

The ϕ -null controllability for stochastic fourth order semi-discrete parabolic equations

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Abstract

This paper is devoted to studying the ϕ -null controllability for a class of stochastic fourth order semi-discrete parabolic equations, where the spatial variable is discretized by a finite difference scheme while the time variable remains continuous. For this purpose, we establish a new global Carleman estimate for a backward stochastic fourth order semi-discrete parabolic operator, in which the large parameter is connected to the mesh size. A relaxed observability estimate is then obtained for backward stochastic fourth order semi-discrete parabolic equations, with an explicit observability constant depending on the discretization parameter and the lower-order coefficients. By the standard duality argument, we finally prove the ϕ -null controllability of the corresponding stochastic fourth order semi-discrete parabolic equations.

Mathematics Subject Classification. Primary 93B05; Secondary 93B07, 93C20

Key Words. Stochastic fourth order semi-discrete parabolic equation, ϕ -null controllability, observability, global Carleman estimate.

1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Write $\mathbb{F}$ for the progressive σ -field with respect to $\mathbf{F}$. Let $\mathcal{H}$ be a Banach space. Denote by $L^2_{\mathbb{F}}(\Omega; \mathcal{H})$ the space of all $\mathcal{F}_\tau$ -measurable random variables ξ such that $\mathbb{E}|\xi|_{\mathcal{H}}^2 < \infty$; by $L^2_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$ -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{L^2(0, T; \mathcal{H})}^2) < \infty$, with the canonical norm; by $L^\infty_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$ -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted essentially bounded processes; and by $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathcal{H}))$ the Banach space consisting of all $\mathcal{H}$ -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{C([0, T]; \mathcal{H})}^2) < \infty$.

Let $G = (0, 1)$ and G_0 be a nonempty open subset of G . As usual, χ_{G_0} denotes the characteristic function of G_0 . Firstly, we introduce the following controlled fourth order stochastic parabolic equation in a continuous framework:

$$\begin{cases} dy + \partial_x^4 y dt = (a_1 y + \chi_{G_0} u) dt + (a_2 y + v) dW(t) & \text{in } (0, T) \times G, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ \partial_x y(t, 0) = \partial_x y(t, 1) = 0 & \text{on } (0, T), \\ y(0) = y_0 & \text{in } G. \end{cases} \quad (1.1)$$

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Here, the initial datum $y_0 \in L^2(G)$, the coefficients a_1, a_2 in $L^\infty(0, T; L^\infty(G))$ and the control $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$. By the classical well-posedness result for stochastic evolution equations, we know that (1.1) admits a unique weak solution $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \times L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G))$ (e.g., [19, Theorem 3.24]). The system (1.1) is said to be null controllable at time T if for any given y_0 , there exist controls $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$ such that the corresponding solution satisfies $y(T, x) = 0$, $\mathbb{P}$ -a.s.

Stochastic fourth order parabolic equation in a continuous framework is widely used to describe some fast diffusion phenomenon under stochastic perturbations (e.g., [9]). Further, it is the linearized version of the stochastic Cahn-Hilliard equation, which is used to model phase separation phenomena in a melted alloy that is quenched to a temperature (e.g., [6]). Compared to second order parabolic equation, to derive controllability for higher order parabolic equation is not an easy task even in the continuous framework. To the best of our knowledge, the controllability of deterministic and stochastic fourth order parabolic equations has been extensively studied in many work (see [8, 12, 13, 20, 30] and the references cited therein).

As the numerical approximation of this control system, it is natural to investigate the controllability properties of the corresponding discrete systems.

For $N \in \mathbb{N}$, put the space discretization parameter $h \triangleq 1/(N + 1)$. We consider the pairs (t, x_i) with $t \in (0, T)$ and $x_i = ih$ for $i = 0, \dots, N + 1$. Utilizing the centered finite difference method to the space variable for the system (1.1), and considering two additional points $x_{-1} = -h$ and $x_{N+2} = (N + 2)h$ to discretize the boundary conditions, we consider the following stochastic semi-discrete system:

$$\begin{cases} dy^i + \frac{1}{h^4}(y^{i+2} - 4y^{i+1} + 6y^i - 4y^{i-1} + y^{i-2})dt, \\ \quad = (a_1^i y^i + \chi_{G_0}^i u^i)dt + (a_2^i y^i + v^i)dW(t) & t \in (0, T), \quad i = 1, \dots, N, \\ y^0(t) = y^{N+1}(t) = 0 & t \in (0, T), \\ y^{-1}(t) = y^{N+2}(t) = 0 & t \in (0, T), \\ y^i(0) = y_0^i & i = 1, \dots, N. \end{cases} \quad (1.2)$$

It is well known that the uniform null controllability property is very delicate for semi-discrete parabolic equations and generally fails. For instance, as demonstrated by the counterexample provided in [31], the uniform null controllability property may not hold even for the deterministic second order semi-discrete equation in the two dimensional case. To the best of our knowledge, there are also only few results on the uniform null controllability of deterministic second order semi-discrete parabolic equations for some special cases; see [1, 14, 18, 22, 25]. The underlying reason is that the semi-discrete operator may admit some eigenfunctions which are not seen from the observation region. As explained in [4], the presence of such modes leads to initial data that are not null controllable for the semi-discrete system. In the stochastic semi-discrete setting, it was also shown in [17] that, in higher dimensions, a semi-discrete approximation of a null-controllable system may fail to preserve null controllability. This mechanism provides a strong motivation for relaxing the notion of null controllability in the semi-discrete setting.

Therefore, rather than seeking uniform null controllability, it is natural to consider weaker controllability notions. There are papers that study ϕ -null controllability instead of uniform null controllability. In [2, 3], a uniform null controllability result for the lower part of the spectrum of deterministic semi-discrete second order parabolic problem is established. Later, [5] proved that the deterministic second order semi-discrete parabolic equations are ϕ -null controllable by establishing a discrete global Carleman estimate directly for semi-discrete parabolic operator. This ϕ -null controllability means that the norm of the semi-discrete solution at time T can be dominated

by $\sqrt{\phi(h)}$, where ϕ is a real-valued function that tends to zero as the space discretization parameter h tends to zero.

Since then, there have been many results on the controllability of deterministic semi-discrete or fully discrete parabolic systems by using discrete Carleman estimates. We refer readers to [7, 15, 16, 21]. Recent works on semi-discrete and fully discrete problems also include stochastic controllability results in [17, 27, 28, 34] and a fully discrete controllability result in [29].

To state our main results, we first introduce some notations. We define the following regular partition of the interval $[0, 1]$ as $\mathcal{K} = \{x_i \mid i = 0, 1, \dots, N, N+1\}$. Considering any set of points $\mathcal{W} \subset \mathcal{K}$, we define the dual meshes $\mathcal{W}'$ and $\mathcal{W}^*$, respectively, by

$$\mathcal{W}' \triangleq \tau_+(\mathcal{W}) \cap \tau_-(\mathcal{W}), \quad \mathcal{W}^* \triangleq \tau_+(\mathcal{W}) \cup \tau_-(\mathcal{W})$$

where

$$\tau_{\pm}(\mathcal{W}) \triangleq \left\{ x \pm \frac{h}{2} \mid x \in \mathcal{W} \right\}$$

We denote $\overline{\mathcal{W}} \triangleq (\mathcal{W}^*)^*$ and $\mathring{\mathcal{W}} \triangleq (\mathcal{W}')'$. Note that for two consecutive points $x_i, x_{i+1} \in \mathcal{W}$, we have $x_{i+1} - x_i = h$ provided $\mathring{\mathcal{W}} = \mathcal{W}$. We call such a subset $\mathcal{W} \subset \mathcal{K}$ a regular mesh. Define the boundary of a regular mesh $\mathcal{W}$ as $\partial\mathcal{W} \triangleq \overline{\mathcal{W}} \setminus \mathcal{W}$.

Then, we introduce the semi-discrete sets $\mathcal{Q} \triangleq (0, T) \times \mathcal{W}$. Moreover, we say that the semi-discrete set is regular if the space variable is a regular mesh. We also define the dual semi-discrete sets by $\mathcal{Q}' \triangleq (0, T) \times \mathcal{W}'$ and $\mathcal{Q}^* \triangleq (0, T) \times \mathcal{W}^*$. Similarly, the semi-discrete boundary is given by $\partial\mathcal{Q} = (0, T) \times \partial\mathcal{W}$.

Next, we define the average operator A_h and the difference operator D_h by

$$\begin{aligned} A_h(u)(t, x) &\triangleq \frac{\tau_+ u(t, x) + \tau_- u(t, x)}{2}, \\ D_h(u)(t, x) &\triangleq \frac{\tau_+ u(t, x) - \tau_- u(t, x)}{h}, \end{aligned}$$

where $\tau_{\pm} u(t, x) \triangleq u(t, x \pm \frac{h}{2})$.

Here and in the sequel, $L(\mathcal{W})$ denotes the set of real-valued functions defined in $\mathcal{W}$. We also denote by $L_h^2(\mathcal{W})$ the Hilbert space with the inner product

$$\langle u, v \rangle_{L_h^2(\mathcal{W})} \triangleq \int_{\mathcal{W}} uv \triangleq h \sum_{x \in \mathcal{W}} u(x)v(x)$$

For $u \in L(\mathcal{W})$, we define its $L_h^\infty(\mathcal{W})$ -norm as

$$\|u\|_{L_h^\infty(\mathcal{W})} \triangleq \max_{x \in \mathcal{W}} \{|u(x)|\}.$$

With above notations, defining $Q \triangleq (0, T) \times \mathcal{M}$ where $\mathcal{M} \triangleq \mathring{\mathcal{K}}$, the controlled semi-discrete system

(1.2) can be written as

$$\begin{cases} dy + D_h^4 y dt = (a_1 y + \chi_{G_0} u) dt + (a_2 y + v) dW(t) & \text{in } Q, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ D_h y(t, -\frac{h}{2}) = D_h y(t, 1 + \frac{h}{2}) = 0 & \text{on } (0, T), \\ y(0, x) = y_0 & \text{in } \mathcal{M}, \end{cases} \quad (1.3)$$

where the initial datum $y_0 \in L_h^2(\mathcal{M})$, the coefficients a_1, a_2 in $L_{\mathbb{F}}^\infty(0, T; L_h^\infty(\mathcal{M}))$ and the control $(u, v) \in L_{\mathbb{F}}^2(0, T; L_h^2(G_0 \cap \mathcal{M})) \times L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M}))$. According to the classical theory of stochastic differential equations (see [19, Theorem 3.2]), we know that there is a unique solution $y \in L_{\mathbb{F}}^2(\Omega; C([0, T]; L_h^2(\mathcal{M})))$ to (1.3).

The main ϕ -null controllability result of this paper is the following.

Theorem 1.1. *There exist positive constants h_0 and C such that for all $T > 0$ and $h \leq \min\{h_0, h_1\}$ with*

$$h_1 = C(1 + T^{-1} + \mathcal{H}^{2/7})^{-1},$$

where $\mathcal{H} = |a_1|_{L_{\mathbb{F}}^\infty(0, T; L_h^\infty(\mathcal{M}))} + |a_2|_{L_{\mathbb{F}}^\infty(0, T; L_h^\infty(\mathcal{M}))}$, there exist $(u, v) \in L_{\mathbb{F}}^2(0, T; L_h^2(G_0 \cap \mathcal{M})) \times L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M}))$ such that the solution y to (1.3) satisfies

$$\mathbb{E} \int_Q |v|^2 dt + \mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} |u|^2 dt \leq C_{obs} \int_{\mathcal{M}} |y_0|^2,$$

and

$$\mathbb{E} \int_{\mathcal{M}} |y(T)|^2 \leq C_{obs} e^{-\frac{C}{h}} \int_{\mathcal{M}} |y_0|^2. \quad (1.4)$$

Here, $C_{obs} = e^{C(1+T^{-1}+\mathcal{H}^{2/7}+T+\mathcal{H}^2 T)}$.

Here and in what follows, unless otherwise stated, C stands for a generic positive constant depending only on G and G_0 whose value may vary from line to line.

Remark 1.1. Choosing $\phi(h)$ to be the right hand side of (1.4), we have $|y(T)|_{L_{\mathcal{F}_T}^2(\Omega; L_h^2(\mathcal{M}))} \leq \sqrt{\phi(h)}$ with $\lim_{h \rightarrow 0} \phi(h) = 0$, which means that system (1.3) is ϕ -null controllable at time T . Compared with the null controllability results in the continuous case [20, Theorem 1.3] and [11, Theorem 1.1], here we obtain only the ϕ -null controllability in Theorem 1.1. It would be quite interesting to study the uniform null controllability; i.e., for all sufficiently small admissible $h > 0$, there exist (u, v) which have uniform bounds, such that the corresponding solution to (1.3) satisfies $y(T, x) = 0$, $\mathbb{P}$ -a.s for any $x \in \mathcal{M}$. It has been done for the deterministic semi-discrete heat equation on a uniform mesh in [1]. However, we currently do not have a method for achieving this for the stochastic case.

Remark 1.2. Similar to proving Theorem 1.1, employing the techniques presented in this paper allows us to obtain ϕ -null controllability results for the backward stochastic fourth order semi-discrete parabolic equation.

Remark 1.3. Here we use an additional control in the diffusion term, as in [24]. Therefore, the control of (1.3) is a pair of function (u, v) . It is an interesting question to study the ϕ -null controllability for system (1.3) with only one control in the drift.

To prove the controllability result of (1.3), we introduce the adjoint equation:

$$\begin{cases} dz - D_h^4 z dt = -(a_1 z + a_2 Z) dt + Z dW(t) & \text{in } Q, \\ z(t, 0) = z(t, 1) = 0 & \text{on } (0, T), \\ D_h z(t, -\frac{h}{2}) = D_h z(t, 1 + \frac{h}{2}) = 0 & \text{on } (0, T), \\ z(T, x) = z_T & \text{in } \mathcal{M}. \end{cases} \quad (1.5)$$

Due to the classical theory of backward stochastic differential equations (see [19, Theorem 4.2]), we know that there is a unique solution $(z, Z) \in L_{\mathbb{F}}^2(\Omega; C([0, T]; L_h^2(\mathcal{M}))) \times L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M}))$ to (1.5).

We have the following observability estimate

Theorem 1.2. *There exist positive constants h_0 and C such that for all $T > 0$ and $h \leq \min\{h_0, h_1\}$, the solution (z, Z) to system (1.5) satisfies*

$$\mathbb{E} \int_{\mathcal{M}} |z(0)|^2 \leq C_{obs} \left(\mathbb{E} \int_Q |Z|^2 dt + \mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} |z|^2 dt + e^{-\frac{C}{h}} \mathbb{E} \int_{\mathcal{M}} |z|^2 \Big|_{t=T} \right),$$

where h_1 and C_{obs} is given in Theorem 1.1.

Remark 1.4. Compared with the continuous observability estimates in [20, Theorem 1.6] and [11, Theorem 1.3], Theorem 1.2 contains the additional last term on the right hand side. This term tends to 0 exponentially as h tends to 0. Currently, we do not know how to remove this term.

