \psi_{x_i x_i x_j} \nu^j + 2 \sum_{i,k,p=1}^n s \lambda \xi \frac{\partial \eta}{\partial \nu} \psi_{x_k x_k x_p} \psi_{x_i x_i x_p} \\
&\quad - 10 \sum_{i,k=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \psi_{x_i x_i} \psi_{x_k x_k} \\
&= -4 s \lambda \xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta \psi \cdot \nu|^2 + 2 s \lambda \xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta \psi|^2 - 10 s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\Delta \psi|^2. \tag{3.8}
\end{aligned}$$

Next, we consider the second term in the right hand side of (3.8). Put $\tau \triangleq \frac{\nabla \Delta \psi - (\nabla \Delta \psi \cdot \nu) \nu}{|\nabla \Delta \psi - (\nabla \Delta \psi \cdot \nu) \nu|}$. We have

$$2 s \lambda \xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta \psi|^2 = 2 s \lambda \xi \frac{\partial \eta}{\partial \nu} (|\nabla \Delta \psi \cdot \nu|^2 + |\nabla \Delta \psi \cdot \tau|^2). \tag{3.9}$$

Using the interpolation inequality, we obtain

$$|\Delta \psi|_{H^1(\Gamma)} \leq |\Delta \psi|_{L^2(\Gamma)}^{1/3} |\Delta \psi|_{H^{3/2}(\Gamma)}^{2/3},$$

which implies

$$\begin{aligned}
&\mathbb{E} \int_{\Sigma} 2 s \lambda \xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta \psi \cdot \tau|^2 d\Gamma dt \\
&\geq -C \mathbb{E} \int_0^T s \lambda \xi |\Delta \psi|_{H^1(\Gamma)}^2 dt \\
&\geq -C \mathbb{E} \int_0^T s \lambda \xi |\Delta \psi|_{L^2(\Gamma)}^{2/3} |\Delta \psi|_{H^{3/2}(\Gamma)}^{4/3} dt \\
&\geq \mathbb{E} \int_{\Sigma} s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\Delta \psi|^2 d\Gamma dt - C \mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt \\
&= \mathbb{E} \int_{\Sigma} s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 \psi|^2 d\Gamma dt - C \mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt, \tag{3.10}
\end{aligned}$$

where we use the the fact that $\frac{\partial \eta}{\partial \nu} \leq -C$ on Σ and (3.7). Combining (3.8)–(3.10), we obtain

$$\begin{aligned}
&\mathbb{E} \int_Q \operatorname{div} V_1 dx dt = \mathbb{E} \int_{\Sigma} V_1 \cdot \nu d\Gamma dt \\
&\geq \mathbb{E} \int_{\Sigma} \left[-2 s \lambda \xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta w \cdot \nu|^2 - 9 s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 w|^2 \right] d\Gamma dt - C \mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt \\
&\geq -C \mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt. \tag{3.11}
\end{aligned}$$

Now, we estimate V_2 . From (2.1) and (3.1) and $\psi = |\nabla \psi| = 0$ on the boundary, there exists a constant $s_2 \geq CT$, such that for $s \geq s_2$, we deduce

$$\mathbb{E} \int_Q \operatorname{div} V_2 dx dt = \mathbb{E} \int_{\Sigma} V_2 \cdot \nu d\Gamma dt$$

$$\begin{aligned}
&= \mathbb{E} \int_{\Sigma} \sum_{j=1}^n \left[\sum_{i,l,r,m=1}^n \Theta_1^{ijlrm} \psi_{x_i x_l} \psi_{x_r x_m} + \sum_{i,k,l,r=1}^n \Theta_2^{ijklr} \psi_{x_i x_k} \psi_{x_l x_r} \right] \nu^j d\Gamma dt \\
&\geq -C\mathbb{E} \int_{\Sigma} \sum_{i,k,p,r,q=1}^n s\lambda^2 \xi |\psi_{x_i x_k x_p} \psi_{x_r x_q}| d\Gamma dt - C\mathbb{E} \int_{\Sigma} s\lambda^3 \xi |\nabla^2 \psi|^2 d\Gamma dt \\
&\geq -C\mathbb{E} \int_{\Sigma} [s^2 \lambda^3 \xi^2 |\nabla^2 \psi|^2 + \lambda |\nabla^3 \psi|^2] d\Gamma dt - C\mathbb{E} \int_{\Sigma} s\lambda^3 \xi |\nabla^2 \psi|^2 d\Gamma dt \\
&\geq -C\mathbb{E} \int_{\Sigma} [s^2 \lambda^3 \xi^2 |\nabla^2 \psi|^2 + \lambda |\nabla^3 \psi|^2] d\Gamma dt \\
&\geq \mathcal{B}_1 + \mathcal{B}_2.
\end{aligned} \tag{3.12}$$

Hence, from (3.11) and (3.12), we obtain

$$\mathbb{E} \int_Q \operatorname{div}(V_1 + V_2) dx dt \geq -C\mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt + \mathcal{B}_1 + \mathcal{B}_2 \tag{3.13}$$

Step 2. In this step, we deal with the interior term $\mathbb{E} \int_Q M dx dt$ and stochastic terms via integration by parts.

From Theorem 2.1, we have

$$\begin{aligned}
M &= 8s\lambda^2 \xi |\nabla \Delta \psi \cdot \nabla \eta|^2 + 32s^3 \lambda^4 \xi^3 |\nabla^2 \psi \nabla \eta \nabla \eta|^2 + 48s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 \psi \nabla \eta) (\nabla^2 \psi \nabla \eta) \\
&\quad + 16s^3 \lambda^3 \xi^3 (\nabla^2 \psi \nabla \eta \nabla \eta) \sum_{i,j=1}^n \eta_{x_i x_j} \psi_{x_i x_j} - 16s^3 \lambda^4 \xi^3 |\nabla \eta|^2 |\nabla^2 \psi \nabla \eta|^2 + 6s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\nabla^2 \psi|^2 \\
&\quad + 4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla^2 \psi|^2 + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta |\nabla^2 \psi|^2 + 2s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta \psi|^2 \\
&\quad - 4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\Delta \psi|^2 - 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta |\Delta \psi|^2 + 40s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 \\
&\quad + 64s^5 \lambda^5 \xi^5 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla \psi \cdot \nabla \eta|^2 - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla \psi|^2 + 8s^7 \lambda^8 \xi^7 |\nabla \eta|^8 \psi^2 + \lambda |\nabla \Delta \psi|^2 \\
&\quad - s^3 \lambda^{\frac{7}{2}} \xi^3 |\nabla \eta|^4 |\Delta \psi|^2 + 4s^5 \lambda^{\frac{11}{2}} \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 + 2s^5 \lambda^{\frac{11}{2}} \xi^5 |\nabla \eta|^6 |\nabla \psi|^2 - s^7 \lambda^{\frac{15}{2}} \xi^7 |\nabla \eta|^8 \psi^2 \\
&\quad - 4s^2 \lambda^3 \xi^2 |\nabla^2 \psi \nabla \eta|^2 - 2s^2 \lambda^3 \xi^2 |\nabla \eta|^2 |\Delta \psi|^2 + s^4 \lambda^5 \xi^4 |\nabla \eta|^4 |\nabla \psi|^2.
\end{aligned} \tag{3.14}$$

Thanks to (3.1), it holds that

$$\mathbb{E} \int_Q (-4s^2 \lambda^3 \xi^2 |\nabla^2 \psi \nabla \eta|^2 - 2s^2 \lambda^3 \xi^2 |\nabla \eta|^2 |\Delta \psi|^2 + s^4 \lambda^5 \xi^4 |\nabla \eta|^4 |\nabla \psi|^2) dx dt \geq -\mathcal{A}_2 - \mathcal{A}_3. \tag{3.15}$$

Combining (2.2) and (3.1), using integration by parts, for $s \geq s_2$, we obtain

$$\begin{aligned}
&- \mathbb{E} \int_Q 16s^3 \lambda^4 \xi^3 |\nabla \eta|^2 |\nabla^2 \psi \nabla \eta|^2 dx dt \\
&\geq -4\mathbb{E} \int_Q s\lambda^2 \xi |\nabla \Delta \psi \cdot \nabla \eta|^2 dx dt - 16\mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 dx dt - \mathcal{A}_2.
\end{aligned} \tag{3.16}$$

Thanks to (1.4), there exists $\lambda_1 \geq C$, such that for all $\lambda \geq \lambda_1$, it holds that

$$\begin{aligned} & \mathbb{E} \int_Q 64s^5 \lambda^5 \xi^5 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla \psi \cdot \nabla \eta|^2 dx dt \\ & \geq -C \mathbb{E} \int_{(0,T) \times G_3} s^5 \lambda^5 \xi^5 |\nabla \psi|^2 dx dt - \mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 dx dt. \end{aligned} \quad (3.17)$$