We will use discrete Carleman estimates to prove Theorem 1.2. To state our Carleman estimate, we first introduce the weight functions. Let $\tilde{G}$ be an open interval contains $[0, 1]$, $G_1 \subset \overline{G_1} \subset G_0$ and $G_2 \subset \overline{G_2} \subset G_1$ be two open sets. By [10, Lemma 1.1], we know that there exists a function $\psi \in C^\infty(\tilde{G})$ satisfies

$$\psi > 0 \text{ in } \tilde{G}, \quad |\partial_x \psi| \geq C_0 > 0 \text{ in } \overline{G \setminus G_2}, \quad \partial_x \psi(0) > 0, \text{ and } \partial_x \psi(1) < 0. \quad (1.6)$$

Furthermore, we assume $\overline{\mathcal{M} \cap G_2} \subset \mathcal{M} \cap G_1$. For any positive parameters λ and μ , choosing $0 < \delta < \frac{1}{2}$, put

$$\begin{cases} \phi(x) = e^{\mu(|2\psi|_{C(\overline{G})} + \psi(x))} - e^{4\mu|\psi|_{C(\overline{G})}}, & \varphi(x) = e^{\mu(|2\psi|_{C(\overline{G})} + \psi(x))}, \\ s(t) = \lambda\theta(t), & \theta(t) = \frac{1}{(t+\delta T)(T+\delta T-t)}, \\ r(t, x) = e^{s(t)\phi(x)}, & \rho(t, x) = (r(t, x))^{-1}. \end{cases} \quad (1.7)$$

We have the following Carleman estimate.

Theorem 1.3. *Let the function s and ϕ be defined according to (1.7). There exists a positive constant μ_0 such that for any $\mu \geq \mu_0$, one can find positive constants $C = C(\mu)$, h_0 and ε_0 with $0 < \varepsilon_0 \leq 1$ and a sufficiently large $\lambda_0 \geq 1$ such that for all $\lambda \geq \lambda_0$, $0 < h \leq h_0$ and $\lambda h (\delta T^2)^{-1} \leq \varepsilon_0$, it holds that*

$$\begin{aligned} & \mathbb{E} \int_Q s^7 e^{2s\phi} |w|^2 dt + \mathbb{E} \int_{Q^*} s^5 e^{2s\phi} |D_h w|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 e^{2s\phi} |D_h^2 w|^2 dt + \mathbb{E} \int_{Q^*} s e^{2s\phi} |D_h^3 w|^2 dt \\ & \leq C \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} s^7 e^{2s\phi} |w|^2 dt + \mathbb{E} \int_Q e^{2s\phi} |f|^2 dt + \mathbb{E} \int_Q s^4 e^{2s\phi} |g|^2 dt \right. \\ & \quad \left. + h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |w|^2 \Big|_{t=0} + h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |w|^2 \Big|_{t=T} \right), \end{aligned} \quad (1.8)$$

for all $w \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_h(\mathcal{M})))$ and $f, g \in L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$ satisfying

$$-dw + D_h^4 w dt = f dt + g dW(t) \quad (1.9)$$

with $w = 0$ on $\partial\mathcal{M}$ and $D_h w = 0$ on $\partial\mathcal{M}^*$.

Remark 1.5. For continuous stochastic differential operators, Carleman estimates are often derived from weighted pointwise identities (see e.g. [24]). In the semi-discrete setting, however, we do not establish such an identity, but instead derive the estimate directly. There are two reasons for this approach. First, the stochastic structure of the operator makes the corresponding identity extremely complicated. Second, a pointwise formulation would lose essential information at individual mesh points and would therefore obscure the argument.

Remark 1.6. By combining the approach in [26, Theorem 1.3] with the method of the present paper, one can also derive a Carleman estimate for the forward equation associated with the discrete system.

Remark 1.7. Starting from the Carleman estimate [Theorem 1.3](#), one can include first order, second order, and third order spatial derivative terms of the state in the drift term of [\(1.1\)](#). Moreover, one can also include first order spatial derivative terms of the state in the diffusion term. We have not written these lower order terms explicitly only for the sake of notational simplicity. However, we currently do not know how to treat the case where second order or higher order spatial derivative terms of the state appear in the diffusion term.

Remark 1.8. Since the control v acts on the whole domain, the term $a_2 y$ in the diffusion can be absorbed into the control by setting $\tilde{v} = v + a_2 y$ (see [\[27, Remark 1.6\]](#)). Therefore, the problem can be reduced to the case $a_2 \equiv 0$. We retain the more general form only for completeness.

The rest of this paper is organized as follows. In [section 2](#), we give some preliminaries. The Carleman estimate [\(1.8\)](#) is proved in [section 3](#). In [section 4](#), we prove [Theorem 1.2](#). Finally, we prove the main controllability result [Theorem 1.1](#) based on the Hilbert uniqueness method in [section 5](#).

2 Some preliminary results

This section is devoted to presenting some preliminary results we needed.

Lemma 2.1 ([\[7, Lemma 2.1\]](#)). *For any $u, v \in C(\overline{\mathcal{W}})$, we have for the difference operator*

$$D_h(uv) = D_h u A_h v + A_h u D_h v, \text{ on } \mathcal{W}^*. \quad (2.1)$$

Similarly, the average of the product gives

$$A_h(uv) = A_h u A_h v + \frac{h^2}{4} D_h u D_h v, \text{ on } \mathcal{W}^*. \quad (2.2)$$

Finally, on $\mathring{\mathcal{W}}$ we have

$$u = A_h^2 u - \frac{h^2}{4} D_h^2 u. \quad (2.3)$$

Corollary 2.2 ([\[7, Corollary 2.2\]](#)). *Let $\overline{\mathcal{W}} \subseteq \mathcal{M}$ be a regular mesh. For $u \in L(\overline{\mathcal{W}})$,*

$$A_h(u^2) = (A_h u)^2 + \frac{h^2}{4} (D_h u)^2, \text{ on } \mathcal{W}^*. \quad (2.4)$$

In particular, for all $u \in L(\overline{\mathcal{W}})$,

$$A_h(u^2) \geq (A_h u)^2, \text{ on } \mathcal{W}^*. \quad (2.5)$$

For $u \in L(\overline{\mathcal{W}})$

$$D_h(u^2) = 2D_h u A_h u. \quad (2.6)$$

To deal with the boundary conditions, we note that, in the one-dimensional setting considered here, $\mathcal{W}$ is a regular mesh and therefore, for each $x \in \partial\mathcal{W}$, exactly one of the following two cases occurs. We define the outward normal for $x \in \partial\mathcal{W}$ by

$$\nu(x) \triangleq \begin{cases} 1, & \text{if } \tau_-(x) \in \mathcal{W}^* \text{ and } \tau_+(x) \notin \mathcal{W}^*, \\ -1, & \text{if } \tau_-(x) \notin \mathcal{W}^* \text{ and } \tau_+(x) \in \mathcal{W}^*. \end{cases} \quad (2.7)$$

We also introduce the trace operator for $u \in L(\mathcal{W}^*)$ by

$$\forall x \in \partial\mathcal{W}, \quad t_r(u)(x) \triangleq \begin{cases} \tau_- u(x), & \text{if } \nu(x) = 1, \\ \tau_+ u(x), & \text{if } \nu(x) = -1. \end{cases}$$

Then by the definition of trace operator, for $x \in \partial\mathcal{W}$ and $u \in L(\overline{\mathcal{W}})$, it is easy to check that

$$t_r(|A_h u|^2)\nu - \frac{h^2}{4} t_r(|D_h u|^2)\nu = |u|^2\nu - h u t_r(D_h u). \quad (2.8)$$

Finally, let us introduce the discrete integration on the boundary for $u \in L(\partial\mathcal{W})$ as

$$\int_{\partial\mathcal{W}} u \triangleq \sum_{x \in \partial\mathcal{W}} u(x).$$

Proposition 2.3 ([7, Proposition 2.3]). *Let $\mathcal{W} \subset \mathcal{K}$ be a regular mesh. For $u \in L(\overline{\mathcal{W}})$ and $v \in L(\mathcal{W}^*)$ we have*

$$\int_{\mathcal{W}} u D_h v = - \int_{\mathcal{W}^*} D_h u v + \int_{\partial\mathcal{W}} u t_r(v) \nu \quad (2.9)$$

and

$$\int_{\mathcal{W}} u A_h v = \int_{\mathcal{W}^*} A_h u v - \frac{h}{2} \int_{\partial\mathcal{W}} u t_r(v). \quad (2.10)$$

Corollary 2.4 ([7, Corollary 2.4]). *Let $\mathcal{W} \subset \mathcal{K}$ be a regular mesh. For $u \in L(\overline{\mathcal{W}})$ and $v \in L(\overline{\mathcal{W}})$ we have*

$$\int_{\mathcal{W}} u D_h^2 v = \int_{\overline{\mathcal{W}}} v D_h^2 u - \int_{\partial\mathcal{W}^*} D_h u t_r(v) \nu + \int_{\partial\mathcal{W}} u t_r(D_h v) \nu, \quad (2.11)$$

$$\int_{\mathcal{W}} u A_h^2 v = \int_{\overline{\mathcal{W}}} v A_h^2 u - \frac{h}{2} \int_{\partial\mathcal{W}^*} A_h u t_r(v) - \frac{h}{2} \int_{\partial\mathcal{W}} u t_r(A_h v), \quad (2.12)$$

and

$$\begin{aligned} \int_{\mathcal{W}} u A_h D_h v &= - \int_{\overline{\mathcal{W}}} v A_h D_h u + \frac{h}{2} \int_{\partial \mathcal{W}^*} D_h u t_r(v) + \int_{\partial \mathcal{W}} u t_r(A_h v) \nu \\ &= - \int_{\overline{\mathcal{W}}} v A_h D_h u - \frac{h}{2} \int_{\partial \mathcal{W}} u t_r(D_h v) + \int_{\partial \mathcal{W}^*} A_h u t_r(v) \nu. \end{aligned} \quad (2.13)$$

Proposition 2.5 ([7, Corollary 2.7]). *Let f be a $(n+4)$ -times differentiable function defined on $\mathbb{R}$ and $m, n \in \mathbb{N}$. Then*

$$A_h^m D_h^n f = f^{(n)} + R_{A_h^m}(f^{(n)}) + R_{D_h^n}(f) + R_{A_h^m D_h^n}(f),$$

where

$$\begin{aligned} R_{D_h^n}(f) &:= h^2 \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{(n-2k)}{2} \right)^{n+2} \int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!} f^{(n+2)} \left(\cdot + \frac{(n-2k)h}{2} \sigma \right) d\sigma \\ R_{A_h^n}(f) &:= \frac{h^2}{2^{n+2}} \sum_{k=0}^n \binom{n}{k} (n-2k)^2 \int_0^1 (1-\sigma) f^{(2)} \left(\cdot + \frac{(n-2k)h}{2} \sigma \right) d\sigma, \end{aligned}$$

and $R_{A_h^m D_h^n} := R_{A_h^m} \circ R_{D_h^n}$. Here, $f^{(n)}$ denotes the n -th derivative of f with respect to x .

In the sequel, for any $n \in \mathbb{N}$, we denote by $\mathcal{O}(s^n)$ a function of order s^n for sufficiently large s (which is independent of μ and h), by $\mathcal{O}_\mu(s^n)$ a function of order s^n for fixed μ and sufficiently large s (which is independent of h), and by $\mathcal{O}_\mu((sh)^n)$ a function of order $(sh)^n$ for fixed μ and sufficiently large s , sufficiently small h .

Lemma 2.6 ([16, Lemma 4.3]). *For $\alpha, \beta \in \mathbb{N}$, we have*

$$\begin{aligned} \partial_x^\beta (r \partial_x^\alpha \rho) &= \alpha^\beta (-s\varphi)^\alpha \mu^{\alpha+\beta} (\partial_x \psi)^{\alpha+\beta} + \alpha \beta (s\varphi)^\alpha \mu^{\alpha+\beta-1} \mathcal{O}(1) + s^{\alpha-1} \alpha (\alpha-1) \mathcal{O}_\mu(1) \\ &= s^\alpha \mathcal{O}_\mu(1). \end{aligned} \quad (2.14)$$

Corollary 2.7 ([16, Corollary 4.4]). *For $\alpha, \beta, \gamma \in \mathbb{N}$, we have*

$$\begin{aligned} \partial_x^\gamma (r^2 (\partial_x^\alpha \rho) \partial_x^\beta \rho) &= (\alpha + \beta)^\gamma (-s\varphi)^{\alpha+\beta} \mu^{\alpha+\beta+\gamma} (\partial_x \psi)^{\alpha+\beta+\gamma} + \gamma (\alpha + \beta) (s\varphi)^{\alpha+\beta} \mu^{\alpha+\beta+\gamma-1} \mathcal{O}(1) \\ &\quad + s^{\alpha+\beta-1} [\alpha(\alpha-1) + \beta(\beta-1)] \mathcal{O}_\mu(1) \\ &= s^{\alpha+\beta} \mathcal{O}_\mu(1). \end{aligned} \quad (2.15)$$

By (1.7) and (2.14), it follows that the following result holds.

Lemma 2.8. *For $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, we have*

$$\begin{aligned} \partial_x^\sigma (r \partial_x^\alpha \rho \partial_x^\beta (r \partial_x^\gamma \rho)) &= \gamma^\beta (\alpha + \gamma)^\sigma (-s\varphi)^{\alpha+\gamma} \mu^{\alpha+\beta+\gamma+\sigma} (\partial_x \psi)^{\alpha+\beta+\gamma+\sigma} + (s\varphi)^{\alpha+\gamma} \mu^{\alpha+\beta+\gamma+\sigma-1} \mathcal{O}(1) \\ &\quad + s^{\alpha+\gamma-1} \mathcal{O}_\mu(1) \\ &= s^{\alpha+\gamma} \mathcal{O}_\mu(1). \end{aligned} \quad (2.16)$$

Lemma 2.9. [16, Lemma 4.8] *Let $\alpha, j, k, m, n \in \mathbb{N}$. Provided $\lambda h (\delta T^2)^{-1} \leq 1$, we have*

$$A_h^j D_h^k \partial_x^\alpha (r A_h^m D_h^n \rho) = \partial_x^k \partial_x^\alpha (r \partial_x^n \rho) + s^n \mathcal{O}_\mu((sh)^2) = s^n \mathcal{O}_\mu(1). \quad (2.17)$$

Theorem 2.10. [16, Theorem 4.9] Let $\alpha, \beta, j, k, l, m, n, p \in \mathbb{N}$. Provided $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned} A_h^p D_h^l \partial_x^\beta (r^2 A_h^j D_h^k (\partial_x^\alpha \rho) A_h^m D_h^n (\rho)) &= \partial_x^l \partial_x^\beta (r^2 \partial_x^k \partial_x^\alpha \rho \partial_x^n \rho) + s^{n+k+\alpha} \mathcal{O}_\mu((sh)^2) \\ &= s^{n+k+\alpha} \mathcal{O}_\mu(1). \end{aligned} \quad (2.18)$$

Theorem 2.11. [7, Theorem 2.9] Let $\alpha, j, k, m, n \in \mathbb{N}$. Provided $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\partial_t A_h^j D_h^k (r A_h^m D_h^n (\partial_x^\alpha \rho)) = T \theta s^{\alpha+n} \mathcal{O}_\mu(1). \quad (2.19)$$

By Theorem 2.5 and (2.17), it is easy to check that the following estimate holds.

Lemma 2.12. Let $i, j, k, l, m, n, p, q \in \mathbb{N}$. Provided $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned} A_h^i D_h^j (r A_h^k D_h^l \rho A_h^m D_h^n (r A_h^p D_h^q \rho)) &= \partial_x^j (r \partial_x^l \rho \partial_x^n (r \partial_x^q \rho)) + s^{l+q} \mathcal{O}_\mu((sh)^2) \\ &= s^{l+q} \mathcal{O}_\mu(1). \end{aligned} \quad (2.20)$$

Remark 2.1. Several asymptotic estimates collected in this section can alternatively be deduced from [23, Proposition 3.3 and Theorems 3.5, 3.7, 3.13]. We keep the present preliminaries in order to make the paper self-contained and to fix the notation used later in the proof of the Carleman estimate.

3 Proof of the Carleman estimate

Proof of Theorem 1.3. Before entering the computations, let us emphasize two features that are specific to the stochastic semi-discrete fourth-order setting. Compared with the deterministic case in [7], the conjugated operator now produces Itô terms involving dv , $D_h(dv)$, and $D_h^2(dv)$, which must be balanced with the diffusion variable g . Moreover, the fourth-order discrete operator gives rise to correction terms that must be combined carefully; in particular, the contributions corresponding to I_{41} and I_{51} will be absorbed into the estimates of I_{11} , I_{21} , and I_{31} after discrete integrations by parts. For readability, we divide the proof into six clearly separated parts.

Step 1. Conjugated operator and decomposition. At first, we define

$$\mathcal{P}_h w = -dw + D_h^4 w. \quad (3.1)$$

Letting $v = rw$ and recalling that $\rho = r^{-1}$, utilizing Itô's formula, we have

$$-rdw = -rd(\rho v) = -dv + \lambda \phi \partial_t \theta v dt. \quad (3.2)$$

Thanks to Theorem 2.1 (see also [23, Proposition 2.2] for a general discrete Leibniz rule) and $w = \rho v$, it holds that

$$D_h^4 w = D_h^4 \rho A_h^4 v + 4A_h D_h^3 \rho A_h^3 D_h v + 6A_h^2 D_h^2 \rho A_h^2 D_h^2 v + 4A_h^3 D_h \rho A_h D_h^3 v + A_h^4 \rho D_h^4 v. \quad (3.3)$$

Combining (3.1)–(3.3), we deduce that

$$r \mathcal{P}_h w = I_1 dt + I_2 + I_3 dt, \quad (3.4)$$

where

$$I_1 = 6r A_h^2 D_h^2 \rho A_h^2 D_h^2 v + r D_h^4 \rho A_h^4 v + r A_h^4 \rho D_h^4 v + 6A_h D_h (r A_h^2 D_h^2 \rho) A_h D_h v + \Phi(v),$$

$$\begin{aligned}
I_2 &= -dv + 4rA_h D_h^3 \rho A_h^3 D_h v dt + 4rA_h^3 D_h \rho A_h D_h^3 v dt + 2A_h D_h(rA_h D_h^3 \rho) v dt, \\
I_3 &= \lambda \phi \partial_t \theta v - 6A_h D_h(rA_h^2 D_h^2 \rho) A_h D_h v - 2A_h D_h(rA_h D_h^3 \rho) v - \Phi(v) \\
\Phi(v) &= 3h^2 D_h^2[A_h D_h(rA_h^2 D_h^2 \rho) A_h D_h v] + \frac{h^2}{2} A_h^2[A_h D_h(rD_h^4 \rho) A_h D_h v] + \frac{h^2}{4} D_h^2(rD_h^4 \rho) A_h^2 v \\
&\quad - D_h^2(rA_h^4 \rho) D_h^2 v + 2A_h D_h[A_h D_h(rA_h^4 \rho) D_h^2 v].
\end{aligned} \tag{3.5}$$

Using (3.4), it follows that

$$2I_1 r \mathcal{P}_h w = 2I_1^2 dt + 2I_1 I_2 + 2I_1 I_3 dt,$$

which, combined with Cauchy-Schwarz inequality, (1.9) and (3.1), implies

$$\mathbb{E} \int_Q |rf|^2 dt \geq 2\mathbb{E} \int_Q I_1 I_2 - \mathbb{E} \int_Q |I_3|^2 dt. \tag{3.6}$$

The next steps provide an estimate for the right hand side of (3.6). For convenience, we denote

$$\mathbb{E} \int_Q I_1 I_2 = \sum_{i=1}^5 \sum_{j=1}^4 I_{ij}, \tag{3.7}$$

where I_{ij} stands for the inner product in $L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M}))$ between the i -th term of I_1 and j -th term of I_2 .

In order to shorten the formulas used in the sequel, we define the following lower order terms

$$\begin{aligned}
\mathcal{A}_1 &= \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2) + s^7 \mu^7 \varphi^7 \mathcal{O}(1)] |v|^2 dt \\
&\quad + \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2) + s^5 \mu^5 \varphi^5 \mathcal{O}(1)] |D_h v|^2 dt \\
&\quad + \mathbb{E} \int_{\overline{Q}} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |D_h^2 v|^2 dt \\
&\quad + \mathbb{E} \int_{Q^*} [\mathcal{O}_\mu(1) + s \mathcal{O}_\mu((sh)^2) + s \mu \varphi \mathcal{O}(1)] |D_h^3 v|^2 dt, \\
\mathcal{A}_2 &= \mathbb{E} \int_Q s^4 \mathcal{O}_\mu(1) |dv|^2 + \mathbb{E} \int_{\overline{Q}} h^2 \mathcal{O}_\mu(1) |D_h^2(dv)|^2, \\
\mathcal{A}_3 &= \mathbb{E} \int_{\mathcal{M}} s^4 \mathcal{O}_\mu(1) |v|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}^*} s^2 \mathcal{O}_\mu(1) |D_h v|^2 \Big|_{t=0} + \mathbb{E} \int_{\overline{\mathcal{M}}} \mathcal{O}_\mu(1) |D_h^2 v|^2 \Big|_{t=0} \\
&\quad + \mathbb{E} \int_{\mathcal{M}} s^4 \mathcal{O}_\mu(1) |v|^2 \Big|_{t=T} + \mathbb{E} \int_{\mathcal{M}^*} s^2 \mathcal{O}_\mu(1) |D_h v|^2 \Big|_{t=T} + \mathbb{E} \int_{\overline{\mathcal{M}}} \mathcal{O}_\mu(1) |D_h^2 v|^2 \Big|_{t=T}, \\
\mathcal{B} &= \mathbb{E} \int_{\partial Q} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] t_r (|D_h v|^2) dt + \mathbb{E} \int_{\partial Q} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu(sh)] |D_h^2 v|^2 dt \\
&\quad + \mathbb{E} \int_{\partial Q} [\mathcal{O}_\mu(1) + s \mathcal{O}_\mu(sh)] t_r (|D_h^3 v|^2) dt.
\end{aligned}$$

Step 2. Estimate of the terms I_{i1} with $i = 1, \dots, 5$. We begin with the terms involving $-dv$. The deterministic part of the computation is parallel to [7], while the stochastic setting requires additional applications of Itô's formula to control the terms involving $D_h(dv)$ and $D_h^2(dv)$.

Since $w = 0$ on $\partial\mathcal{M}$ and $D_hw = 0$ on $\partial\mathcal{M}^*$, noting that $v = rw$, we obtain

$$v = 0 \text{ on } \partial Q, \quad D_h v = A_h v = 0 \text{ on } \partial Q^*, \quad (3.8)$$

and

$$t_r(v) = 0 \text{ on } \partial Q^* \text{ and } \partial \overline{Q}^*, \quad t_r(D_h v) = t_r(A_h v) = 0 \text{ on } \partial \overline{Q}. \quad (3.9)$$

From (3.5), (2.13), (3.8), (2.1), (2.2) and (2.3), letting $q_{11} = rA_h^2 D_h^2 \rho$, we have

$$\begin{aligned} I_{11} &= -6\mathbb{E} \int_Q q_{11} A_h^2 D_h^2 v dv = 6\mathbb{E} \int_Q A_h D_h v A_h D_h (q_{11} dv) \\ &= 6\mathbb{E} \int_Q A_h D_h q_{11} A_h^2 (dv) A_h D_h v + 6\mathbb{E} \int_Q A_h^2 q_{11} A_h D_h (dv) A_h D_h v \\ &\quad + \frac{3}{2} h^2 \mathbb{E} \int_Q D_h^2 q_{11} A_h D_h (dv) A_h D_h v + \frac{3}{2} h^2 \mathbb{E} \int_Q A_h D_h q_{11} D_h^2 (dv) A_h D_h v \\ &= 6\mathbb{E} \int_Q A_h D_h q_{11} A_h D_h v dv + 6\mathbb{E} \int_Q \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) A_h D_h v A_h D_h (dv) \\ &\quad + 3h^2 \mathbb{E} \int_Q A_h D_h q_{11} A_h D_h v D_h^2 (dv). \end{aligned} \quad (3.10)$$

By Itô's formula, it follows that

$$\begin{aligned} &6\mathbb{E} \int_Q \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) A_h D_h v A_h D_h (dv) \\ &= 3\mathbb{E} \int_{\overline{\mathcal{M}}} \left[\left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) |A_h D_h v|^2 \right] \Big|_0^T - 3\mathbb{E} \int_Q \partial_t \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) |A_h D_h v|^2 dt \\ &\quad - 3\mathbb{E} \int_Q \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) |A_h D_h (dv)|^2. \end{aligned} \quad (3.11)$$

To derive the next estimate, (2.17) yields the coefficient bound for $q_{11} + \frac{h^2}{2} D_h^2 q_{11}$. Next, (2.5) gives

$$|A_h D_h v|^2 \leq A_h (|D_h v|^2).$$

Finally, (2.10) together with (3.8) transfers the average from $\overline{\mathcal{M}}$ to $\mathcal{M}^*$ without boundary contributions. Hence, for $\lambda h (\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned} &3\mathbb{E} \int_{\overline{\mathcal{M}}} \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) |A_h D_h v|^2 \geq -\mathbb{E} \int_{\overline{\mathcal{M}}} [s^2 \mathcal{O}_\mu(1) + \mathcal{O}_\mu((sh)^2)] |A_h D_h v|^2 \\ &\geq -\mathbb{E} \int_{\overline{\mathcal{M}}} [s^2 \mathcal{O}_\mu(1) + \mathcal{O}_\mu((sh)^2)] A_h (|D_h v|^2) \\ &\geq -\mathbb{E} \int_{\mathcal{M}^*} [s^2 \mathcal{O}_\mu(1) + \mathcal{O}_\mu((sh)^2)] |D_h v|^2. \end{aligned} \quad (3.12)$$

Therefore, similar to (3.12), by (2.19), it holds that

$$-3\mathbb{E} \int_Q \partial_t \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) |A_h D_h v|^2 dt - 3\mathbb{E} \int_Q \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) |A_h D_h (dv)|^2$$

$$\geq -\mathbb{E} \int_{Q^*} T\theta[s^2\mathcal{O}_\mu(1) + \mathcal{O}_\mu((sh)^2)]|D_h v|^2 dt - \mathbb{E} \int_{Q^*} [s^2\mathcal{O}_\mu(1) + \mathcal{O}_\mu((sh)^2)]|D_h(dv)|^2. \quad (3.13)$$

By Cauchy-Schwarz inequality and combining (3.11)–(3.13), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, we obtain

$$\begin{aligned} & 6\mathbb{E} \int_{\overline{Q}} \left(q_{11} + \frac{h^2}{2} D_h^2 q_{11} \right) A_h D_h v d(A_h D_h v) \\ & \geq -\mathbb{E} \int_{Q^*} [s^2\mathcal{O}_\mu(1) + \mathcal{O}_\mu((sh)^2)]|D_h(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_3 \\ & = \mathbb{E} \int_Q [s^2\mathcal{O}_\mu(1) + \mathcal{O}_\mu((sh)^2)] D_h^2(dv) dv - \mathcal{A}_1 - \mathcal{A}_3 \\ & \geq -\frac{1}{8}\mathbb{E} \int_Q |D_h^2(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \end{aligned} \quad (3.14)$$

Combining (3.10), (3.14), (2.11), (3.9) and (3.8), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, we have

$$\begin{aligned} I_{11} & \geq 6\mathbb{E} \int_Q A_h D_h(r A_h^2 D_h^2 \rho) A_h D_h v dv + 3h^2 \mathbb{E} \int_{\overline{Q}} D_h^2[A_h D_h(r A_h^2 D_h^2 \rho) A_h D_h v] dv \\ & \quad - \frac{1}{8}\mathbb{E} \int_Q |D_h^2(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \end{aligned} \quad (3.15)$$

From (2.12), (3.8), (2.2) and (2.3), letting $q_{21} = r D_h^4 \rho$, we obtain

$$\begin{aligned} I_{21} & = -\mathbb{E} \int_Q q_{21} A_h^4 v dv = -\mathbb{E} \int_{\overline{Q}} A_h^2(q_{21} dv) A_h^2 v \\ & = -\mathbb{E} \int_{\overline{Q}} A_h^2 q_{21} A_h^2(dv) A_h^2 v - \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h D_h q_{21} A_h D_h(dv) A_h^2 v - \frac{h^2}{4} \mathbb{E} \int_{\overline{Q}} D_h^2 q_{21} A_h^2(dv) A_h^2 v \\ & \quad + \frac{h^2}{4} \mathbb{E} \int_{\overline{Q}} D_h^2 q_{21} A_h^2 v dv. \end{aligned} \quad (3.16)$$

Thanks to Itô's formula and similar to (3.12), from (2.17) and (2.19), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, we have

$$\begin{aligned} & -\mathbb{E} \int_{\overline{Q}} A_h^2 q_{21} A_h^2(dv) A_h^2 v \\ & = -\frac{1}{2} \mathbb{E} \int_{\overline{\mathcal{M}}} (A_h^2 q_{21} |A_h^2 v|^2) \Big|_0^T + \frac{1}{2} \mathbb{E} \int_{\overline{Q}} \partial_t(A_h^2 q_{21}) |A_h^2 v|^2 dt + \frac{1}{2} \mathbb{E} \int_{\overline{Q}} A_h^2 q_{21} |A_h^2(dv)|^2 \\ & \geq -\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \end{aligned} \quad (3.17)$$

Using the identity $A_h D_h(dv) A_h^2 v = \frac{1}{2} d[D_h(|A_h v|^2)] - A_h D_h v A_h^2(dv) - A_h D_h(dv) A_h^2(dv)$ and Cauchy-Schwarz inequality, from (2.5), (3.9), (2.9), (2.10), (2.17) and (2.19), for $\lambda h(\delta T^2)^{-1} \leq 1$, we deduce that

$$\begin{aligned} & -\frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h D_h q_{21} A_h D_h(dv) A_h^2 v \\ & = -\frac{h^2}{4} \mathbb{E} \int_{\overline{Q}} d[A_h D_h q_{21} D_h(|A_h v|^2)] + \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} \partial_t(A_h D_h q_{21}) A_h D_h v A_h^2 v dt \end{aligned}$$

$$\begin{aligned}
& + \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h D_h q_{21} A_h D_h v A_h^2(dv) + \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h D_h q_{21} A_h D_h(dv) A_h^2(dv) \\
& = -\frac{h^2}{4} \mathbb{E} \int_{\overline{M}} A_h D_h q_{21} D_h(|A_h v|^2) \Big|_0^T + \mathbb{E} \int_{\overline{Q}} T \theta s^2 \mathcal{O}_\mu((sh)^2) A_h D_h v A_h^2 v dt \\
& \quad + \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h D_h q_{21} A_h D_h v A_h^2(dv) - \frac{h^2}{4} \mathbb{E} \int_{\overline{Q}^*} A_h D_h^2 q_{21} |A_h(dv)|^2 \\
& \geq \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h D_h q_{21} A_h D_h v A_h^2(dv) - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 \\
& = \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h^2 (A_h D_h q_{21} A_h D_h v) dv - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3.
\end{aligned} \tag{3.18}$$