Combining (2.2) and (3.1), using integration by parts, for $s \geq s_2$, we have

$$\mathbb{E} \int_Q [4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta] (|\nabla^2 \psi|^2 - |\Delta \psi|^2) dx dt \geq -\mathcal{A}_2, \quad (3.18)$$

and

$$6\mathbb{E} \int_Q s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\nabla^2 \psi|^2 dx dt \geq 6\mathbb{E} \int_Q s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta \psi|^2 dx dt - \mathcal{A}_2. \quad (3.19)$$

Similarly, for $s \geq s_2$, we deduce

$$\begin{aligned} & \mathbb{E} \int_Q 16s^3 \lambda^3 \xi^3 (\nabla^2 \psi \nabla \eta \nabla \eta) \sum_{i,j=1}^n \eta_{x_i x_j} \psi_{x_i x_j} dx dt \\ & \geq \mathbb{E} \int_Q 16s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 \psi \nabla \eta) (\nabla^2 \psi \nabla \eta) dx dt - (\mathcal{A}_2 + \mathcal{A}_3). \end{aligned} \quad (3.20)$$

Combining (2.2) and (3.1), there exists $\lambda_2 \geq C$, such that for all $\lambda \geq \lambda_2$ and $s \geq s_2$, it holds that

$$\begin{aligned} & \mathbb{E} \int_Q 64s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 \psi \nabla \eta) (\nabla^2 \psi \nabla \eta) dx dt \\ & \geq -\mathbb{E} \int_Q s \lambda^2 \xi |\nabla \Delta \psi \cdot \nabla \eta|^2 dx dt - \mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 dx dt - \mathcal{A}_2. \end{aligned} \quad (3.21)$$

Next, we estimate the stochastic terms. Thanks to the divergence theorem, we obtain

$$-2\mathbb{E} \int_Q \sum_{i,j=1}^n (\psi_{x_i x_i x_j} d\psi - \psi_{x_i x_j} d\psi_{x_i} + \Psi_2 \psi_{x_i} \delta_{ij} d\psi + \Psi_3^{ij} \psi_{x_i} d\psi)_{x_j} dx = 0 \quad (3.22)$$

From (1.5), for $s \geq s_2$, it holds that

$$\begin{aligned} & -\mathbb{E} \int_Q \theta^2 \sum_{i,j=1}^n [d\varphi_{x_i x_j} + \ell_{x_i} d\varphi_{x_j} + \ell_{x_j} d\varphi_{x_i} + (\ell_{x_i x_j} + \ell_{x_i} \ell_{x_j}) d\varphi]^2 dx - \mathbb{E} \int_Q \Psi_5 \theta^2 (d\varphi)^2 dx \\ & + \mathbb{E} \int_Q \theta^2 \Psi_2 |d\nabla \varphi + \nabla \ell d\varphi|^2 dx + 4\mathbb{E} \int_Q s^2 \lambda^2 \xi^2 \theta^2 |\nabla \eta d\nabla \varphi + \nabla \eta \nabla \ell d\varphi|^2 dx \\ & \geq -C \left(s^4 \mathbb{E} \int_Q \lambda^4 e^{2s\alpha} \xi^4 g^2 dx dt + s^2 \mathbb{E} \int_Q \lambda^2 e^{2s\alpha} \xi^2 |\nabla g|^2 dx dt + \mathbb{E} \int_Q e^{2s\alpha} |\nabla^2 g|^2 dx dt \right). \end{aligned} \quad (3.23)$$

Combining (3.2) and (3.13)–(3.23), there exist $\lambda_3 \geq \max\{\lambda_1, \lambda_2\}$ and $s_3 \geq \max\{s_1, s_2\}$, such that for all $\lambda \geq \lambda_3$ and $s \geq s_3$, we obtain

$$\begin{aligned}
& 2\mathbb{E} \int_Q \theta K_1 (d\varphi + \Delta^2 \varphi dt) dx + C\mathbb{E} \int_Q \theta^2 (f^2 + s^4 \lambda^4 \xi^4 g^2 + s^2 \lambda^2 \xi^2 |\nabla g|^2 + |\nabla^2 g|^2) dx dt \\
& + C\mathbb{E} \int_{(0,T) \times G_3} s^5 \lambda^5 \xi^5 |\nabla \psi|^2 dx dt + C\mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt + \mathcal{A} + \mathcal{B} \\
& \geq 2\mathbb{E} \int_Q K_1^2 dx dt + 2\mathbb{E} \int_Q K_1 K_3 dx \\
& + 2\mathbb{E} \int_Q (3s\lambda^2 \xi |\nabla \Delta \psi \cdot \nabla \eta|^2 + 32s^3 \lambda^4 \xi^3 |\nabla^2 \psi \nabla \eta \nabla \eta|^2 + 8s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta \psi|^2 \\
& + 22s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla \psi|^2 + 8s^7 \lambda^8 \xi^7 |\nabla \eta|^8 \psi^2 + \lambda |\nabla \Delta \psi|^2 \\
& + 2s^5 \lambda^{\frac{11}{2}} \xi^5 |\nabla \eta|^6 |\nabla \psi|^2 - s^7 \lambda^{\frac{15}{2}} \xi^7 |\nabla \eta|^8 \psi^2 - s^3 \lambda^{\frac{7}{2}} \xi^3 |\nabla \eta|^4 |\Delta \psi|^2) dx dt. \tag{3.24}
\end{aligned}$$

Step 3. In this step, we get the estimate of φ .

Using integration by parts, we have

$$\begin{aligned}
& \mathbb{E} \int_Q [(8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^4 |\Delta \psi|^2 - 2(8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^6 s^2 \lambda^2 \xi^2 |\nabla \psi|^2 \\
& + (8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^8 s^4 \lambda^4 \xi^4 |\psi|^2] dx dt \\
& \geq \mathbb{E} \int_Q (8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^4 (\Delta \psi + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \psi)^2 dx dt - \mathcal{A}_1, \tag{3.25}
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\nabla \eta|^4 |\Delta \varphi|^2 dx dt \\
& = \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 |\nabla \eta|^4 (\Delta \psi - 2s\lambda\xi \nabla \eta \cdot \nabla \psi + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \psi - s\lambda^2 \xi |\nabla \eta|^2 \psi - s\lambda\xi \Delta \eta \psi)^2 dx dt \\
& \leq 3\mathbb{E} \int_Q s^3 \lambda^4 \xi^3 |\nabla \eta|^4 (\Delta \psi + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \psi)^2 dx dt + 12\mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 + \mathcal{A}_1. \tag{3.26}
\end{aligned}$$

We also obtain

$$\begin{aligned}
& \mathbb{E} \int_Q \lambda |\nabla \Delta \psi|^2 dx dt + C\mathbb{E} \int_Q (s^2 \lambda^3 \xi^2 \theta^2 |\nabla^2 \varphi|^2 + s^4 \lambda^5 \xi^4 \theta^2 |\nabla \varphi|^2 + s^6 \lambda^7 \xi^6 \theta^2 |\varphi|^2) dx dt \\
& \geq \mathbb{E} \int_Q \lambda \theta^2 |\nabla \Delta \varphi|^2 dx dt. \tag{3.27}
\end{aligned}$$

Using (3.1), there exists $s_4 \geq C(T^{1/2} + T)$, such that for $s \geq s_4$, it holds that

$$2\mathbb{E} \int_Q K_1 K_3 dx \geq -\mathbb{E} \int_Q K_1^2 dx dt - \mathbb{E} \int_Q K_3^2 dx dt \geq -\mathbb{E} \int_Q K_1^2 dx dt - \mathcal{A}. \tag{3.28}$$

Thanks to (1.4), we obtain

$$\mathbb{E} \int_{(0,T) \times G_3} s^5 \lambda^5 \xi^5 |\nabla \psi|^2 dx dt \leq C \mathbb{E} \int_{(0,T) \times G_3} (s^7 \lambda^7 \xi^7 \theta^2 |\varphi|^2 + s^5 \lambda^5 \xi^5 \theta^2 |\nabla \varphi|^2) dx dt. \quad (3.29)$$

and

$$2\mathbb{E} \int_Q \theta K_1 (d\varphi + \Delta^2 \varphi dt) dx \leq \mathbb{E} \int_Q K_1^2 dx dt + \mathbb{E} \int_Q \theta^2 f^2 dx dt. \quad (3.30)$$

Using Lemma 2.3, there exist $\lambda_4 \geq \hat{\lambda}$ and $s_4 \geq \hat{s}$, such that for all $\lambda \geq \lambda_4$ and $s \geq s_4$, we obtain

$$\begin{aligned} & \mathbb{E} \int_Q s^6 \lambda^8 \xi^6 \theta^2 |\varphi|^2 dx dt + \mathbb{E} \int_Q s^4 \lambda^6 \xi^4 \theta^2 |\nabla \varphi|^2 dx dt \\ & \leq C \left(\mathbb{E} \int_{(0,T) \times G_3} s^6 \lambda^8 \xi^6 \theta^2 |\varphi|^2 dx dt + \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\Delta \varphi|^2 dx dt \right). \end{aligned} \quad (3.31)$$

By means of Lemma 2.4 and (2.2), we can absorb $\mathcal{B}$ and $|e^{s\alpha_*} \varphi|_{L_{\mathbb{F}}^2(0,T;H^4(G))}^2$.