Thanks to Itô's formula, from (2.5), (2.17), (2.10) and (2.19), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, it holds that

$$\begin{aligned}
& -\frac{h^2}{4} \mathbb{E} \int_{\overline{Q}} D_h^2 q_{21} A_h^2(dv) A_h^2 v \\
& = -\frac{h^2}{8} \mathbb{E} \int_{\overline{M}} D_h^2 q_{21} |A_h^2 v|^2 \Big|_0^T + \frac{h^2}{8} \mathbb{E} \int_{\overline{Q}} \partial_t(D_h^2 q_{21}) |A_h^2 v|^2 dt + \frac{h^2}{8} \mathbb{E} \int_{\overline{Q}} D_h^2 q_{21} |A_h^2(dv)|^2 \\
& \geq -\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3.
\end{aligned} \tag{3.19}$$

Combining (3.16)–(3.19), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, it follows that

$$I_{21} \geq \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h^2 [A_h D_h(r D_h^4 \rho) A_h D_h v] dv + \frac{h^2}{4} \mathbb{E} \int_{\overline{Q}} D_h^2(r D_h^4 \rho) A_h^2 v dv - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \tag{3.20}$$

From (2.11), (3.8), (2.1), (2.3), (2.13) and (3.9), letting $q_{31} = r A_h^4 \rho$, we have

$$\begin{aligned}
I_{31} & = -\mathbb{E} \int_{\overline{Q}} q_{31} D_h^4 v dv = -\mathbb{E} \int_{\overline{Q}} D_h^2(q_{31} dv) D_h^2 v \\
& = -\mathbb{E} \int_{\overline{Q}} D_h^2 q_{31} A_h^2(dv) D_h^2 v - 2\mathbb{E} \int_{\overline{Q}} A_h D_h q_{31} A_h D_h(dv) D_h^2 v - \mathbb{E} \int_{\overline{Q}} A_h^2 q_{31} D_h^2(dv) D_h^2 v \\
& = -\mathbb{E} \int_{\overline{Q}} D_h^2 q_{31} D_h^2 v dv + 2\mathbb{E} \int_{\overline{Q}} A_h D_h (A_h D_h q_{31} D_h^2 v) dv \\
& \quad - \mathbb{E} \int_{\overline{Q}} \left(A_h^2 q_{31} + \frac{h^2}{4} D_h^2 q_{31} \right) D_h^2(dv) D_h^2 v.
\end{aligned} \tag{3.21}$$

Choose $\varepsilon_* = \varepsilon_*(\mu) \in (0, 1]$ sufficiently small. Since ϕ is smooth, every ratio $r(x)\rho(x+jh)$ with $|j| \leq 2$ lies between $e^{-C(\mu)sh}$ and $e^{C(\mu)sh}$. Thus, if $\lambda h(\delta T^2)^{-1} \leq \varepsilon_*$, then

$$A_h^2 q_{31} + \frac{h^2}{4} D_h^2 q_{31} = \frac{q_{31}(x+h) + q_{31}(x-h)}{2} \geq \frac{3}{4}.$$

Thanks to Itô's formula, (2.17) and (2.19), for $h \leq 1$ and $\lambda \geq T + T^2$, it follows that

$$-\mathbb{E} \int_{\overline{Q}} \left(A_h^2 q_{31} + \frac{h^2}{4} D_h^2 q_{31} \right) D_h^2(dv) D_h^2 v$$

$$\begin{aligned}
&= -\frac{1}{2}\mathbb{E} \int_{\bar{Q}} \left(A_h^2 q_{31} + \frac{h^2}{4} D_h^2 q_{31} \right) |D_h^2 v|^2 \Big|_0^T + \frac{1}{2}\mathbb{E} \int_{\bar{Q}} \partial_t \left(A_h^2 q_{31} + \frac{h^2}{4} D_h^2 q_{31} \right) |D_h^2 v|^2 dt \\
&\quad + \frac{1}{2}\mathbb{E} \int_{\bar{Q}} \left(A_h^2 q_{31} + \frac{h^2}{4} D_h^2 q_{31} \right) |D_h^2(dv)|^2 \\
&\geq \frac{3}{8}\mathbb{E} \int_{\bar{Q}} |D_h^2(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3.
\end{aligned} \tag{3.22}$$

Combining (3.22) and (3.21), for $\lambda h(\delta T^2)^{-1} \leq \varepsilon_*$, $h \leq 1$ and $\lambda \geq T + T^2$, we have

$$\begin{aligned}
I_{31} &\geq -\mathbb{E} \int_{\bar{Q}} D_h^2(r A_h^4 \rho) D_h^2 v dv + 2\mathbb{E} \int_{\bar{Q}} A_h D_h [A_h D_h(r A_h^4 \rho) D_h^2 v] dv \\
&\quad + \frac{3}{8}\mathbb{E} \int_{\bar{Q}} |D_h^2(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3.
\end{aligned} \tag{3.23}$$

The terms I_{41} and I_{51} are not treated separately. Indeed, after the discrete integrations by parts in the estimates of I_{11} , I_{21} , and I_{31} , the correction terms appearing in (3.15), (3.20) and (3.23) are exactly the contributions corresponding to I_{41} and I_{51} . Hence (3.24) is obtained by collecting all these terms together.

Hence, combining (3.15), (3.20), (3.23) and (3.5), for $\lambda h(\delta T^2)^{-1} \leq \varepsilon_*$, $h \leq 1$ and $\lambda \geq T + T^2$, we obtain

$$I_{11} + I_{21} + I_{31} + I_{41} + I_{51} \geq \frac{1}{4}\mathbb{E} \int_{\bar{Q}} |D_h^2(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \tag{3.24}$$

Noting that (1.9) and $v = rw$, by Proposition 2.5, (1.7) and (2.1), for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned}
\mathbb{E} \int_{\bar{Q}} |D_h^2(dv)|^2 &= \mathbb{E} \int_{\bar{Q}} |D_h^2(rg)|^2 dt \\
&\geq \frac{1}{3}\mathbb{E} \int_{\bar{Q}} r^2 |D_h^2 g|^2 dt - \mathbb{E} \int_{\bar{Q}} \mathcal{O}_\mu((sh)^4) r^2 |D_h^2 g|^2 dt - \mathbb{E} \int_{Q^*} s^2 \mathcal{O}_\mu(1) r^2 |D_h g|^2 dt \\
&\quad - \mathbb{E} \int_Q s^4 \mathcal{O}_\mu(1) r^2 |g|^2 dt.
\end{aligned} \tag{3.25}$$

From (2.9), (2.1), Proposition 2.5 and (1.7), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$-\mathbb{E} \int_{Q^*} s^2 \mathcal{O}_\mu(1) r^2 |D_h g|^2 dt \geq -\frac{1}{12}\mathbb{E} \int_{\bar{Q}} r^2 |D_h^2 g|^2 dt - \mathbb{E} \int_Q s^4 \mathcal{O}_\mu(1) r^2 |g|^2 dt. \tag{3.26}$$

Decrease $h_1 \leq 1$ so that the coefficient $h^2 \mathcal{O}_\mu(1)$ in $\mathcal{A}_2$ is at most $1/12$. Then (3.24) retains at least $1/6$ of $\mathbb{E} \int_{\bar{Q}} |D_h^2(dv)|^2$. Decrease ε_* further so that the coefficient $\mathcal{O}_\mu((sh)^4)$ in (3.25) is at most $1/8$. By (3.26), the remaining coefficient of $\mathbb{E} \int_{\bar{Q}} r^2 |D_h^2 g|^2 dt$ is at least $1/3 - 1/12 - 1/8 = 1/8$. Consequently, for $\lambda \geq T + T^2$, $h \leq h_1$ and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_*$, we obtain

$$I_{11} + I_{21} + I_{31} + I_{41} + I_{51} \geq \frac{1}{48}\mathbb{E} \int_{\bar{Q}} r^2 |D_h^2 g|^2 dt - C(\mu)\mathbb{E} \int_Q s^4 r^2 |g|^2 dt - \mathcal{A}_1 - \mathcal{A}_3. \tag{3.27}$$

Step 3. Estimate of the terms I_{ij} with $i = 1, \dots, 4$ and $j = 2, 3, 4$. The estimates in

this step follow the same pattern as in the deterministic fourth-order semi-discrete case treated in [7]. We include them for completeness and to keep the notation consistent with the present stochastic framework; readers familiar with [7] may proceed directly to Step 4. The genuinely new contributions among the terms with $j = 2, 3, 4$ are the terms I_{5j} , which will be treated separately in Step 4.

Noting that (2.6), (2.9), (2.3), (2.18), (2.15), (1.6) and (2.7), letting $q_{12} = r^2 A_h^2 D_h^2 \rho A_h D_h^3 \rho$, for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned}
I_{12} &= 24\mathbb{E} \int_Q q_{12} A_h^2 D_h^2 v A_h^3 D_h v dt = 12\mathbb{E} \int_Q q_{12} D_h [(A_h^2 D_h v)^2] dt \\
&= -12\mathbb{E} \int_{Q^*} D_h q_{12} |A_h^2 D_h v|^2 dt + 12\mathbb{E} \int_{\partial Q} q_{12} t_r (|A_h^2 D_h v|^2) \nu dt \\
&= -12\mathbb{E} \int_{Q^*} D_h q_{12} \left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 dt + 12\mathbb{E} \int_{\partial Q} q_{12} t_r \left(\left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 \right) \nu dt \\
&\geq 60\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt - \mathbb{E} \int_{Q^*} [s \mathcal{O}_\mu((sh)^2) + s \mathcal{O}_\mu((sh)^4)] |D_h^3 v|^2 dt \\
&\quad - \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2) + s^5 \mu^5 \varphi^5 \mathcal{O}(1)] |D_h v|^2 dt \\
&\quad - 12\mathbb{E} \int_{\partial Q} s^5 \mu^5 \varphi^5 (\partial_x \psi)^5 t_r (|D_h v|^2) \nu dt - \mathbb{E} \int_{\partial Q} [s \mathcal{O}_\mu((sh)^2) + s \mathcal{O}_\mu((sh)^4)] t_r (|D_h^3 v|^2) dt \\
&\quad - \mathbb{E} \int_{\partial Q} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] t_r (|D_h v|^2) dt \\
&\geq 60\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt + 12\mathbb{E} \int_{\partial Q} s^5 \mu^5 \varphi^5 |\partial_x \psi|^5 t_r (|D_h v|^2) dt - \mathcal{A}_1 - \mathcal{B}. \tag{3.28}
\end{aligned}$$

From (2.6), (2.9), (2.4) and (2.10), letting $q_{13} = r^2 A_h^2 D_h^2 \rho A_h^3 D_h \rho$, we obtain

$$\begin{aligned}
I_{13} &= 24\mathbb{E} \int_Q q_{13} A_h^2 D_h^2 v A_h D_h^3 v dt = 12\mathbb{E} \int_Q q_{13} D_h [|A_h D_h^2 v|^2] dt \\
&= -12\mathbb{E} \int_{Q^*} D_h q_{13} |A_h D_h^2 v|^2 dt + 12\mathbb{E} \int_{\partial Q} q_{13} t_r (|A_h D_h^2 v|^2) \nu dt \\
&= -12\mathbb{E} \int_{Q^*} D_h q_{13} A_h (|D_h^2 v|^2) dt + 3h^2 \mathbb{E} \int_{Q^*} D_h q_{13} |D_h^3 v|^2 dt + 12\mathbb{E} \int_{\partial Q} q_{13} t_r (|A_h D_h^2 v|^2) \nu dt \\
&= -12\mathbb{E} \int_{\overline{Q}} A_h D_h q_{13} |D_h^2 v|^2 dt + 6h \mathbb{E} \int_{\partial Q^*} D_h q_{13} t_r (|D_h^2 v|^2) dt + 3h^2 \mathbb{E} \int_{Q^*} D_h q_{13} |D_h^3 v|^2 dt \\
&\quad + 12\mathbb{E} \int_{\partial Q} q_{13} t_r (|A_h D_h^2 v|^2) \nu dt. \tag{3.29}
\end{aligned}$$

From (2.18) and (2.15), for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned}
-12\mathbb{E} \int_{\overline{Q}} A_h D_h q_{13} |D_h^2 v|^2 dt &\geq 36\mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt \\
&\quad - \mathbb{E} \int_{\overline{Q}} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |D_h^2 v|^2 dt, \tag{3.30}
\end{aligned}$$

and

$$\begin{aligned}
& 6h\mathbb{E} \int_{\partial Q^*} D_h q_{13} t_r (|D_h^2 v|^2) dt + 3h^2 \mathbb{E} \int_{Q^*} D_h q_{13} |D_h^3 v|^2 dt \\
& \geq -\mathbb{E} \int_{\partial Q} s^2 \mathcal{O}_\mu(1) |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt.
\end{aligned} \tag{3.31}$$

From (2.18), (2.15), (2.8), and the Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned}
& 12\mathbb{E} \int_{\partial Q} q_{13} t_r (|A_h D_h^2 v|^2) \nu dt \\
& = 12\mathbb{E} \int_{\partial Q} q_{13} |D_h^2 v|^2 \nu dt + 3h^2 \mathbb{E} \int_{\partial Q} q_{13} t_r (|D_h^3 v|^2) \nu dt - 12h \mathbb{E} \int_{\partial Q} q_{13} D_h^2 v t_r (D_h^3 v) dt \\
& \geq -12\mathbb{E} \int_{\partial Q} s^3 \mu^3 \varphi^3 (\partial_x \psi)^3 |D_h^2 v|^2 \nu dt - \mathbb{E} \int_{\partial Q} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu(sh)] |D_h^2 v|^2 dt \\
& \quad - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu(sh) t_r (|D_h^3 v|^2) dt.
\end{aligned} \tag{3.32}$$

Combining (3.29)–(3.32), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$I_{13} \geq 36\mathbb{E} \int_{\bar{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt + 12\mathbb{E} \int_{\partial Q} s^3 \mu^3 \varphi^3 |\partial_x \psi|^3 |D_h^2 v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \tag{3.33}$$