Next, we estimate $\mathcal{A}_3$. Letting $\tilde{\varphi} = s\lambda^2 \xi e^{s\alpha_*} \varphi$, we obtain

$$\mathbb{E} \int_Q |\Delta \tilde{\varphi}|^2 dx dt \leq C \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |\varphi|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla \varphi|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\Delta \varphi|^2) dx dt. \quad (3.32)$$

Note that

$$|\tilde{\varphi}|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 \leq C |\Delta \tilde{\varphi}|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2, \quad (3.33)$$

and

$$\mathbb{E} \int_Q s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 \varphi|^2 dx dt \leq C |\tilde{\varphi}|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 + C \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |\varphi|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla \varphi|^2) dx dt. \quad (3.34)$$

Combining (3.24), (3.25) and (3.27)–(3.34), there exist $\lambda_5 \geq \max\{\lambda_3, \lambda_4\}$ and $s_5 \geq \max\{s_3, s_4\}$, such that for all $\lambda \geq \lambda_5$ and $s \geq s_5$, it holds that

$$\begin{aligned} & C \mathbb{E} \int_Q \theta^2 f^2 dx dt + C \mathbb{E} \int_Q \theta^2 (s^5 \lambda^5 \xi^5 g^2 + s^3 \lambda^3 \xi^3 |\nabla g|^2 + s \lambda \xi |\nabla^2 g|^2) dx dt \\ & + C \left[\mathbb{E} \int_{(0,T) \times G_3} (s^7 \lambda^7 \xi^7 \theta^2 |\varphi|^2 + s^5 \lambda^5 \xi^5 \theta^2 |\nabla \varphi|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta \varphi|^2) dx dt \right] \\ & + C \mathbb{E} \int_{(0,T) \times G_2} s^6 \lambda^8 \xi^6 \theta^2 |\varphi|^2 dx dt \\ & \geq \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |\varphi|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla \varphi|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta \varphi|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 \varphi|^2 + \lambda \theta^2 |\nabla \Delta \varphi|^2) dx dt. \end{aligned} \quad (3.35)$$

Now, we estimate the local terms on $\nabla \varphi$ and $\Delta \varphi$. Let us introduce a cut-off function ρ such that

$$\rho \in C_0^\infty(G_2), \quad \rho = 1 \text{ in } G_3, \quad 0 \leq \rho \leq 1. \quad (3.36)$$

Using Itô's formula and (1.5), we get

$$d(\rho^4 \xi^3 \theta^2 \varphi^2) = (2s\rho^4 \xi^3 \alpha_t + 3\rho^4 \xi^2 \xi_t) \theta^2 \varphi^2 dt + \rho^4 \xi^3 \theta^2 g^2 dt + 2\rho^4 \xi^3 \theta^2 \varphi d\varphi.$$

Integrating it on Q_2 , taking mathematical expectation on the both sides, we arrive

$$\begin{aligned} 0 = & \mathbb{E} \int_{Q_2} (2s\rho^4 \xi^3 \alpha_t + 3\rho^4 \xi^2 \xi_t) \theta^2 \varphi^2 dxdt + \mathbb{E} \int_{Q_2} \rho^4 \xi^3 \theta^2 g^2 dxdt - \mathbb{E} \int_{Q_2} 2\rho^4 \xi^3 \theta^2 \varphi \Delta^2 \varphi dxdt \\ & + \mathbb{E} \int_{Q_2} 2\rho^4 \xi^3 \theta^2 \varphi f dxdt. \end{aligned} \quad (3.37)$$

From (3.36) and integration by parts, there exist $\lambda_6 \geq C$ and $s_6 \geq CT$, such that for all $\lambda \geq \lambda_6$ and $s \geq s_6$, it follows that

$$\begin{aligned} & \mathbb{E} \int_{Q_2} s^3 \lambda^4 \xi^3 \rho^4 \theta^2 |\Delta \varphi|^2 dxdt \\ & \leq \mathbb{E} \int_{Q_2} s^3 \lambda^4 \xi^3 \rho^4 \theta^2 \varphi \Delta^2 \varphi dxdt + C \mathbb{E} \int_{Q_2} s^5 \lambda^6 \xi^5 \rho^4 \theta^2 |\nabla \varphi|^2 dxdt \\ & \quad + C \mathbb{E} \int_{Q_2} s^7 \lambda^8 \xi^7 \theta^2 \varphi^2 dxdt + C \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 (|\varphi|^2 + |\nabla \varphi|^2) dxdt. \end{aligned} \quad (3.38)$$

For $s \geq s_6$, we also have

$$\left| \mathbb{E} \int_{Q_2} 2s^3 \lambda^4 \xi^3 \rho^4 \theta^2 \varphi f dxdt \right| \leq C \mathbb{E} \int_{Q_2} s^6 \lambda^8 \xi^6 \theta^2 \varphi^2 dxdt + C \mathbb{E} \int_Q \theta^2 f^2 dxdt \quad (3.39)$$

and

$$\left| \mathbb{E} \int_{Q_2} s^3 \lambda^4 (2s\rho^4 \xi^3 \alpha_t + 3\rho^4 \xi^2 \xi_t) \theta^2 \varphi^2 dxdt \right| \leq C \mathbb{E} \int_{Q_2} s^7 \lambda^8 \xi^7 \theta^2 \varphi^2 dxdt. \quad (3.40)$$

Using integration by parts, for $\varepsilon > 0$, $\lambda \geq \lambda_6$ and $s \geq s_6$, we obtain

$$\mathbb{E} \int_{Q_2} s^5 \lambda^6 \xi^5 \rho^4 \theta^2 |\nabla \varphi|^2 dxdt \leq \varepsilon \mathbb{E} \int_{Q_2} s^3 \lambda^4 \xi^3 \rho^4 \theta^2 |\Delta \varphi|^2 dxdt + C\varepsilon^{-1} \mathbb{E} \int_{Q_2} s^7 \lambda^8 \xi^7 \theta^2 \varphi^2 dxdt. \quad (3.41)$$

From (3.36)–(3.41), letting $\varepsilon > 0$ small enough, for $\lambda \geq \lambda_6$ and $s \geq s_6$, it follows that

$$\begin{aligned} & \mathbb{E} \int_{(0,T) \times G_3} (s^5 \lambda^5 \xi^5 \theta^2 |\nabla \varphi|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta \varphi|^2) dxdt \\ & \leq C \left[\mathbb{E} \int_Q \theta^2 (f^2 + s^4 \lambda^4 \xi^4 g^2) dxdt + \mathbb{E} \int_{Q_2} s^7 \lambda^8 \xi^7 \theta^2 \varphi^2 dxdt + \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 (|\varphi|^2 + |\nabla \varphi|^2) dxdt \right] \end{aligned} \quad (3.42)$$

From (3.35) and (3.42), for $\lambda \geq \max\{\lambda_5, \lambda_6\} \geq C$ and $s \geq \max\{s_5, s_6\} \geq C(T + T^{1/2})$, we deduce (1.6). $\square$

4 Proof of the observability estimate

In this section, we use the Carleman estimate [Theorem 1.3](#) to prove [Theorem 1.2](#).

From [Theorem 1.3](#), we obtain the following proposition.

Proposition 4.1. *Let (z_1, z_2) be a solution to the system [\(1.2\)](#). Then for all $\lambda \geq \lambda_0$ and $s \geq s_0$, it holds that*

$$\mathbb{E} \int_Q s^6 \lambda^8 \xi^6 e^{2s\alpha} (z_1^2 + z_2^2) dx dt \leq C \mathbb{E} \int_{Q_2} s^7 \lambda^8 \xi^7 e^{2s\alpha} (z_1^2 + z_2^2) dx dt.$$

Lemma 4.2. *For any constants $c_0 > 0$ and $p, q, c_1 \geq 0$, there exists a constant $\tilde{s}_0 > 0$ such that for all $s \geq \tilde{s}_0$, we have*

$$|c_1 s^p \xi^q e^{c_0 s \alpha}| \leq 1. \quad (4.1)$$

Proof. Letting $\gamma(t) = \frac{1}{t^{1/2}(T-t)^{1/2}}$, for $t \in (0, T)$, it clearly holds that

$$\frac{2}{T} \leq \gamma(t) < \infty. \quad (4.2)$$

By means of [\(1.3\)](#) and [\(1.4\)](#), the inequality

$$c_1 s^p \xi^q e^{c_0 s \alpha} \leq c_1 s^p c_2(\lambda) \gamma^q e^{-c_0 c_3(\lambda) s \gamma} \leq 1$$

holds if

$$s \geq \frac{\ln(c_1 c_2(\lambda))}{c_0 c_3(\lambda) \gamma} + \frac{q \ln \gamma}{c_0 c_3(\lambda) \gamma} + \frac{p \ln s}{c_0 c_3(\lambda) \gamma}.$$

Thanks to [\(4.2\)](#) and $\frac{\ln \gamma}{\gamma} \leq c_4 < \infty$, it is sufficient that

$$s \geq \frac{T \ln(c_1 c_2(\lambda))}{2 c_0 c_3(\lambda)} + \frac{q c_4}{c_0 c_3(\lambda)} + \frac{T p \ln s}{2 c_0 c_3(\lambda)}.$$

Hence, there exists $\tilde{s}_0 = \tilde{s}_0(p, q, c_1, c_0, \lambda, T)$ such that [\(4.1\)](#) holds. $\square$

Proposition 4.3. *Fix $\lambda = \lambda_0$. Let (z_1, z_2) be a solution to the system [\(1.2\)](#). There exists constants $C, \tilde{s}_1 > 0$ such that for all $s \geq \tilde{s}_1$, it holds that*

$$\mathbb{E} \int_Q \xi^6 e^{2s\alpha} (z_1^2 + z_2^2) dx dt \leq C \mathbb{E} \int_{Q_0} e^{\frac{3}{2}s\alpha} z_2^2 dx dt.$$

Proof. From [Assumption 1.1](#), without loss of generality, we assume that $a_2(x, t) \geq \sigma$, a.e. $(x, t) \in G_1 \times (0, T)$, $\mathbb{P}$ -a.s.

Thanks to [Proposition 4.1](#) and [Lemma 4.2](#), there exists a constant $\tilde{s}_2 > 0$ such that for all $s \geq \tilde{s}_2$, we have

$$\mathbb{E} \int_Q s^6 \xi^6 e^{2s\alpha} (z_1^2 + z_2^2) dx dt \leq C \mathbb{E} \int_{Q_2} s^7 \xi^7 e^{\frac{1}{3}s\alpha} e^{\frac{5}{3}s\alpha} (z_1^2 + z_2^2) dx dt$$

$$\leq C\mathbb{E} \int_{Q_2} e^{\frac{5}{3}s\alpha}(z_1^2 + z_2^2)dxdt. \quad (4.3)$$

Let $\zeta \in C_0^\infty(G_1)$ be a cut-off function satisfying $\zeta = 1$ in G_2 and $0 \leq \zeta \leq 1$ in G_1 . Put $\rho = \zeta^{12}$. Let β, k, l, r and τ be positive numbers which will be specified later.

Let

$$\mathcal{P} \triangleq e^{k\tau\alpha} \rho^{\frac{4}{3}} z_1^2 + \beta e^{2\tau\alpha} \rho z_1 z_2 + e^{l\tau\alpha} \rho^{\frac{2}{3}} z_2^2.$$

Using Itô's formula, we have

$$\begin{aligned} d\mathcal{P} &= k\tau\alpha_t e^{k\tau\alpha} \rho^{\frac{4}{3}} z_1^2 + 2e^{k\tau\alpha} \rho^{\frac{4}{3}} z_1 dz_1 + e^{k\tau\alpha} \rho^{\frac{4}{3}} (dz_1)^2 \\ &\quad + 2\tau\beta\alpha_t e^{2\tau\alpha} \rho z_1 z_2 + \beta e^{2\tau\alpha} \rho (z_1 dz_2 + z_2 dz_1 + dz_1 dz_2) \\ &\quad + l\tau\alpha_t e^{l\tau\alpha} \rho^{\frac{2}{3}} z_2^2 + 2e^{l\tau\alpha} \rho^{\frac{2}{3}} z_2 dz_2 + e^{l\tau\alpha} \rho^{\frac{2}{3}} (dz_2)^2. \end{aligned}$$

From (1.2), noting that $\lim_{t \rightarrow 0+} \theta(x, t) = \lim_{t \rightarrow T-} \theta(x, t) = 0$ for all $x \in G$, we have

$$\begin{aligned} &\beta \mathbb{E} \int_Q e^{2\tau\alpha} \rho a_2 z_1^2 dxdt \\ &= \mathbb{E} \int_Q (k\tau e^{k\tau\alpha} \alpha_t \rho^{\frac{4}{3}} z_1^2 - 2e^{k\tau\alpha} \rho^{\frac{4}{3}} a_1 z_1^2 + e^{k\tau\alpha} \rho^{\frac{4}{3}} a_3^2 z_1^2) dxdt \\ &\quad - \mathbb{E} \int_Q [2e^{k\tau\alpha} \rho^{\frac{4}{3}} b_1 z_1 z_2 - 2\beta\tau e^{2\tau\alpha} \alpha_t \rho z_1 z_2 + \beta e^{2\tau\alpha} \rho (a_1 + b_2 - a_3 b_3) z_1 z_2 + 2e^{l\tau\alpha} \rho^{\frac{2}{3}} a_2 z_1 z_2] dxdt \\ &\quad + \mathbb{E} \int_Q (l\tau e^{l\tau\alpha} \alpha_t \rho^{\frac{2}{3}} z_2^2 - \beta e^{2\tau\alpha} \rho b_1 z_2^2 - 2e^{l\tau\alpha} \rho^{\frac{2}{3}} b_2 z_2^2 + e^{l\tau\alpha} \rho^{\frac{2}{3}} b_3^2 z_2^2) dxdt \\ &\quad - \mathbb{E} \int_Q (2e^{k\tau\alpha} \rho^{\frac{4}{3}} z_1 \Delta^2 z_1 + \beta e^{2\tau\alpha} \rho (z_2 \Delta^2 z_1 + z_1 \Delta^2 z_2) + 2e^{l\tau\alpha} \rho^{\frac{2}{3}} z_2 \Delta^2 z_2) dxdt \\ &\triangleq \sum_{i=1}^4 I_i. \end{aligned} \quad (4.4)$$

Next, we estimate I_i for $i = 1, 2, 3, 4$.

Thanks to (2.2) and (4.4), assuming $k > 2$ and noting Lemma 4.2, there exists $\tau_1 > 0$ such that for all $\tau \geq \tau_1$, it holds that

$$\begin{aligned} I_1 &= \mathbb{E} \int_Q (k\tau e^{k\tau\alpha} \alpha_t \rho^{\frac{4}{3}} z_1^2 - 2e^{k\tau\alpha} \rho^{\frac{4}{3}} a_1 z_1^2 + e^{k\tau\alpha} \rho^{\frac{4}{3}} a_3^2 z_1^2) dxdt \\ &= \mathbb{E} \int_Q (k\tau e^{(k-2)\tau\alpha} \alpha_t \rho^{\frac{1}{3}} - 2e^{(k-2)\tau\alpha} \rho^{\frac{1}{3}} a_1 + e^{(k-2)\tau\alpha} \rho^{\frac{1}{3}} a_3^2) e^{2\tau\alpha} \rho z_1^2 dxdt \\ &\leq \mathbb{E} \int_Q e^{2\tau\alpha} \rho z_1^2 dxdt. \end{aligned} \quad (4.5)$$

Using (2.2) and (4.4) again, assuming $k > 2$, $2 > r$ and $l > 1 + \frac{r}{2}$, noting Lemma 4.2 and Cauchy-Schwarz inequality, there exists $\tau_2 > 0$ such that for all $\tau \geq \tau_2$, it holds that

$$I_2 = -\mathbb{E} \int_Q [2e^{k\tau\alpha} \rho^{\frac{4}{3}} b_1 z_1 z_2 - 2\beta\tau e^{2\tau\alpha} \alpha_t \rho z_1 z_2 + \beta e^{2\tau\alpha} \rho (a_1 + b_2 - a_3 b_3) z_1 z_2 + 2e^{l\tau\alpha} \rho^{\frac{2}{3}} a_2 z_1 z_2] dxdt$$

$$\leq \mathbb{E} \int_Q e^{2\tau\alpha} \rho z_1^2 dxdt + (1 + \beta^2) \mathbb{E} \int_Q e^{r\tau\alpha} \rho^{\frac{1}{3}} z_2^2 dxdt. \quad (4.6)$$