From (2.3), (3.8), (3.9), (2.9), (2.6), (2.1), (2.20) and (2.16), noting that Cauchy-Schwarz inequality, letting $q_{14} = r A_h^2 D_h^2 \rho A_h D_h (r A_h D_h^3 \rho)$, for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$\begin{aligned}
I_{14} & = 12\mathbb{E} \int_Q q_{14} A_h^2 D_h^2 v v dt = 12\mathbb{E} \int_Q q_{14} D_h^2 v v dt + 3h^2 \mathbb{E} \int_Q q_{14} D_h^4 v v dt \\
& = -12\mathbb{E} \int_{Q^*} D_h q_{14} A_h v D_h v dt - 12\mathbb{E} \int_{Q^*} A_h q_{14} |D_h v|^2 dt - 3h^2 \mathbb{E} \int_{Q^*} D_h q_{14} A_h v D_h^3 v dt \\
& \quad - 3h^2 \mathbb{E} \int_{Q^*} A_h q_{14} D_h v D_h^3 v dt \\
& = 6\mathbb{E} \int_{\bar{Q}} D_h^2 q_{14} |v|^2 dt - 12\mathbb{E} \int_{Q^*} A_h q_{14} |D_h v|^2 dt - 3h^2 \mathbb{E} \int_{Q^*} D_h q_{14} A_h v D_h^3 v dt \\
& \quad - 3h^2 \mathbb{E} \int_{Q^*} A_h q_{14} D_h v D_h^3 v dt \\
& \geq 36\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt + \mathbb{E} \int_Q s^5 \mathcal{O}_\mu(1) |v|^2 dt + \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt \\
& \quad - \mathbb{E} \int_{Q^*} [s^5 \mu^5 \varphi^5 \mathcal{O}(1) + s^5 \mathcal{O}_\mu((sh)^2) + s^4 \mathcal{O}_\mu(1)] |D_h v|^2 dt \\
& \geq 36\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt - \mathcal{A}_1.
\end{aligned} \tag{3.34}$$

From (2.2), (2.6), (2.3), (2.13) and (2.9), letting $q_{22} = r^2 D_h^4 \rho A_h D_h^3 \rho$, we have

$$I_{22} = 4\mathbb{E} \int_Q q_{22} A_h^4 v A_h^3 D_h v dt = 4\mathbb{E} \int_Q q_{22} A_h (A_h^3 v A_h^2 D_h v) dt - h^2 \mathbb{E} \int_Q q_{22} A_h^3 D_h v A_h^2 D_h^2 v dt$$

$$\begin{aligned}
&= 2\mathbb{E} \int_Q q_{22} A_h D_h (|A_h^2 v|^2) dt - \frac{h^2}{2} \mathbb{E} \int_Q q_{22} D_h (|A_h^2 D_h v|^2) dt \\
&= 2\mathbb{E} \int_Q q_{22} A_h D_h \left(\left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) dt - \frac{h^2}{2} \mathbb{E} \int_Q q_{22} D_h \left(\left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 \right) dt \\
&= -2\mathbb{E} \int_{\bar{Q}} A_h D_h q_{22} \left| v + \frac{h^2}{4} D_h^2 v \right|^2 dt + h\mathbb{E} \int_{\partial Q^*} D_h q_{22} t_r \left(\left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) dt \\
&\quad + 2\mathbb{E} \int_{\partial Q} q_{22} t_r \left(A_h \left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) \nu dt + \frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{22} \left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 dt \\
&\quad - \frac{h^2}{2} \mathbb{E} \int_{\partial Q} q_{22} t_r \left(\left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 \right) \nu dt.
\end{aligned} \tag{3.35}$$

From (2.18) and (2.15), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned}
&-2\mathbb{E} \int_{\bar{Q}} A_h D_h q_{22} \left| v + \frac{h^2}{4} D_h^2 v \right|^2 dt \\
&\geq 14\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |\partial_x \psi|^8 |v|^2 dt - \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2) + s^7 \mu^7 \varphi^7 \mathcal{O}(1)] |v|^2 dt \\
&\quad - \mathbb{E} \int_{\bar{Q}} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt.
\end{aligned} \tag{3.36}$$

Thanks to (2.18) and (3.9), for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned}
&h\mathbb{E} \int_{\partial Q^*} D_h q_{22} t_r \left(\left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) dt - \frac{h^2}{2} \mathbb{E} \int_{\partial Q} q_{22} t_r \left(\left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 \right) \nu dt \\
&\geq -\mathbb{E} \int_{\partial Q} s^2 \mathcal{O}_\mu((sh)^5) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s^5 \mathcal{O}_\mu((sh)^2) t_r (|D_h v|^2) dt \\
&\quad - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^6) t_r (|D_h^3 v|^2) dt.
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
&2\mathbb{E} \int_{\partial Q} q_{22} t_r \left(A_h \left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) \nu dt \\
&\geq -\mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s^5 \mathcal{O}_\mu((sh)^2) t_r (|D_h v|^2) dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^2) t_r (|D_h^3 v|^2) dt.
\end{aligned} \tag{3.38}$$

From (2.18), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{22} \left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 dt \geq -\mathbb{E} \int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^6) |D_h^3 v|^2 dt, \tag{3.39}$$

Hence, combining (3.35)–(3.39), for all $\mu \geq 1$, one can find constant $h'_1 \leq h_1$, such that for all

$\lambda \geq T + T^2$, $h \leq h'_1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$I_{22} \geq 14\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |\partial_x \psi|^8 |v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \quad (3.40)$$

From (2.3), letting $q_{23} = r^2 D_h^4 \rho A_h^3 D_h \rho$, for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$I_{23} = 4\mathbb{E} \int_Q q_{23} A_h^4 v A_h D_h^3 v dt = 4\mathbb{E} \int_Q q_{23} \left(v + \frac{h^2}{4} D_h^2 v + \frac{h^2}{4} A_h^2 D_h^2 v \right) A_h D_h^3 v dt. \quad (3.41)$$

From (2.9), (3.8), (3.9), (2.1), (2.10), (2.6), (2.18), (2.15), (2.4) and (2.11), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned} 4\mathbb{E} \int_Q q_{23} v A_h D_h^3 v dt &= 4\mathbb{E} \int_Q D_h^2(q_{23} v) A_h D_h v dt \\ &= 4\mathbb{E} \int_Q D_h^2 q_{23} A_h^2 v A_h D_h v dt + 8\mathbb{E} \int_Q A_h D_h q_{23} |A_h D_h v|^2 dt + 4\mathbb{E} \int_Q A_h^2 q_{23} A_h D_h v D_h^2 v dt \\ &= -2\mathbb{E} \int_{Q^*} D_h^3 q_{23} |A_h v|^2 dt + 6\mathbb{E} \int_{Q^*} A_h^2 D_h q_{23} |D_h v|^2 dt - 2h^2 \mathbb{E} \int_Q A_h D_h q_{23} |D_h^2 v|^2 dt \\ &\geq -30\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt - \mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_Q s^5 \mathcal{O}_\mu(1) |v|^2 dt \\ &\quad - \mathbb{E} \int_{Q^*} [s^5 \mu^5 \varphi^5 \mathcal{O}(1) + s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] |D_h v|^2 dt. \end{aligned} \quad (3.42)$$

From (2.10), (2.2), (2.6), (2.9) and (2.18), for $\lambda h(\delta T^2)^{-1} \leq 1$, we get

$$\begin{aligned} h^2 \mathbb{E} \int_Q q_{23} D_h^2 v A_h D_h^3 v dt &= h^2 \mathbb{E} \int_{Q^*} A_h q_{23} A_h D_h^2 v D_h^3 v dt + \frac{h^4}{4} \mathbb{E} \int_{Q^*} D_h q_{23} |D_h^3 v|^2 dt - \frac{h^3}{2} \mathbb{E} \int_{\partial Q} q_{23} D_h^2 v t_r(D_h^3 v) dt \\ &= -\frac{h^2}{2} \mathbb{E} \int_Q A_h D_h q_{23} |D_h^2 v|^2 dt + \frac{h^2}{2} \mathbb{E} \int_{\partial Q^*} A_h q_{23} t_r(|D_h^2 v|^2) \nu dt + \frac{h^4}{4} \mathbb{E} \int_{Q^*} D_h q_{23} |D_h^3 v|^2 dt \\ &\quad - \frac{h^3}{2} \mathbb{E} \int_{\partial Q} q_{23} D_h^2 v t_r(D_h^3 v) dt \\ &\geq -\mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^4) |D_h^3 v|^2 dt - \mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt \\ &\quad - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^4) t_r(|D_h^3 v|^2) dt. \end{aligned} \quad (3.43)$$

From (2.6), (2.9), (2.18), (2.5), (2.10) and (2.8), for $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned} h^2 \mathbb{E} \int_Q q_{23} A_h^2 D_h^2 v A_h D_h^3 v dt &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{23} |A_h D_h^2 v|^2 dt + \frac{h^2}{2} \mathbb{E} \int_{\partial Q} q_{23} t_r(|A_h D_h^2 v|^2) \nu dt \\ &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{23} |A_h D_h^2 v|^2 dt + \frac{h^2}{2} \mathbb{E} \int_{\partial Q} q_{23} \left(|D_h^2 v|^2 \nu - h D_h^2 v t_r(D_h^3 v) + \frac{h^2}{4} t_r(|D_h^3 v|^2) \nu \right) dt \end{aligned}$$

$$\geq -\mathbb{E} \int_{\bar{Q}} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^4) t_r (|D_h^3 v|^2) dt. \quad (3.44)$$

Combining (3.41)–(3.44), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$I_{23} \geq -30\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \quad (3.45)$$

From (2.3), letting $q_{24} = r D_h^4 \rho A_h D_h (r A_h D_h^3 \rho)$, we have

$$I_{24} = 2\mathbb{E} \int_Q q_{24} A_h^4 v v dt = 2\mathbb{E} \int_Q q_{24} \left(v + \frac{h^2}{4} D_h^2 v + \frac{h^2}{4} A_h^2 D_h^2 v \right) v dt. \quad (3.46)$$

From (2.20) and (2.16), for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$2\mathbb{E} \int_Q q_{24} v^2 dt \geq -6\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |\partial_x \psi|^8 v^2 dt - \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2) + s^7 \mu^7 \varphi^7 \mathcal{O}(1)] |v|^2 dt. \quad (3.47)$$

From (3.8), (3.9), (2.9), (2.1), (2.6) and (2.17), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned} \frac{h^2}{2} \mathbb{E} \int_Q q_{24} D_h^2 v v dt &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{24} A_h v D_h v dt - \frac{h^2}{2} \mathbb{E} \int_{Q^*} A_h q_{24} |D_h v|^2 dt \\ &= \frac{h^2}{4} \mathbb{E} \int_{\bar{Q}} D_h^2 q_{24} v^2 dt - \frac{h^2}{2} \mathbb{E} \int_{Q^*} A_h q_{24} |D_h v|^2 dt \\ &\geq -\mathbb{E} \int_Q s^5 \mathcal{O}_\mu(1) v^2 dt - \mathbb{E} \int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt. \end{aligned} \quad (3.48)$$

From (2.9), (3.8), (2.1), (2.3), (2.6), (2.17), (2.10) and (2.5), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned} &\frac{h^2}{2} \mathbb{E} \int_Q q_{24} A_h^2 D_h^2 v v dt \\ &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{24} A_h v A_h^2 D_h v dt - \frac{h^2}{2} \mathbb{E} \int_{Q^*} A_h q_{24} D_h v A_h^2 D_h v dt \\ &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{24} A_h v D_h v dt - \frac{h^4}{8} \mathbb{E} \int_{Q^*} D_h q_{24} A_h v D_h^3 v dt - \frac{h^2}{2} \mathbb{E} \int_{Q^*} A_h q_{24} |D_h v|^2 dt \\ &\quad - \frac{h^4}{8} \mathbb{E} \int_{Q^*} A_h q_{24} D_h v D_h^3 v dt \\ &\geq -\mathbb{E} \int_Q s^5 \mathcal{O}_\mu((sh)^2) |v|^2 dt - \mathbb{E} \int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^4) |D_h^3 v|^2 dt. \end{aligned} \quad (3.49)$$

Combining (3.46)–(3.49), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$I_{24} \geq -6\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |\partial_x \psi|^8 v^2 dt - \mathcal{A}_1. \quad (3.50)$$

From (2.9), (2.1) and (2.3), letting $q_{32} = r^2 A_h^4 \rho A_h D_h^3 \rho$, we have

$$\begin{aligned}
I_{32} &= 4\mathbb{E} \int_Q q_{32} D_h^4 v A_h^3 D_h v dt = 4\mathbb{E} \int_Q q_{32} D_h^4 v A_h D_h v dt + h^2 \mathbb{E} \int_Q q_{32} D_h^4 v A_h D_h^3 v dt \\
&= -4\mathbb{E} \int_{Q^*} D_h q_{32} A_h^2 D_h v D_h^3 v dt - 4\mathbb{E} \int_{Q^*} A_h q_{32} A_h D_h^2 v D_h^3 v dt + 4\mathbb{E} \int_{\partial Q} q_{32} A_h D_h v t_r (D_h^3 v) \nu dt \\
&\quad + h^2 \mathbb{E} \int_Q q_{32} D_h^4 v A_h D_h^3 v dt.
\end{aligned} \tag{3.51}$$

From (2.3), (2.9), (2.1), (3.8), (3.9), (2.6), (2.18) and (2.15), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned}
&-4\mathbb{E} \int_{Q^*} D_h q_{32} A_h^2 D_h v D_h^3 v dt \\
&= -4\mathbb{E} \int_{Q^*} D_h q_{32} D_h v D_h^3 v dt - h^2 \mathbb{E} \int_{Q^*} D_h q_{32} |D_h^3 v|^2 dt \\
&= 4\mathbb{E} \int_{\bar{Q}} D_h^2 q_{32} A_h D_h v D_h^2 v dt + 4\mathbb{E} \int_{\bar{Q}} A_h D_h q_{32} |D_h^2 v|^2 dt - h^2 \mathbb{E} \int_{Q^*} D_h q_{32} |D_h^3 v|^2 dt \\
&= -2\mathbb{E} \int_{\bar{Q}^*} D_h^3 q_{32} |D_h v|^2 dt + 4\mathbb{E} \int_{\bar{Q}} A_h D_h q_{32} |D_h^2 v|^2 dt - h^2 \mathbb{E} \int_{Q^*} D_h q_{32} |D_h^3 v|^2 dt \\
&\geq -12\mathbb{E} \int_{\bar{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} s^3 \mathcal{O}_\mu(1) |D_h v|^2 dt - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt \\
&\quad - \mathbb{E} \int_{\bar{Q}} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |D_h^2 v|^2 dt.
\end{aligned} \tag{3.52}$$

From (2.6), (2.9), (2.18) and (2.15), for $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned}
&-4\mathbb{E} \int_{Q^*} A_h q_{32} A_h D_h^2 v D_h^3 v dt \\
&= 2\mathbb{E} \int_{\bar{Q}} A_h D_h q_{32} |D_h^2 v|^2 dt - 2\mathbb{E} \int_{\partial Q^*} A_h q_{32} t_r (|D_h^2 v|^2) \nu dt \\
&\geq -6\mathbb{E} \int_{\bar{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt + 2\mathbb{E} \int_{\partial Q^*} s^3 \mu^3 \varphi^3 (\partial_x \psi)^3 t_r (|D_h^2 v|^2) \nu dt \\
&\quad - \mathbb{E} \int_{\bar{Q}} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |D_h^2 v|^2 dt \\
&\quad - \mathbb{E} \int_{\partial Q^*} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2) + s^3 \mu^2 \varphi^3 \mathcal{O}(1)] t_r (|D_h^2 v|^2) dt.
\end{aligned} \tag{3.53}$$

From (3.8), (2.18) and (2.15), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned}
&4\mathbb{E} \int_{\partial Q} q_{32} A_h D_h v t_r (D_h^3 v) \nu dt \geq -\mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu(1) |A_h D_h v| t_r (|D_h^3 v|) dt \\
&= -\frac{h}{2} \mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu(1) |D_h^2 v| t_r (|D_h^3 v|) dt \\
&\geq -\mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu(sh) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu(sh) t_r (|D_h^3 v|^2) dt.
\end{aligned} \tag{3.54}$$