Utilizing (2.2) and (4.4), assuming $l > r$ and $r < 2$, noting Lemma 4.2, there exists $\tau_3 > 0$ such that for all $\tau \geq \tau_3$, it holds that

$$\begin{aligned} I_3 &= \mathbb{E} \int_Q (l\tau e^{l\tau\alpha} \alpha_t \rho^{\frac{2}{3}} z_2^2 - \beta e^{2\tau\alpha} \rho b_1 z_2^2 - 2e^{l\tau\alpha} \rho^{\frac{2}{3}} b_2 z_2^2 + e^{l\tau\alpha} \rho^{\frac{2}{3}} b_3^2 z_2^2) dxdt \\ &= \mathbb{E} \int_Q (l\tau e^{(l-r)\tau\alpha} \alpha_t \rho^{\frac{1}{3}} - \beta e^{(2-r)\tau\alpha} \rho^{\frac{2}{3}} b_1 - 2e^{(l-r)\tau\alpha} \rho^{\frac{1}{3}} b_2 + e^{(l-r)\tau\alpha} \rho^{\frac{1}{3}} b_3^2) e^{r\tau\alpha} \rho^{\frac{1}{3}} z_2^2 dxdt \\ &\leq (1 + \beta) \mathbb{E} \int_Q e^{r\tau\alpha} \rho^{\frac{1}{3}} z_2^2 dxdt. \end{aligned} \quad (4.7)$$

In order to estimate I_4 , through the direct calculation, for the $C^2(G)$ function $\mathcal{R}$, u and v , we have the following equality

$$\begin{aligned} &\sum_{i,j=1}^n \mathcal{R} u v_{x_i x_i x_j x_j} \\ &= \sum_{i,j=1}^n (\mathcal{R} u v_{x_i x_i x_j} - \mathcal{R}_{x_j} u v_{x_i x_i} - \mathcal{R} u_{x_j} v_{x_i x_i})_{x_j} + \Delta \mathcal{R} u \Delta v + 2 \nabla \mathcal{R} \cdot \nabla u \Delta v + \mathcal{R} \Delta u \Delta v. \end{aligned} \quad (4.8)$$

Put $Q_\tau = 1 + \tau^2 \xi^2$. We retain this factor when estimating spatial derivatives of the exponential weights. With λ fixed, its derivatives of order at most two are bounded by CQ_τ . The strict gaps $k > 2$, $l > r$, and $4 - k - l > 0$ allow every remaining polynomial in $\tau\xi$ to be absorbed by the corresponding exponential gap, uniformly for sufficiently large τ . From (4.8), choosing $u = v = z_1$ and $\mathcal{R} = -2e^{k\tau\alpha} \rho^{\frac{4}{3}}$, utilizing the divergence theorem, we have

$$\begin{aligned} -\mathbb{E} \int_Q 2e^{k\tau\alpha} \rho^{\frac{4}{3}} z_1 \Delta^2 z_1 dxdt &= -2\mathbb{E} \int_Q \Delta(e^{k\tau\alpha} \rho^{\frac{4}{3}}) z_1 \Delta z_1 dxdt - 4\mathbb{E} \int_Q \nabla(e^{k\tau\alpha} \rho^{\frac{4}{3}}) \nabla z_1 \Delta z_1 dxdt \\ &\quad - 2\mathbb{E} \int_Q e^{k\tau\alpha} \rho^{\frac{4}{3}} |\Delta z_1|^2 dxdt. \end{aligned} \quad (4.9)$$

Thanks to Cauchy-Schwarz inequality, (1.3) and Lemma 4.2, recalling that $k > 2$, there exists $\tau_4 > 0$ such that for all $\tau \geq \tau_4$, it holds that

$$-2\mathbb{E} \int_Q \Delta(e^{k\tau\alpha} \rho^{\frac{4}{3}}) z_1 \Delta z_1 dxdt \leq \frac{1}{8} \mathbb{E} \int_Q e^{k\tau\alpha} \rho^{\frac{4}{3}} |\Delta z_1|^2 dxdt + \mathbb{E} \int_Q e^{2\tau\alpha} \rho |z_1|^2 dxdt, \quad (4.10)$$

and

$$-4\mathbb{E} \int_Q \nabla(e^{k\tau\alpha} \rho^{\frac{4}{3}}) \nabla z_1 \Delta z_1 dxdt \leq C \mathbb{E} \int_Q Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}} |\nabla z_1|^2 dxdt + \frac{1}{8} \mathbb{E} \int_Q e^{k\tau\alpha} \rho^{\frac{4}{3}} |\Delta z_1|^2 dxdt. \quad (4.11)$$

Combining (4.9)–(4.11), for $\tau \geq \tau_4$, we have

$$-\mathbb{E} \int_Q 2e^{k\tau\alpha} \rho^{\frac{4}{3}} z_1 \Delta^2 z_1 dxdt \leq \mathbb{E} \int_Q e^{2\tau\alpha} \rho |z_1|^2 dxdt + C \mathbb{E} \int_Q Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}} |\nabla z_1|^2 dxdt$$

$$-\frac{7}{4}\mathbb{E}\int_Q e^{k\tau\alpha}\rho^{\frac{4}{3}}|\Delta z_1|^2 dxdt. \quad (4.12)$$

From (4.8), choosing $u = v = z_2$ and $\mathcal{R} = -2e^{l\tau\alpha}\rho^{\frac{2}{3}}$, recalling $l > r$, similarly to (4.12), there exists $\tau_5 > 0$ such that for all $\tau \geq \tau_5$, we have

$$\begin{aligned} -\mathbb{E}\int_Q 2e^{l\tau\alpha}\rho^{\frac{2}{3}}z_2\Delta^2 z_2 dxdt &\leq \mathbb{E}\int_Q e^{r\tau\alpha}\rho^{\frac{1}{3}}|z_2|^2 dxdt + C\mathbb{E}\int_Q Q_\tau e^{l\tau\alpha}\rho^{\frac{1}{2}}|\nabla z_2|^2 dxdt \\ &\quad -\frac{7}{4}\mathbb{E}\int_Q e^{l\tau\alpha}\rho^{\frac{2}{3}}|\Delta z_2|^2 dxdt. \end{aligned} \quad (4.13)$$

From (4.8), choosing $\mathcal{R} = -\beta e^{2\tau\alpha}\rho$, we have

$$\begin{aligned} &-\mathbb{E}\int_Q \beta e^{2\tau\alpha}\rho(z_2\Delta^2 z_1 + z_1\Delta^2 z_2) dxdt \\ &= -\beta\mathbb{E}\int_Q \Delta(e^{2\tau\alpha}\rho)(z_1\Delta z_2 + z_2\Delta z_1) dxdt - 2\beta\mathbb{E}\int_Q \nabla(e^{2\tau\alpha}\rho)(\nabla z_1\Delta z_2 + \nabla z_2\Delta z_1) dxdt \\ &\quad - 2\beta\mathbb{E}\int_Q e^{2\tau\alpha}\rho\Delta z_1\Delta z_2 dxdt. \end{aligned} \quad (4.14)$$

Thanks to Cauchy-Schwarz inequality, (1.3) and Lemma 4.2, assuming that $k + l < 4$, there exists $\tau_6 > 0$ such that for all $\tau \geq \tau_6$, it holds that

$$\begin{aligned} &-\beta\mathbb{E}\int_Q \Delta(e^{2\tau\alpha}\rho)(z_1\Delta z_2 + z_2\Delta z_1) dxdt \\ &\leq \frac{1}{6}\mathbb{E}\int_Q e^{k\tau\alpha}\rho^{\frac{4}{3}}|\Delta z_1|^2 dxdt + \frac{1}{6}\mathbb{E}\int_Q e^{l\tau\alpha}\rho^{\frac{2}{3}}|\Delta z_2|^2 dxdt + \frac{1}{4}\mathbb{E}\int_Q e^{k\tau\alpha}\rho|z_1|^2 dxdt \\ &\quad + \frac{1}{4}\mathbb{E}\int_Q e^{l\tau\alpha}\rho^{\frac{1}{3}}|z_2|^2 dxdt, \end{aligned} \quad (4.15)$$

$$\begin{aligned} &-2\beta\mathbb{E}\int_Q \nabla(e^{2\tau\alpha}\rho)(\nabla z_1\Delta z_2 + \nabla z_2\Delta z_1) dxdt \\ &\leq \frac{1}{6}\mathbb{E}\int_Q e^{k\tau\alpha}\rho^{\frac{4}{3}}|\Delta z_1|^2 dxdt + \frac{1}{6}\mathbb{E}\int_Q e^{l\tau\alpha}\rho^{\frac{2}{3}}|\Delta z_2|^2 dxdt + \frac{1}{4}\mathbb{E}\int_Q Q_\tau e^{k\tau\alpha}\rho^{\frac{7}{6}}|\nabla z_1|^2 dxdt \\ &\quad + \frac{1}{4}\mathbb{E}\int_Q Q_\tau e^{l\tau\alpha}\rho^{\frac{1}{2}}|\nabla z_2|^2 dxdt, \end{aligned} \quad (4.16)$$

and

$$-2\beta\mathbb{E}\int_Q e^{2\tau\alpha}\rho\Delta z_1\Delta z_2 dxdt \leq \frac{1}{6}\mathbb{E}\int_Q e^{k\tau\alpha}\rho^{\frac{4}{3}}|\Delta z_1|^2 dxdt + \frac{1}{6}\mathbb{E}\int_Q e^{l\tau\alpha}\rho^{\frac{2}{3}}|\Delta z_2|^2 dxdt. \quad (4.17)$$

Combining (4.14)–(4.17), for $\tau \geq \tau_6$, it holds that

$$-\mathbb{E}\int_Q \beta e^{2\tau\alpha}\rho(z_2\Delta^2 z_1 + z_1\Delta^2 z_2) dxdt$$

$$\begin{aligned}
&\leq \frac{1}{2}\mathbb{E} \int_Q e^{k\tau\alpha} \rho^{\frac{4}{3}} |\Delta z_1|^2 dxdt + \frac{1}{2}\mathbb{E} \int_Q e^{l\tau\alpha} \rho^{\frac{2}{3}} |\Delta z_2|^2 dxdt + \frac{1}{4}\mathbb{E} \int_Q Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}} |\nabla z_1|^2 dxdt \\
&\quad + \frac{1}{4}\mathbb{E} \int_Q Q_\tau e^{l\tau\alpha} \rho^{\frac{1}{2}} |\nabla z_2|^2 dxdt \\
&\quad + \frac{1}{4}\mathbb{E} \int_Q e^{k\tau\alpha} \rho z_1^2 dxdt + \frac{1}{4}\mathbb{E} \int_Q e^{l\tau\alpha} \rho^{1/3} z_2^2 dxdt.
\end{aligned} \tag{4.18}$$

Hence, from (4.4), (4.12), (4.13) and (4.18), for $\tau \geq \tau_7 = \max\{\tau_4, \tau_5, \tau_6\}$, we have

$$\begin{aligned}
I_4 &= -\mathbb{E} \int_Q (2e^{k\tau\alpha} \rho^{\frac{4}{3}} z_1 \Delta^2 z_1 + \beta e^{2\tau\alpha} \rho (z_2 \Delta^2 z_1 + z_1 \Delta^2 z_2) + 2e^{l\tau\alpha} \rho^{\frac{2}{3}} z_2 \Delta^2 z_2) dxdt \\
&\leq -\frac{5}{4}\mathbb{E} \int_Q e^{k\tau\alpha} \rho^{\frac{4}{3}} |\Delta z_1|^2 dxdt - \frac{5}{4}\mathbb{E} \int_Q e^{l\tau\alpha} \rho^{\frac{2}{3}} |\Delta z_2|^2 dxdt + C\mathbb{E} \int_Q Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}} |\nabla z_1|^2 dxdt \\
&\quad + C\mathbb{E} \int_Q Q_\tau e^{l\tau\alpha} \rho^{\frac{1}{2}} |\nabla z_2|^2 dxdt + \mathbb{E} \int_Q e^{2\tau\alpha} \rho |z_1|^2 dxdt + \mathbb{E} \int_Q e^{r\tau\alpha} \rho^{\frac{1}{3}} |z_2|^2 dxdt \\
&\quad + \frac{1}{4}\mathbb{E} \int_Q e^{k\tau\alpha} \rho z_1^2 dxdt + \frac{1}{4}\mathbb{E} \int_Q e^{l\tau\alpha} \rho^{1/3} z_2^2 dxdt.
\end{aligned} \tag{4.19}$$

Through the direct calculation, for $C^2(G)$ functions $\mathcal{R}$ and u , we have the following equality

$$\mathcal{R}|\nabla u|^2 = \sum_{i=1}^n \left(\mathcal{R} u_{x_i} u - \frac{1}{2} \mathcal{R}_{x_i} u^2 \right)_{x_i} + \frac{1}{2} \Delta \mathcal{R} u^2 - \mathcal{R} \Delta u u.$$

Choosing $\mathcal{R} = Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}}$ and $u = z_1$, from (1.2) and Lemma 4.2, there exists $\tau_8 > 0$ such that for all $\tau \geq \tau_8$, it holds that

$$\begin{aligned}
\mathbb{E} \int_Q Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}} |\nabla z_1|^2 dxdt &= \frac{1}{2}\mathbb{E} \int_Q \Delta \left(Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}} \right) |z_1|^2 dxdt - \mathbb{E} \int_Q Q_\tau e^{k\tau\alpha} \rho^{\frac{7}{6}} \Delta z_1 z_1 dxdt \\
&\leq \mathbb{E} \int_Q e^{2\tau\alpha} \rho |z_1|^2 dxdt + \frac{1}{4C} \mathbb{E} \int_Q e^{k\tau\alpha} \rho^{\frac{4}{3}} |\Delta z_1|^2 dxdt.
\end{aligned} \tag{4.20}$$

Similarly, there exists $\tau_9 > 0$ such that for all $\tau \geq \tau_9$, we have

$$\begin{aligned}
\mathbb{E} \int_Q Q_\tau e^{l\tau\alpha} \rho^{\frac{1}{2}} |\nabla z_2|^2 dxdt &= \frac{1}{2}\mathbb{E} \int_Q \Delta \left(Q_\tau e^{l\tau\alpha} \rho^{\frac{1}{2}} \right) |z_2|^2 dxdt - \mathbb{E} \int_Q Q_\tau e^{l\tau\alpha} \rho^{\frac{1}{2}} \Delta z_2 z_2 dxdt \\
&\leq \mathbb{E} \int_Q e^{r\tau\alpha} \rho^{\frac{1}{3}} |z_2|^2 dxdt + \frac{1}{4C} \mathbb{E} \int_Q e^{l\tau\alpha} \rho^{\frac{2}{3}} |\Delta z_2|^2 dxdt.
\end{aligned} \tag{4.21}$$

Absorbing the two retained zero-order terms using $k > 2$ and $l > r$, and combining (4.19)–(4.21), for $\tau \geq \tau_{10} = \max\{\tau_7, \tau_8, \tau_9\}$, we obtain

$$\begin{aligned}
I_4 &\leq -\mathbb{E} \int_Q e^{k\tau\alpha} \rho^{\frac{4}{3}} |\Delta z_1|^2 dxdt - \mathbb{E} \int_Q e^{l\tau\alpha} \rho^{\frac{2}{3}} |\Delta z_2|^2 dxdt + C\mathbb{E} \int_Q e^{2\tau\alpha} \rho |z_1|^2 dxdt \\
&\quad + C\mathbb{E} \int_Q e^{r\tau\alpha} \rho^{\frac{1}{3}} |z_2|^2 dxdt.
\end{aligned} \tag{4.22}$$

From (4.4)–(4.7) and (4.22), for $\tau \geq \tau_{11} = \max\{\tau_1, \tau_2, \tau_3, \tau_{10}\}$, it holds that

$$\beta \mathbb{E} \int_Q e^{2\tau\alpha} \rho a_2 z_1^2 dx dt \leq C \mathbb{E} \int_Q e^{2\tau\alpha} \rho |z_1|^2 dx dt + C(1 + \beta^2) \mathbb{E} \int_Q e^{r\tau\alpha} \rho^{\frac{1}{3}} |z_2|^2 dx dt.$$

Choosing $\beta = 2C/\sigma$, recalling that $a_2(x, t) \geq \sigma$, a.e. $(x, t) \in G_1 \times (0, T)$, $\mathbb{P}$ -a.s., for $\tau \geq \tau_{11}$, we have

$$\mathbb{E} \int_Q e^{2\tau\alpha} \rho z_1^2 dx dt \leq C \mathbb{E} \int_Q e^{r\tau\alpha} \rho^{\frac{1}{3}} |z_2|^2 dx dt.$$