From (2.6), (2.9) and (2.18), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned} h^2 \mathbb{E} \int_Q q_{32} D_h^4 v A_h D_h^3 v dt &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{32} |D_h^3 v|^2 dt + \frac{h^2}{2} \mathbb{E} \int_{\partial Q} q_{32} t_r (|D_h^3 v|^2) \nu dt \\ &\geq -\mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^2) t_r (|D_h^3 v|^2) dt. \end{aligned} \quad (3.55)$$

Combining (3.51)–(3.55), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$I_{32} \geq -18 \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt - 2 \mathbb{E} \int_{\partial Q} s^3 \mu^3 \varphi^3 |\partial_x \psi|^3 |D_h^2 v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \quad (3.56)$$

From (2.6), (2.9), (2.18) and (2.14), letting $q_{33} = r^2 A_h^4 \rho A_h^3 D_h \rho$, for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned} I_{33} &= 4 \mathbb{E} \int_Q q_{33} D_h^4 v A_h D_h^3 v dt = -2 \mathbb{E} \int_{Q^*} D_h q_{33} |D_h^3 v|^2 dt + 2 \mathbb{E} \int_{\partial Q} q_{33} t_r (|D_h^3 v|^2) \nu dt \\ &\geq 2 \mathbb{E} \int_{Q^*} s \mu^2 \varphi |\partial_x \psi|^2 |D_h^3 v|^2 dt + 2 \mathbb{E} \int_{\partial Q} s \mu \varphi |\partial_x \psi| t_r (|D_h^3 v|^2) dt \\ &\quad - \mathbb{E} \int_{Q^*} [\mathcal{O}_\mu(1) + s \mathcal{O}_\mu((sh)^2) + s \mu \varphi \mathcal{O}(1)] |D_h^3 v|^2 dt \\ &\quad - \mathbb{E} \int_{\partial Q} [\mathcal{O}_\mu(1) + s \mathcal{O}_\mu(sh) + s \varphi \mathcal{O}(1)] t_r (|D_h^3 v|^2) dt. \end{aligned} \quad (3.57)$$

Letting $q_{34} = r A_h^4 \rho A_h D_h (r A_h D_h^3 \rho)$, from (2.11), (3.8), (3.9), (2.1), (2.3), (2.6), (2.9), (2.20) and (2.16), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$\begin{aligned} I_{34} &= 2 \mathbb{E} \int_Q q_{34} D_h^4 v dt = 2 \mathbb{E} \int_{\overline{Q}} D_h^2 (q_{34} v) D_h^2 v dt \\ &= 2 \mathbb{E} \int_{\overline{Q}} D_h^2 q_{34} A_h^2 v D_h^2 v dt + 4 \mathbb{E} \int_{\overline{Q}} A_h D_h q_{34} A_h D_h v D_h^2 v dt + 2 \mathbb{E} \int_{\overline{Q}} A_h^2 q_{34} |D_h^2 v|^2 dt \\ &= 2 \mathbb{E} \int_{\overline{Q}} D_h^2 q_{34} v D_h^2 v dt + \frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} D_h^2 q_{34} |D_h^2 v|^2 dt - 2 \mathbb{E} \int_{Q^*} A_h D_h^2 q_{34} |D_h v|^2 dt \\ &\quad + 2 \mathbb{E} \int_{\overline{Q}} A_h^2 q_{34} |D_h^2 v|^2 dt \\ &\geq -6 \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} s^3 \mathcal{O}_\mu(1) |D_h v|^2 dt - \mathbb{E} \int_Q s^4 \mathcal{O}_\mu(1) |v|^2 dt \\ &\quad - \mathbb{E} \int_{\overline{Q}} [s^2 \mathcal{O}_\mu(1) + (s + s^3) \mathcal{O}_\mu((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |D_h^2 v|^2 dt \\ &\geq -6 \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt - \mathcal{A}_1. \end{aligned} \quad (3.58)$$

Letting $q_{42} = A_h D_h (r A_h^2 D_h^2 \rho) r A_h D_h^3 \rho$ from (2.3), (2.4), (2.10), (2.5), (2.17) and (2.14), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$I_{42} = 24 \mathbb{E} \int_Q q_{42} A_h D_h v A_h^3 D_h v dt = 24 \mathbb{E} \int_Q q_{42} |A_h D_h v|^2 dt + 6 h^2 \mathbb{E} \int_Q q_{42} A_h D_h v A_h D_h^3 v dt$$

$$\begin{aligned}
&= 24\mathbb{E} \int_Q q_{42} A_h (|D_h v|^2) dt - 6h^2 \mathbb{E} \int_Q q_{42} |D_h^2 v|^2 dt + 6h^2 \mathbb{E} \int_Q q_{42} A_h D_h v A_h D_h^3 v dt \\
&= 24\mathbb{E} \int_{Q^*} A_h q_{42} |D_h v|^2 dt - 12h \mathbb{E} \int_{\partial Q} q_{42} t_r (|D_h v|^2) dt - 6h^2 \mathbb{E} \int_Q q_{42} |D_h^2 v|^2 dt \\
&\quad + 6h^2 \mathbb{E} \int_Q q_{42} A_h D_h v A_h D_h^3 v dt \\
&\geq -48\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt - \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2) + s^5 \mu^5 \varphi^5 \mathcal{O}(1)] |D_h v|^2 dt \\
&\quad - \mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt - \mathbb{E} \int_{\partial Q} s^4 \mathcal{O}_\mu((sh)^2) t_r (|D_h v|^2) dt \\
&\geq -48\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \tag{3.59}
\end{aligned}$$

Letting $q_{43} = A_h D_h (r A_h^2 D_h^2 \rho) r A_h^3 D_h \rho$, from (2.9) and (2.1), we have

$$\begin{aligned}
I_{43} &= 24\mathbb{E} \int_Q q_{43} A_h D_h v A_h D_h^3 v dt \\
&= -24\mathbb{E} \int_{Q^*} D_h q_{43} A_h^2 D_h v A_h D_h^2 v dt - 24\mathbb{E} \int_{Q^*} A_h q_{43} |A_h D_h^2 v|^2 dt \\
&\quad + 24\mathbb{E} \int_{\partial Q} q_{43} A_h D_h v t_r (A_h D_h^2 v) \nu dt. \tag{3.60}
\end{aligned}$$

From (2.6), (2.9), (2.4), (2.10), (3.8), (3.9), (2.17) and (2.8), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned}
&-24\mathbb{E} \int_{Q^*} D_h q_{43} A_h^2 D_h v A_h D_h^2 v dt \\
&= 12\mathbb{E} \int_{\bar{Q}} D_h^2 q_{43} |A_h D_h v|^2 dt - 12\mathbb{E} \int_{\partial Q^*} D_h q_{43} t_r (|A_h D_h v|^2) \nu dt \\
&= 12\mathbb{E} \int_{\bar{Q}^*} A_h D_h^2 q_{43} |D_h v|^2 dt - 3h^2 \mathbb{E} \int_{\bar{Q}} D_h^2 q_{43} |D_h^2 v|^2 dt - 3h^2 \mathbb{E} \int_{\partial Q^*} D_h q_{43} t_r (|D_h^2 v|^2) \nu dt \\
&\geq -\mathbb{E} \int_{Q^*} s^3 \mathcal{O}_\mu(1) |D_h v|^2 dt - \mathbb{E} \int_{\bar{Q}} s \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt. \tag{3.61}
\end{aligned}$$

From (2.4), (2.10), (2.17) and (2.14), for $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned}
&-24\mathbb{E} \int_{Q^*} A_h q_{43} |A_h D_h^2 v|^2 dt \\
&= -24\mathbb{E} \int_{\bar{Q}} A_h^2 q_{43} |D_h^2 v|^2 dt + 12h \mathbb{E} \int_{\partial Q^*} A_h q_{43} t_r (|D_h^2 v|^2) dt + 6h^2 \mathbb{E} \int_{Q^*} A_h q_{43} |D_h^3 v|^2 dt \\
&\geq 48\mathbb{E} \int_{\bar{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt - \mathbb{E} \int_{\bar{Q}} [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |D_h^2 v|^2 dt \\
&\quad - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt - \mathbb{E} \int_{\partial Q} s^2 \mathcal{O}_\mu(sh) |D_h^2 v|^2 dt. \tag{3.62}
\end{aligned}$$

From (3.8) and Cauchy-Schwarz inequality, by $A_h D_h v = -\frac{h}{2} \nu D_h^2 v$, $t_r(A_h D_h^2 v) = D_h^2 v - \frac{h}{2} t_r(D_h^3 v) \nu$

on ∂Q , then for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned} 24\mathbb{E} \int_{\partial Q} q_{43} A_h D_h v t_r (A_h D_h^2 v) \nu dt &\geq -\mathbb{E} \int_{\partial Q} s^2 \mathcal{O}_\mu(sh) |D_h^2 v|^2 dt \\ &\quad - \mathbb{E} \int_{\partial Q} \mathcal{O}_\mu((sh)^2) t_r (|D_h^3 v|^2) dt. \end{aligned} \quad (3.63)$$

Combining (3.60)–(3.63), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we obtain

$$I_{43} \geq 48\mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \quad (3.64)$$

From (2.17), (2.14), (2.5) and (2.10), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$\begin{aligned} I_{44} &= 12\mathbb{E} \int_Q A_h D_h (r A_h^2 D_h^2 \rho) A_h D_h (r A_h D_h^3 \rho) A_h D_h v v dt \\ &\geq -\mathbb{E} \int_{Q^*} s^4 \mathcal{O}_\mu(1) |D_h v|^2 dt - \mathbb{E} \int_Q s^6 \mathcal{O}_\mu(1) |v|^2 dt \\ &\geq -\mathcal{A}_1. \end{aligned} \quad (3.65)$$

Step 4. Estimate of the terms I_{5j} with $j = 2, 3, 4$. These terms come from the correction term $\Phi(v)$ in the decomposition of I_1 . They do not appear in the deterministic argument of [7], so we isolate them here.

From (3.5), (2.1), (2.2), (2.3), (2.17) and (2.14), letting

$$\begin{cases} q_1 = A_h D_h (r A_h^2 D_h^2 \rho), & q_2 = A_h D_h (r D_h^4 \rho), & q_3 = D_h^2 (r D_h^4 \rho), \\ q_4 = D_h^2 (r A_h^4 \rho), & q_5 = A_h D_h (r A_h^4 \rho), \end{cases} \quad (3.66)$$

for $\lambda h(\delta T^2)^{-1} \leq 1$, $h \leq 1$, we obtain

$$\begin{aligned} \Phi(v) &= 3h^2 D_h^2 (q_1 A_h D_h v) + \frac{h^2}{2} A_h^2 (q_2 A_h D_h v) + \frac{h^2}{4} q_3 A_h^2 v - q_4 D_h^2 v + 2A_h D_h (q_5 D_h^2 v) \\ &= 3h^2 (D_h^2 q_1 A_h^3 D_h v + 2A_h D_h q_1 A_h^2 D_h^2 v + A_h^2 q_1 A_h D_h^3 v) + \frac{h^2}{2} q_2 A_h D_h v \\ &\quad + \frac{h^4}{8} (D_h^2 q_2 A_h^3 D_h v + 2A_h D_h q_2 A_h^2 D_h^2 v + A_h^2 q_2 A_h D_h^3 v) + \frac{h^2}{4} q_3 A_h^2 v - q_4 D_h^2 v \\ &\quad + 2A_h D_h q_5 A_h^2 D_h^2 v + \frac{h^2}{2} D_h^2 q_5 A_h D_h^3 v + 2A_h (A_h q_5 D_h^3 v) \\ &\geq -\mathcal{O}_\mu((sh)^2) (|A_h^3 D_h v| + |A_h^2 D_h^2 v| + |A_h D_h^3 v| + |D_h^2 v|) - s^2 \mathcal{O}_\mu((sh)^2) (|A_h D_h v| + |A_h^2 v|) \\ &\quad + 2A_h (A_h q_5 D_h^3 v). \end{aligned} \quad (3.67)$$

From (3.66), (3.67), (2.17), (2.14), (2.5) and (2.10), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, $h \leq 1$ and $\lambda \geq T^2$, we have

$$I_{52} = 4\mathbb{E} \int_Q r A_h D_h^3 \rho A_h^3 D_h v \Phi(v) dt$$

$$\begin{aligned}
&\geq -\mathbb{E} \int_Q s^5 \mathcal{O}_\mu((sh)^2) |v|^2 dt - \mathbb{E} \int_{Q^*} [(s^3 + s^5) \mathcal{O}_\mu((sh)^2) + s^4 \mathcal{O}_\mu(1) + s^5 \mu^5 \varphi^5 \mathcal{O}(1)] |D_h v|^2 dt \\
&\quad - \mathbb{E} \int_{\overline{Q}} [s^3 \mathcal{O}_\mu((sh)^2) + s^2 \mathcal{O}_\mu(1)] |D_h^2 v|^2 dt \\
&\quad - \mathbb{E} \int_{Q^*} [\mathcal{O}_\mu(1) + s \mathcal{O}_\mu((sh)^2) + s \mu \varphi \mathcal{O}(1)] |D_h^3 v|^2 dt \\
&\geq -\mathcal{A}_1,
\end{aligned} \tag{3.68}$$

and

$$\begin{aligned}
I_{53} &= 4\mathbb{E} \int_Q r A_h^3 D_h \rho A_h D_h^3 v \Phi(v) dt \\
&\geq -\mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) |v|^2 dt - \mathbb{E} \int_{Q^*} [s \mathcal{O}_\mu((sh)^2) + s^3 \mathcal{O}_\mu((sh)^2)] |D_h v|^2 dt \\
&\quad - \mathbb{E} \int_{\overline{Q}} s \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} [\mathcal{O}_\mu(1) + s \mathcal{O}_\mu((sh)^2) + s \mu \varphi \mathcal{O}(1)] |D_h^3 v|^2 dt \\
&\geq -\mathcal{A}_1.
\end{aligned} \tag{3.69}$$

Further,

$$\begin{aligned}
I_{54} &= 2\mathbb{E} \int_Q A_h D_h (r A_h D_h^3 \rho) v \Phi(v) dt \\
&\geq -\mathbb{E} \int_Q (s^5 + s^6) \mathcal{O}_\mu(1) |v|^2 dt - \mathbb{E} \int_{Q^*} (\mathcal{O}_\mu(1) + s^4 \mathcal{O}_\mu(1)) |D_h v|^2 dt \\
&\quad - \mathbb{E} \int_{\overline{Q}} \mathcal{O}_\mu(1) |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} \mathcal{O}_\mu(1) |D_h^3 v|^2 dt \\
&\geq -\mathcal{A}_1.
\end{aligned} \tag{3.70}$$

Combining the estimates obtained in Steps 3 and 4, namely (3.28), (3.33), (3.34), (3.40), (3.45), (3.50), (3.56)–(3.58), (3.59), (3.64), (3.65) and (3.68)–(3.70), for $\lambda h(\delta T^2)^{-1} \leq 1$, $h \leq 1$ and $\lambda \geq T^2$, we have

$$\begin{aligned}
\sum_{i=1}^5 \sum_{j=2}^4 I_{ij} &\geq 8\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |\partial_x \psi|^8 |v|^2 dt + 18\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt \\
&\quad + 60\mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt + 2\mathbb{E} \int_{Q^*} s \mu^2 \varphi |\partial_x \psi|^2 |D_h^3 v|^2 dt \\
&\quad + 12\mathbb{E} \int_{\partial Q} s^5 \mu^5 \varphi^5 |\partial_x \psi|^5 t_r (|D_h v|^2) dt + 10\mathbb{E} \int_{\partial Q} s^3 \mu^3 \varphi^3 |\partial_x \psi|^3 |D_h^2 v|^2 dt \\
&\quad + 2\mathbb{E} \int_{\partial Q} s \mu \varphi |\partial_x \psi| t_r (|D_h^3 v|^2) dt - \mathcal{A}_1 - \mathcal{B}.
\end{aligned} \tag{3.71}$$

Step 5. Return to the variable w .

From (3.5), (2.19), (2.18), (2.5), (2.10) and (3.67), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, we have

$$\mathbb{E} \int_Q |I_3|^2 dt \leq \mathbb{E} \int_Q (s^4 + s^6) \mathcal{O}_\mu(1) |v|^2 dt + \mathbb{E} \int_{Q^*} (s^4 \mathcal{O}_\mu(1) + \mathcal{O}_\mu(1)) |D_h v|^2 dt$$

$$\begin{aligned}
& + \mathbb{E} \int_Q \mathcal{O}_\mu(1) |D_h^2 v|^2 dt + \mathbb{E} \int_{Q^*} \mathcal{O}_\mu(1) |D_h^3 v|^2 dt \\
& \leq \mathcal{A}_1.
\end{aligned} \tag{3.72}$$

Combining (3.27), (3.71), (3.6), (3.7), (3.72) and (1.6), for all $\mu \geq 1$, there exists a positive constant $\varepsilon_1 \leq \varepsilon_*$ and a sufficiently large $\lambda_1 \geq 1$, such that for $\lambda \geq \lambda_1(T + T^2)$, $h \leq h'_1$, and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_1$, we have

$$\begin{aligned}
& C\mathbb{E} \int_Q r^2 |f|^2 dt + C(\mu)\mathbb{E} \int_Q s^4 r^2 |g|^2 dt + C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_2} s^7 \mu^8 \varphi^7 |v|^2 dt \\
& + C \left[\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_2} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s \mu^2 \varphi |D_h^3 v|^2 dt \right] \\
& \geq \mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |v|^2 dt + \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt + \mathbb{E} \int_{Q^*} s \mu^2 \varphi |D_h^3 v|^2 dt \\
& + \mathbb{E} \int_{\overline{Q}} r^2 |D_h^2 g|^2 dt - \mathcal{A}_1 - \mathcal{A}_3.
\end{aligned} \tag{3.73}$$

Noting $w = \rho v$, combining (2.1), (2.18) and (2.15), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned}
& \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt = \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 r^2 |D_h(\rho v)|^2 dt \\
& \leq 2\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 r^2 (|D_h \rho A_h v|^2 + |A_h \rho D_h v|^2) dt \\
& \leq C\mathbb{E} \int_{Q^*} s^7 \mu^8 \varphi^7 |A_h v|^2 dt + 2\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2)] |v|^2 dt \\
& + \mathbb{E} \int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt.
\end{aligned} \tag{3.74}$$

From (2.5), (2.10), (1.7), Proposition 2.5 and (3.8), we have

$$\begin{aligned}
& \mathbb{E} \int_{Q^*} s^7 \mu^8 \varphi^7 |A_h v|^2 dt \leq \mathbb{E} \int_{Q^*} s^7 \mu^8 \varphi^7 A_h(v^2) dt \\
& = \mathbb{E} \int_Q s^7 \mu^8 A_h(\varphi^7) v^2 dt \\
& \leq \mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |v|^2 dt + \mathbb{E} \int_Q s^5 \mathcal{O}_\mu((sh)^2) |v|^2 dt.
\end{aligned} \tag{3.75}$$

Combining (3.74) and (3.75), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, it holds that

$$\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt \leq C\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |v|^2 dt + 2\mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathcal{A}_1. \tag{3.76}$$

Since

$$r^2 |D_h^2 w|^2 = r^2 |D_h^2(\rho v)|^2 \leq Cr^2 (|D_h^2 \rho A_h^2 v|^2 + |A_h D_h \rho A_h D_h v|^2 + |A_h^2 \rho D_h^2 v|^2),$$

similar to (3.76), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$\begin{aligned}
& \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 r^2 |D_h^2 w|^2 dt \\
& \leq C \left[\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |v|^2 dt + \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt \right] \\
& \quad + \mathbb{E} \int_{\overline{Q}} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt + \mathbb{E} \int_{Q^*} [s^5 \mathcal{O}_\mu((sh)^2) + s^4 \mathcal{O}_\mu(1)] |D_h v|^2 dt \\
& \quad + \mathbb{E} \int_Q [s^7 \mathcal{O}_\mu((sh)^2) + s^6 \mathcal{O}_\mu(1)] |v|^2 dt \\
& \leq C \left[\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |v|^2 dt + \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt \right] + \mathcal{A}_1. \tag{3.77}
\end{aligned}$$

Noting that

$$r^2 |D_h^3 w|^2 = r^2 |D_h^3(\rho v)|^2 \leq C r^2 (|D_h^3 \rho A_h^3 v|^2 + |A_h D_h^2 \rho A_h^2 D_h v|^2 + |A_h^2 D_h \rho A_h D_h^2 v|^2 + |A_h^3 \rho D_h^3 v|^2),$$

similar to (3.76), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$\begin{aligned}
& \mathbb{E} \int_{Q^*} s \mu^2 \varphi r^2 |D_h^3 w|^2 dt \\
& \leq C \left[\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |v|^2 dt + \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt + \mathbb{E} \int_{Q^*} s \mu^2 \varphi |D_h^3 v|^2 dt \right] \\
& \quad + \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt + \mathbb{E} \int_{\overline{Q}} [s^3 \mathcal{O}_\mu((sh)^2) + s^2 \mathcal{O}_\mu(1)] |D_h^2 v|^2 dt \\
& \quad + \mathbb{E} \int_{Q^*} [s^5 \mathcal{O}_\mu((sh)^2) + s^4 \mathcal{O}_\mu(1)] |D_h v|^2 dt + \mathbb{E} \int_Q [s^7 \mathcal{O}_\mu((sh)^2) + s^6 \mathcal{O}_\mu(1)] |v|^2 dt \\
& \leq C \left[\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |v|^2 dt + \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt + \mathbb{E} \int_{Q^*} s \mu^2 \varphi |D_h^3 v|^2 dt \right] \\
& \quad + \mathcal{A}_1. \tag{3.78}
\end{aligned}$$

Hence, combining (3.76), (3.73), (3.77) and (3.78), for $\mu \geq 1$, $\lambda \geq \lambda_1(T + T^2)$, $h \leq h_1$, and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_1$, we have

$$\begin{aligned}
& C \mathbb{E} \int_Q r^2 |f|^2 dt + C(\mu) \mathbb{E} \int_Q s^4 r^2 |g|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_2} s^7 \mu^8 \varphi^7 |v|^2 dt \\
& + C \left[\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s^5 \mu^6 \varphi^5 |D_h v|^2 dt + \mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_2} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s \mu^2 \varphi |D_h^3 v|^2 dt \right] \\
& \geq \mathbb{E} \int_Q s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 r^2 |D_h^2 w|^2 dt \\
& + \mathbb{E} \int_{Q^*} s \mu^2 \varphi r^2 |D_h^3 w|^2 dt + \mathbb{E} \int_{\overline{Q}} r^2 |D_h^2 g|^2 dt - \mathcal{A}_1 - \mathcal{A}_3. \tag{3.79}
\end{aligned}$$

From (2.1) and (2.17), for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$r D_h w = r D_h \rho A_h v + r A_h \rho D_h v = r D_h \rho A_h v + D_h v + \mathcal{O}_\mu((sh)^2) D_h v,$$

which implies

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s^5 \mu^6 \varphi^5 |D_h v|^2 dt \\
& \leq C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s^5 \mu^6 \varphi^5 (r D_h \rho)^2 |A_h v|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt \\
& \quad + \mathbb{E} \int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt \\
& \leq C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_1} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt \\
& \quad + \mathbb{E} \int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt + \mathbb{E} \int_Q [s^7 \mathcal{O}_\mu((sh)^2) + s^6 \mathcal{O}_\mu(1)] |v|^2 dt \\
& \leq C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_1} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt + \mathcal{A}_1, \tag{3.80}
\end{aligned}$$

where we use $|A_h v|^2 \leq C(|\tau_+ v|^2 + |\tau_- v|^2)$.

Similarly, for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$D_h^2 v = r D_h^2 w - r D_h^2 \rho A_h^2 v - 2r A_h D_h \rho A_h D_h v - \mathcal{O}_\mu((sh)^2) D_h^2 v,$$

which implies

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_2} s^3 \mu^4 \varphi^3 |D_h^2 v|^2 dt \\
& \leq C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_1} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt \\
& \quad + C \mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_1} s^3 \mu^4 \varphi^3 r^2 |D_h^2 w|^2 dt + \mathcal{A}_1. \tag{3.81}
\end{aligned}$$

For $\lambda h(\delta T^2)^{-1} \leq 1$, we also obtain

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s \mu^2 \varphi |D_h^3 v|^2 dt \\
& \leq C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_1} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt \\
& \quad + C \mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_1} s^3 \mu^4 \varphi^3 r^2 |D_h^2 w|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s \mu^2 \varphi r^2 |D_h^3 w|^2 dt + \mathcal{A}_1. \tag{3.82}
\end{aligned}$$

Noting that

$$|D_h v|^2 \leq C h^{-2} (|\tau_- v|^2 + |\tau_+ v|^2), \tag{3.83}$$

and

$$|D_h^2 v|^2 \leq C h^{-4} (|\tau_-^2 v|^2 + |\tau_+^2 v|^2 + |v|^2),$$

for all $\mu \geq 1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\mathcal{A}_3 \leq C(\mu)h^{-4} \left(\mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=T} \right) \quad (3.84)$$

Combining (3.79)–(3.82) and (3.84), there exists a positive constant $\mu_1 \geq 1$ such that for all $\mu \geq \mu_1$, one can find positive constants $\varepsilon_2 \leq \varepsilon_1$, $h_2 \leq h'_1$ and $\lambda_2 \geq \lambda_1$, such that for $\lambda \geq \lambda_2(T+T^2)$, $h \leq h_2$, and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_2$, we have

$$\begin{aligned} & C\mathbb{E} \int_Q r^2 |f|^2 dt + C(\mu)\mathbb{E} \int_Q s^4 r^2 |g|^2 dt + C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_1} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt \\ & + C\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt + C\mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_1} s^3 \mu^4 \varphi^3 r^2 |D_h^2 w|^2 dt \\ & + C\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s \mu^2 \varphi r^2 |D_h^3 w|^2 dt + C(\mu)h^{-4} \left(\mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=T} \right) \\ & \geq \mathbb{E} \int_Q s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt + \mathbb{E} \int_{\overline{Q}} s^3 \mu^4 \varphi^3 r^2 |D_h^2 w|^2 dt + \mathbb{E} \int_{Q^*} s \mu^2 \varphi r^2 |D_h^3 w|^2 dt \\ & + \mathbb{E} \int_{\overline{Q}} r^2 |D_h^2 g|^2 dt. \end{aligned} \quad (3.85)$$

Step 6. Local estimates on $D_h w$, $D_h^2 w$, and $D_h^3 w$.

Choose a function $\xi \in C_0^\infty(G_0; [0, 1])$ such that $\xi = 1$ in G_1 . Thanks to Itô's formula and (1.9), we have

$$d(s^3 \varphi^3 \xi^4 r^2 w^2) = \varphi^3 \xi^4 \partial_t (s^3 r^2) w^2 dt + 2s^3 \varphi^3 \xi^4 r^2 w dw + s^3 \varphi^3 \xi^4 r^2 (dw)^2.$$

Hence, we get

$$\begin{aligned} & \mathbb{E} \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 w^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 w^2 \Big|_{t=0} \\ & = \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \varphi^3 \xi^4 \partial_t (s^3 r^2) w^2 dt + 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 w (D_h^4 w - f) dt \\ & + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 |g|^2 dt. \end{aligned} \quad (3.86)$$

For all $\mu \geq 1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\mathbb{E} \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 w^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 w^2 \Big|_{t=0} \leq Ch^{-3} \left(\mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=T} \right). \quad (3.87)$$

For $\lambda \geq T + T^2$, we obtain

$$\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \varphi^3 \xi^4 \partial_t (s^3 r^2) w^2 dt \leq \mathbb{E} \int_Q s^6 \mathcal{O}_\mu(1) r^2 w^2 dt. \quad (3.88)$$

Thanks to Cauchy-Schwarz inequality, it holds that

$$\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 w f dt \leq \mathbb{E} \int_Q s^6 \mathcal{O}_\mu(1) r^2 w^2 dt + \mathbb{E} \int_Q r^2 f^2 dt. \quad (3.89)$$

From (2.11) and (2.1), we have

$$\begin{aligned} & 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 w D_h^4 w dt \\ &= 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 D_h^2(\varphi^3 \xi^4 r^2 w) D_h^2 w dt \\ &= 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 [D_h^2(\varphi^3 \xi^4 r^2) A_h^2 w + 2A_h D_h(\varphi^3 \xi^4 r^2) A_h D_h w + A_h^2(\varphi^3 \xi^4 r^2) D_h^2 w] D_h^2 w dt. \end{aligned} \quad (3.90)$$

Thanks to Proposition 2.5, (2.5), (2.10), and the Cauchy-Schwarz inequality, for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned} & 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 D_h^2(\varphi^3 \xi^4 r^2) A_h^2 w D_h^2 w dt \\ &\leq \varepsilon \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 |D_h^2 w|^2 dt + C(\varepsilon) \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^7 \mu^4 \varphi^7 \xi^4 r^2 |w|^2 dt \\ &\quad + \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2)] r^2 |D_h^2 w|^2 dt + \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2)] r^2 |w|^2 dt, \end{aligned} \quad (3.91)$$

and

$$\begin{aligned} & 4\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 A_h D_h(\varphi^3 \xi^4 r^2) A_h D_h w D_h^2 w dt \\ &\leq \varepsilon \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 |D_h^2 w|^2 dt + C(\varepsilon) \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s^5 \mu^2 \varphi^5 \xi^4 r^2 |D_h w|^2 dt \\ &\quad + \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2)] r^2 |D_h^2 w|^2 dt + \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] r^2 |D_h w|^2 dt. \end{aligned} \quad (3.92)$$

Similarly, for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we get

$$\begin{aligned} & 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 A_h^2(\varphi^3 \xi^4 r^2) |D_h^2 w|^2 dt \\ &\geq 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \varphi^3 \xi^4 r^2 |D_h^2 w|^2 dt - \mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) r^2 |D_h^2 w|^2 dt. \end{aligned} \quad (3.93)$$