Choosing $\tau = \frac{5}{6}s$ and $r = \frac{9}{5}$, recalling that $\zeta = 1$ in G_2 and $\zeta \in C_0^\infty(G_1)$, for $s \geq \frac{6}{5}\tau_{11}$, it holds that

$$\mathbb{E} \int_{Q_2} e^{\frac{5}{3}s\alpha} z_1^2 dx dt \leq C \mathbb{E} \int_{Q_0} e^{\frac{3}{2}s\alpha} |z_2|^2 dx dt.$$

Combining it with (4.3), letting $\tilde{s}_1 = \max\{\tilde{s}_2, \frac{6}{5}\tau_{11}\}$, we complete the proof. $\square$

Now, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Thanks to Proposition 4.3, fixing $\lambda = \lambda_0$ and $s = \tilde{s}_1$, for the solution (z_1, z_2) of the system (1.2), it holds that

$$\mathbb{E} \int_Q \xi^6 e^{2s\alpha} (z_1^2 + z_2^2) dx dt \leq C \mathbb{E} \int_{Q_0} e^{\frac{3}{2}s\alpha} z_2^2 dx dt. \quad (4.23)$$

Recalling (1.3), we have

$$\mathbb{E} \int_Q \xi^6 e^{2s\alpha} (z_1^2 + z_2^2) dx dt \geq \min_{x \in \bar{G}} (\xi^6(x, T/2) e^{2s\alpha(x, T/4)}) \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_G (z_1^2 + z_2^2) dx dt, \quad (4.24)$$

and

$$\mathbb{E} \int_{Q_0} e^{\frac{3}{2}s\alpha} z_2^2 dx dt \leq \max_{(t,x) \in \bar{Q}} e^{\frac{3}{2}s\alpha} \mathbb{E} \int_{Q_0} z_2^2 dx dt. \quad (4.25)$$

Combining (4.23)–(4.25), we get

$$\begin{aligned} \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_G (z_1^2 + z_2^2) dx dt &\leq C \frac{\max_{(t,x) \in \bar{Q}} e^{\frac{3}{2}s\alpha}}{\min_{x \in \bar{G}} (\xi^6(x, T/2) e^{2s\alpha(x, T/4)})} \mathbb{E} \int_{Q_0} z_2^2 dx dt \\ &\leq C \mathbb{E} \int_{Q_0} z_2^2 dx dt. \end{aligned} \quad (4.26)$$

Thanks to Itô's formula, for $0 \leq t_1 \leq t_2 \leq T$, we have

$$\mathbb{E} \int_G (z_1^2(t_2) + z_2^2(t_2)) dx - \mathbb{E} \int_G (z_1^2(t_1) + z_2^2(t_1)) dx = \mathbb{E} \int_{t_1}^{t_2} \int_G d(z_1^2 + z_2^2) dx$$

$$\begin{aligned}
&= -\mathbb{E} \int_{t_1}^{t_2} \int_G [2z_1(\Delta^2 z_1 + a_1 z_1 + b_1 z_2) + 2z_2(\Delta^2 z_2 + a_2 z_1 + b_2 z_2) - a_3^2 z_1^2 - b_3^2 z_2^2] dx dt \\
&\leq C \mathbb{E} \int_{t_1}^{t_2} \int_G (z_1^2 + z_2^2) dx dt.
\end{aligned}$$

Hence, using Gronwall inequality, for all $t \in (0, T)$, we have

$$\mathbb{E} \int_G (z_1^2(T) + z_2^2(T)) dx \leq C \mathbb{E} \int_G (z_1^2(t) + z_2^2(t)) dx.$$

Combining it with (4.26), we complete the proof. $\square$

5 Proof of the null controllability

Proof of Theorem 1.1. We define the linear subspace $\mathcal{Y}$ of $L_{\mathbb{F}}^2(0, T; L^2(G_0))$ to be

$$\mathcal{Y} = \{ \chi_{G_0} z_2 \mid (z_1, z_2) \text{ solves the system (1.2) with } (z_1^0, z_2^0) \in L^2(G) \times L^2(G) \},$$

and define a linear functional F on $\mathcal{Y}$ as follows

$$F(\chi_{G_0} z_2) = \mathbb{E} \int_G (y_1^T z_1(T) + y_2^T z_2(T)) dx.$$

Thanks to Theorem 1.2, it holds that

$$\begin{aligned}
|F(\chi_{G_0} z_2)| &= \left| \mathbb{E} \int_G (y_1^T z_1(T) + y_2^T z_2(T)) dx \right| \\
&\leq \left(\mathbb{E} \int_G (|y_1^T|^2 + |y_2^T|^2) dx \right)^{\frac{1}{2}} \left(\mathbb{E} \int_G (|z_1(T)|^2 + |z_2(T)|^2) dx \right)^{\frac{1}{2}} \\
&\leq C \left(\mathbb{E} \int_G (|y_1^T|^2 + |y_2^T|^2) dx \right)^{\frac{1}{2}} \|z_2\|_{L_{\mathbb{F}}^2(0, T; L^2(G_0))}.
\end{aligned} \tag{5.1}$$

Hence, F is a bounded linear functional on $\mathcal{Y}$. By Hahn-Banach theorem, F can be extended to a bounded linear functional on $L_{\mathbb{F}}^2(0, T; L^2(G_0))$. For simplicity, we also denote F for this extension. From Riesz representation theorem, there exists $u \in L_{\mathbb{F}}^2(0, T; L^2(G_0))$ such that

$$\mathbb{E} \int_G (y_1^T z_1(T) + y_2^T z_2(T)) dx = F(\chi_{G_0} z_2) = \mathbb{E} \int_0^T \int_{G_0} z_2 u dx dt. \tag{5.2}$$

We claim that u is the desired control. In fact, thanks to Itô's formula and (1.1), we know that

$$\begin{aligned}
&\mathbb{E} \int_G y_1^T z_1(T) dx - \int_G y_1(0) z_1^0 dx = \mathbb{E} \int_0^T \int_G d(y_1 z_1) dx \\
&= \mathbb{E} \int_0^T \int_G (z_1 dy_1 + y_1 dz_1 + dy_1 dz_1) dx
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \int_0^T \int_G [(\Delta z_1 \Delta y_1 + a_1 z_1 y_1 + a_2 z_1 y_2 + a_3 Y_1 z_1) + (-\Delta z_1 \Delta y_1 - a_1 z_1 y_1 - b_1 z_2 y_1) - a_3 z_1 Y_1] dx dt \\
&= \mathbb{E} \int_0^T \int_G (a_2 z_1 y_2 - b_1 z_2 y_1) dx dt,
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\mathbb{E} \int_G y_2^T z_2(T) dx - \int_G y_2(0) z_2^0 dx = \mathbb{E} \int_0^T \int_G d(y_2 z_2) dx \\
&= \mathbb{E} \int_0^T \int_G (z_2 dy_2 + y_2 dz_2 + dy_2 dz_2) dx \\
&= \mathbb{E} \int_0^T \int_G [(\Delta z_2 \Delta y_2 + b_1 z_2 y_1 + b_2 z_2 y_2 + b_3 Y_2 z_2 + \chi_{G_0} u z_2) + (-\Delta z_2 \Delta y_2 - a_2 z_1 y_2 - b_2 z_2 y_2) \\
&\quad - b_3 z_2 Y_2] dx dt \\
&= \mathbb{E} \int_0^T \int_G (b_1 z_2 y_1 - a_2 z_1 y_2) dx dt + \mathbb{E} \int_0^T \int_{G_0} u z_2 dx dt.
\end{aligned} \tag{5.4}$$

Combining (5.2)–(5.4), for any $(z_1^0, z_2^0) \in L^2(G) \times L^2(G)$, we have

$$\int_G y_1(0) z_1^0 dx + \int_G y_2(0) z_2^0 dx = 0.$$

Hence, this implies $(y_1(0), y_2(0)) = 0$ in G , $\mathbb{P}$ –a.s., which completes the proof. $\square$

A Proof of Lemma 2.4

We need the following regularity estimate.

Lemma A.1. *There exists a constant $C > 0$, such that for all $h \in L_{\mathbb{F}}^2(0, T; H^4(G))$, $f \in L_{\mathbb{F}}^2(0, T; L^2(G))$ and $g \in L_{\mathbb{F}}^2(0, T; H_0^2(G))$ satisfying*

$$\begin{cases} dh + \Delta^2 h dt = f dt + g dW(t) & \text{in } Q, \\ h = \frac{\partial h}{\partial \nu} = 0 & \text{on } \Sigma, \\ h(0, \cdot) = 0 & \text{in } G, \end{cases}$$

we have

$$|h|_{L_{\mathbb{F}}^2(0, T; H^4(G))}^2 \leq C(|f|_{L_{\mathbb{F}}^2(0, T; L^2(G))}^2 + |g|_{L_{\mathbb{F}}^2(0, T; H^2(G))}^2).$$

Proof. Let $\mathbf{A}$ be an unbounded operator on $L^2(G)$ as follows

$$\begin{cases} \mathcal{D}(\mathbf{A}) = \left\{ \varphi \in H^4(G) \mid \varphi = \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma \right\}, \\ \mathbf{A}\varphi = \Delta^2 \varphi, \quad \forall \varphi \in \mathcal{D}(\mathbf{A}). \end{cases}$$

Denote by $0 < \mu_1 < \mu_2 \leq \dots$ the eigenvalues of $\mathbf{A}$ and by $\{\phi_i\}_{i=1}^\infty \subset \mathcal{D}(\mathbf{A})$ the corresponding eigenfunctions, which serves as a standard orthonormal basis of $L^2(G)$. Since $\mathbf{A}\phi_i = \mu_i\phi_i$, each eigenfunction belongs to $\mathcal{D}(\mathbf{A}^2)$. Both ϕ_i and $\mathbf{A}\phi_i$ satisfy the clamped boundary conditions. All integrations below can equivalently be read as finite spectral sums; no H^8 boundary regularity assertion is needed.

Consider the following stochastic differential equations

$$\begin{cases} dc_i^m = -\mu_i c_i^m dt + (f, \phi_i)_{L^2(G)} dt + (g, \phi_i)_{L^2(G)} dW(t), \\ c_i^m(0) = 0, \end{cases} \quad (\text{A.1})$$

where $i = 1, \dots, m$. Clearly, we have $(f, \phi_i)_{L^2(G)}, (g, \phi_i)_{L^2(G)} \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$. Thanks to the classical theory of stochastic differential equations (see [17, Theorem 3.2]), equation (A.1) exists a unique solution $c_i^m(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}))$ for $i = 1, \dots, m$.

Let $h^m = \sum_{i=1}^m c_i^m(t) \phi_i$. From (A.1), we have

$$dh^m = -\Delta^2 h^m dt + f^m dt + g^m dW(t), \quad (\text{A.2})$$

where $f^m = \sum_{i=1}^m (f, \phi_i)_{L^2(G)} \phi_i$ and $g^m = \sum_{i=1}^m (g, \phi_i)_{L^2(G)} \phi_i$. Note that $h^m(0) \equiv 0$. Thanks to Itô's formula, it holds that

$$d(h^m \Delta^2 h^m) = dh^m \Delta^2 h^m + h^m \Delta^2 (dh^m) + dh^m d(\Delta^2 h^m). \quad (\text{A.3})$$

Integrating it on Q , taking expectation on both sides, we have

$$\mathbb{E} \int_G h^m(T) \Delta^2 h^m(T) dx = \mathbb{E} \int_Q [dh^m \Delta^2 h^m + h^m \Delta^2 (dh^m) + dh^m d(\Delta^2 h^m)] dx. \quad (\text{A.4})$$

From the definition of h^m and ϕ_i , we obtain

$$\mathbb{E} \int_G h^m(T) \Delta^2 h^m(T) dx = \sum_{i=1}^m \mu_i \mathbb{E} (c_i^m(T))^2 \geq 0. \quad (\text{A.5})$$

From (A.2), we get

$$\mathbb{E} \int_Q dh^m \Delta^2 h^m dx \leq -\frac{1}{2} \mathbb{E} \int_Q |\Delta^2 h^m|^2 dx dt + 2 \mathbb{E} \int_Q |f|^2 dx dt, \quad (\text{A.6})$$

$$\begin{aligned} \mathbb{E} \int_Q h^m \Delta^2 (dh^m) dx &= \mathbb{E} \int_Q h^m (-\Delta^4 h^m) dx dt + \mathbb{E} \int_Q h^m (\Delta^2 f^m) dx dt \\ &\leq -\frac{1}{2} \mathbb{E} \int_Q |\Delta^2 h^m|^2 dx dt + 2 \mathbb{E} \int_Q |f|^2 dx dt, \end{aligned} \quad (\text{A.7})$$

and

$$\mathbb{E} \int_Q dh^m d(\Delta^2 h^m) dx = \mathbb{E} \int_Q g^m \Delta^2 g^m dx dt = \mathbb{E} \int_Q |\Delta g^m|^2 dx dt. \quad (\text{A.8})$$

Combining (A.4)–(A.8), we have

$$|\Delta^2 h^m|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 \leq 4|f^m|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 + |\Delta g^m|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2. \quad (\text{A.9})$$

From (A.9), thanks to elliptic regularity theory (see [11, Chapter 4, Theorem 4.1]), we get

$$|h^m|_{L_{\mathbb{F}}^2(0,T;H^4(G))}^2 \leq C|f^m|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 + C|\Delta g^m|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2.$$

Therefore $\{h^m\}_{m=1}^\infty$ exists a subsequence, which is still denoted by $\{h^m\}_{m=1}^\infty$, weakly convergence to $\tilde{h} \in L_{\mathbb{F}}^2(0,T;H^4(G))$ as m tends to ∞ . On the other hand, thanks to the well-posedness of the stochastic evolution equations (see [17, Theorem 3.13]), we know that $\lim_{m \rightarrow \infty} |h^m - h|_{L_{\mathbb{F}}^2(0,T;L^2(G))} = 0$. Hence, $\tilde{h} = h$ for a.e. $(t, x, \omega) \in [0, T] \times G \times \Omega$ and

$$|h|_{L_{\mathbb{F}}^2(0,T;H^4(G))} \leq \liminf_{m \rightarrow \infty} |h^m|_{L_{\mathbb{F}}^2(0,T;H^4(G))} \leq C(|f|_{L_{\mathbb{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathbb{F}}^2(0,T;H^2(G))}).$$

□

Proof of Lemma 2.4. The multiplicative trace inequalities, obtained by integration in a boundary-normal coordinate and Cauchy–Schwarz, give

$$\begin{aligned} \|\nabla^2 \varphi\|_{L^2(\Gamma)}^2 &\leq C\|\varphi\|_{H^2(G)}\|\varphi\|_{H^3(G)}, \\ \|\nabla^3 \varphi\|_{L^2(\Gamma)}^2 &\leq C\|\varphi\|_{H^3(G)}\|\varphi\|_{H^4(G)}. \end{aligned} \quad (\text{A.10})$$

Hence, combining it with (2.5), we have

$$\begin{aligned} &\mathbb{E} \int_{\Sigma} s^{\frac{9}{4}} \lambda^3 \xi^{\frac{9}{4}} \theta^2 |\nabla^2 \varphi|^2 d\Gamma dt + \mathbb{E} \int_{\Sigma} s^{\frac{3}{4}} \lambda \xi^{\frac{3}{4}} \theta^2 |\nabla^3 \varphi|^2 d\Gamma dt \\ &\leq C\mathbb{E} \int_Q s^6 \lambda^8 \xi^6 \theta^2 \varphi^2 dx dt + C\mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt. \end{aligned} \quad (\text{A.11})$$

Let $\varphi_* \triangleq e^{s\alpha_*} \varphi$. From (2.6), we get

$$\begin{cases} d\varphi_* + \Delta^2 \varphi_* dt = (e^{s\alpha_*} f + (e^{s\alpha_*})_t \varphi) dt + e^{s\alpha_*} g dW(t) & \text{in } Q, \\ \varphi_* = \frac{\partial \varphi_*}{\partial \nu} = 0 & \text{on } \Sigma, \\ \varphi_*(0, \cdot) = \varphi_*(T, \cdot) = 0 & \text{in } G. \end{cases}$$

By means of Lemma A.1 and (2.2), for $s \geq CT^{\frac{1}{2}}$, we have

$$\begin{aligned} &\mathbb{E} \int_0^T e^{2s\alpha_*} |\varphi|_{H^4(G)}^2 dt = |\varphi_*|_{L_{\mathbb{F}}^2(0,T;H^4(G))}^2 \\ &\leq C|e^{s\alpha_*} f + (e^{s\alpha_*})_t \varphi|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 + C|e^{s\alpha_*} g|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 \\ &\leq C\mathbb{E} \int_Q \theta^2 (|f|^2 + |\Delta g|^2 + s^2 \lambda^2 \xi^2 |\nabla g|^2 + s^4 \lambda^4 \xi^4 g^2) dx dt + C\mathbb{E} \int_Q s^6 \xi^6 \theta^2 \varphi^2 dx dt. \end{aligned} \quad (\text{A.12})$$

Combining (A.11) and (A.12), we complete the proof. □

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