The bounds (3.88), (3.89), (3.91) and (3.92) also hold with the absolute value of the left-hand side. Solving (3.86) for the fourth-order term, we may discard the nonnegative quadratic-variation term. Combining (3.86)–(3.93), letting ε be sufficiently small, for $\mu \geq 1$, $\lambda \geq T + T^2$, $h \leq 1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_1} s^3 \mu^4 \varphi^3 r^2 |D_h^2 w|^2 dt$$

$$\begin{aligned}
&\leq \mathbb{E} \int_0^T \int_{\overline{\mathcal{M}} \cap G_0} s^3 \mu^4 \varphi^3 \xi^4 r^2 |D_h^2 w|^2 dt \\
&\leq C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s^5 \mu^6 \varphi^5 \xi^4 r^2 |D_h w|^2 dt \\
&\quad + \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2)] r^2 |D_h^2 w|^2 dt + \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2)] r^2 |w|^2 dt \\
&\quad + \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] r^2 |D_h w|^2 dt + C(\mu) \mathbb{E} \int_Q r^2 |f|^2 dt + C(\mu) \mathbb{E} \int_Q s^4 r^2 |g|^2 dt \\
&\quad + C(\mu) h^{-4} \left(\mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=T} \right). \tag{3.94}
\end{aligned}$$

From (2.9), (2.1), and Proposition 2.5, thanks to Cauchy-Schwarz inequality, for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s^5 \mu^6 \varphi^5 \xi^4 r^2 |D_h w|^2 dt \\
&= -\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^5 D_h(\mu^6 \varphi^5 \xi^4 r^2 D_h w) w dt \\
&= -\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^5 D_h(\mu^6 \varphi^5 \xi^4 r^2) A_h D_h w w dt - \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^5 A_h(\mu^6 \varphi^5 \xi^4 r^2) D_h^2 w w dt \\
&\leq \varepsilon \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s^5 \mu^6 \varphi^5 \xi^4 r^2 |D_h w|^2 dt + \varepsilon \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \mu^4 \varphi^3 \xi^4 r^2 |D_h^2 w|^2 dt \\
&\quad + C(\varepsilon) \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] r^2 |D_h w|^2 dt \\
&\quad + \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2)] r^2 |w|^2 dt. \tag{3.95}
\end{aligned}$$

From (3.94) and (3.95), letting ε be sufficiently small, for $\mu \geq 1$, $\lambda \geq T + T^2$, $h \leq 1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s^5 \mu^6 \varphi^5 r^2 |D_h w|^2 dt \\
&\leq \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s^5 \mu^6 \varphi^5 \xi^4 r^2 |D_h w|^2 dt \\
&\leq C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] r^2 |D_h w|^2 dt \\
&\quad + \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2)] r^2 |w|^2 dt + \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2)] r^2 |D_h^2 w|^2 dt \\
&\quad + C(\mu) \mathbb{E} \int_Q r^2 |f|^2 dt + C(\mu) \mathbb{E} \int_Q s^4 r^2 |g|^2 dt + C(\mu) h^{-4} \left(\mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=T} \right). \tag{3.96}
\end{aligned}$$

Thanks to Itô's formula and (1.9), we have

$$d(s\varphi\xi^4r^2|D_hw|^2) = \varphi\xi^4\partial_t(sr^2)|D_hw|^2dt + 2s\varphi\xi^4r^2D_hwd(D_hw) + s\varphi\xi^4r^2|d(D_hw)|^2.$$

Hence, we get

$$\begin{aligned}
& \mathbb{E} \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 |D_h w|^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 |D_h w|^2 \Big|_{t=0} \\
&= \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} \varphi \xi^4 \partial_t (sr^2) |D_h w|^2 dt + 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 D_h w (D_h^5 w - D_h f) dt \\
&+ \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 |D_h g|^2 dt.
\end{aligned} \tag{3.97}$$

From (3.83), for all $\mu \geq 1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned}
& \mathbb{E} \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 |D_h w|^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 |D_h w|^2 \Big|_{t=0} \\
& \leq Ch^{-3} \left(\mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=T} \right).
\end{aligned} \tag{3.98}$$

For $\lambda \geq T + T^2$, we obtain

$$\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} \varphi \xi^4 \partial_t (sr^2) |D_h w|^2 dt \leq \mathbb{E} \int_{Q^*} s^4 \mathcal{O}_\mu(1) r^2 |D_h w|^2 dt. \tag{3.99}$$

Similar to (3.26), for $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 |D_h g|^2 dt \leq \varepsilon \mathbb{E} \int_{\bar{Q}} r^2 |D_h^2 g|^2 dt + C(\varepsilon, \mu) \mathbb{E} \int_{\bar{Q}} s^3 r^2 |g|^2 dt. \tag{3.100}$$

From (2.9), (2.1), Proposition 2.5, (2.5) and (2.10), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned}
& -2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 D_h w D_h f dt \\
&= 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} D_h (s\varphi \xi^4 r^2 D_h w) f dt \\
&\leq \mathbb{E} \int_{\bar{Q}} s^2 \mathcal{O}_\mu(1) r^2 |D_h^2 w|^2 dt + \mathbb{E} \int_{Q^*} s^4 \mathcal{O}_\mu(1) r^2 |D_h w|^2 dt + \mathbb{E} \int_{\bar{Q}} r^2 f^2 dt.
\end{aligned} \tag{3.101}$$

From (2.11) and (2.1), we have

$$\begin{aligned}
& 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^4 r^2 D_h w D_h^5 w dt \\
&= 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} D_h^2 (s\varphi \xi^4 r^2) A_h^2 D_h w D_h^3 w dt + 4\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} A_h D_h (s\varphi \xi^4 r^2) A_h D_h^2 w D_h^3 w dt \\
&+ 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} A_h^2 (s\varphi \xi^4 r^2) |D_h^3 w|^2 dt.
\end{aligned} \tag{3.102}$$

Similar to (3.91), for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned}
& 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} D_h^2(s\varphi\xi^4 r^2) A_h^2 D_h w D_h^3 w dt + 4\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} A_h D_h(s\varphi\xi^4 r^2) A_h D_h^2 w D_h^3 w dt \\
& \leq \varepsilon \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi\xi^4 r^2 |D_h^3 w|^2 dt + C(\varepsilon) \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s^5 \mu^4 \varphi^5 \xi^4 r^2 |D_h w|^2 dt \\
& \quad + C(\varepsilon) \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \mu^2 \varphi^3 \xi^4 r^2 |D_h^2 w|^2 dt + \mathbb{E} \int_{Q^*} [\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2)] r^2 |D_h^3 w|^2 dt \\
& \quad + \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2)] r^2 |D_h^2 w|^2 dt + \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] r^2 |D_h w|^2 dt,
\end{aligned} \tag{3.103}$$

and

$$\begin{aligned}
& 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} A_h^2(s\varphi\xi^4 r^2) |D_h^3 w|^2 dt \\
& \geq 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi\xi^4 r^2 |D_h^3 w|^2 dt - \mathbb{E} \int_{Q^*} s\mathcal{O}_\mu((sh)^2) r^2 |D_h^3 w|^2 dt.
\end{aligned} \tag{3.104}$$

The bounds (3.99), (3.101) and (3.103) also hold with the absolute value of the left-hand side. Again we discard the nonnegative quadratic-variation term when solving (3.97) for the fourth-order term. Combining (3.97)–(3.104), (3.94), (3.95) and (3.96), letting ε be sufficiently small, for $\mu \geq 1$, $\lambda \geq T + T^2$, $h \leq 1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s\mu^2 \varphi r^2 |D_h^3 w|^2 dt \\
& \leq \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\mu^2 \varphi \xi^4 r^2 |D_h^3 w|^2 dt \\
& \leq C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^7 \mu^8 \varphi^7 r^2 |w|^2 dt + \mathbb{E} \int_{Q^*} [s^4 \mathcal{O}_\mu(1) + s^5 \mathcal{O}_\mu((sh)^2)] r^2 |D_h w|^2 dt \\
& \quad + \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2)] r^2 |w|^2 dt + \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2)] r^2 |D_h^2 w|^2 dt \\
& \quad + \mathbb{E} \int_{Q^*} [\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2)] r^2 |D_h^3 w|^2 dt + C(\mu) \mathbb{E} \int_Q r^2 |f|^2 dt + C(\mu) \mathbb{E} \int_Q s^4 r^2 |g|^2 dt \\
& \quad + C(\mu) h^{-4} \left(\mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} r^2 |w|^2 \Big|_{t=T} \right).
\end{aligned} \tag{3.105}$$

Combining (3.85), (3.94), (3.95), (3.96) and (3.105), there exists a positive constant $\mu_0 \geq \mu_1$ such that for all $\mu \geq \mu_0$, one can find positive constants $\varepsilon_0 \leq \varepsilon_2$, $h_0 \leq h_2$ and $\lambda_0 \geq \lambda_2$, such that for $\lambda \geq \lambda_0(T + T^2)$, $0 < h \leq h_0$, and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_0$, we obtain (1.8). $\square$

4 Proof of the observability inequality

This section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. From Theorem 1.3, choosing $\mu = \mu_0$, for $\lambda \geq \lambda_0(T + T^2)$, $0 < h \leq h_0$ and

$\lambda h (\delta T^2)^{-1} \leq \varepsilon_0$, we have

$$\begin{aligned} \mathbb{E} \int_Q s^7 e^{2s\phi} |z|^2 dt &\leq C \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} s^7 e^{2s\phi} |z|^2 dt + \mathbb{E} \int_Q e^{2s\phi} |a_1 z + a_2 Z|^2 dt + \mathbb{E} \int_Q s^4 e^{2s\phi} |Z|^2 dt \right. \\ &\quad \left. + h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |z|^2 \Big|_{t=0} + h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |z|^2 \Big|_{t=T} \right), \end{aligned}$$

From this, there exists $\lambda_1 \geq \lambda_0$ such that for a fixed

$$\lambda = \lambda_1 (T + T^2 + \mathcal{H}^{2/7} T^2) \quad (4.1)$$

we obtain

$$\begin{aligned} \mathbb{E} \int_Q \theta^7 e^{2s\phi} |z|^2 dt &\leq C \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} \theta^7 e^{2s\phi} |z|^2 dt + \mathbb{E} \int_Q \theta^7 e^{2s\phi} |Z|^2 dt + \lambda^{-7} h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |z|^2 \Big|_{t=0} \right. \\ &\quad \left. + \lambda^{-7} h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |z|^2 \Big|_{t=T} \right). \end{aligned} \quad (4.2)$$

Noting that (1.7), we have $\theta(t) \geq \theta(T/2) \geq T^{-2}$ for $t \in [0, T]$ and $\theta(t) \leq \theta(T/4) \leq \frac{16}{3T^2}$ for $t \in [T/4, 3T/4]$. Then

$$\mathbb{E} \int_Q \theta^7 e^{2s\phi} |z|^2 dt \geq \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathcal{M}} \theta^7 e^{2s\phi} |z|^2 dt \geq CT^{-14} e^{-C\lambda T^{-2}} \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathcal{M}} |z|^2 dt. \quad (4.3)$$

From (1.5) and (2.11), thanks to Cauchy-Schwarz inequality, for any $0 \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned} \mathbb{E} \int_{\mathcal{M}} |z(t_2)|^2 - \mathbb{E} \int_{\mathcal{M}} |z(t_1)|^2 &= \mathbb{E} \int_{t_1}^{t_2} \int_{\mathcal{M}} d(z^2) = \mathbb{E} \int_{t_1}^{t_2} \int_{\mathcal{M}} [2z dz + (dz)^2] \\ &= \mathbb{E} \int_{t_1}^{t_2} \int_{\mathcal{M}} [2z(D_h^4 z - a_1 z - a_2 Z) + Z^2] dt \\ &= 2\mathbb{E} \int_{t_1}^{t_2} \int_{\mathcal{M}} |D_h^2 z|^2 dt + \mathbb{E} \int_{t_1}^{t_2} \int_{\mathcal{M}} [-2a_1 z^2 - 2a_2 z Z + Z^2] dt \\ &\geq -C(1 + \mathcal{H}^2) \mathbb{E} \int_{t_1}^{t_2} \int_{\mathcal{M}} |z|^2 dt. \end{aligned}$$

Thus, in terms of Gronwall inequality, for $0 \leq t_1 \leq t_2 \leq T$, we have

$$\mathbb{E} \int_{\mathcal{M}} |z(t_1)|^2 \leq C e^{CT(1+\mathcal{H}^2)} \mathbb{E} \int_{\mathcal{M}} |z(t_2)|^2. \quad (4.4)$$

From it, since $\theta(T) = \theta(0) \geq \frac{1}{2}(\delta T^2)^{-1}$, we obtain

$$\begin{aligned} &h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |z|^2 \Big|_{t=0} + h^{-4} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |z|^2 \Big|_{t=T} \\ &\leq Ch^{-4} e^{-C\lambda(\delta T^2)^{-1}} \left[\mathbb{E} \int_{\mathcal{M}} |z|^2 \Big|_{t=0} + \mathbb{E} \int_{\mathcal{M}} |z|^2 \Big|_{t=T} \right] \\ &\leq Ch^{-4} e^{-C\lambda(\delta T^2)^{-1} + CT(1+\mathcal{H}^2)} \mathbb{E} \int_{\mathcal{M}} |z|^2 \Big|_{t=T}. \end{aligned} \quad (4.5)$$

Since $\sup_{x \geq 0} x^7 e^{-cx} < \infty$ for each $c > 0$, we have

$$\begin{aligned} & \mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} \theta^7 e^{2s\phi} |z|^2 dt + \mathbb{E} \int_Q \theta^7 e^{2s\phi} |Z|^2 dt \\ & \leq C \lambda^{-7} e^{-C\lambda T^{-2}} \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} |z|^2 dt + \mathbb{E} \int_Q |Z|^2 dt \right). \end{aligned} \quad (4.6)$$

Integrating (4.4) over $t_2 \in [T/4, 3T/4]$ gives a factor $2/T$. Together with (4.3) this yields the prefactor $T^{13} \lambda^{-7}$, which is bounded uniformly because $\lambda \geq \lambda_1(T + T^2)$. Combining (4.2)–(4.6), for $0 < h \leq h_0$ and $\lambda h (\delta T^2)^{-1} \leq \varepsilon_0$, it holds that

$$\mathbb{E} \int_{\mathcal{M}} |z(0)|^2 \leq C e^{C\lambda T^{-2} + CT(1+\mathcal{H}^2)} \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} |z|^2 dt + \mathbb{E} \int_Q |Z|^2 dt + h^{-4} e^{-C\lambda(\delta T^2)^{-1}} \mathbb{E} \int_{\mathcal{M}} |z|^2 \Big|_{t=T} \right) \quad (4.7)$$

Since we required $\lambda h (\delta T^2)^{-1} \leq \varepsilon_0$ and (4.1), define

$$h_1 = \frac{\varepsilon_0 T^2}{4\lambda_1(T + T^2 + \mathcal{H}^{2/7} T^2)}.$$

Then, for every $0 < h \leq \min\{h_0, h_1\}$, choose $\delta = \lambda h / (\varepsilon_0 T^2) \in (0, 1/4]$. This gives $\lambda h / (\delta T^2) = \varepsilon_0$ and $\lambda / (\delta T^2) = \varepsilon_0 / h$. After decreasing h_0 if necessary, the factor h^{-4} is absorbed by half of this decaying exponential. From (4.1) and (4.7), we obtain

$$\mathbb{E} \int_{\mathcal{M}} |z(0)|^2 \leq e^{C(1+T^{-1}+\mathcal{H}^{2/7}+T+\mathcal{H}^2 T)} \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} |z|^2 dt + \mathbb{E} \int_Q |Z|^2 dt + e^{-\frac{C}{h}} \mathbb{E} \int_{\mathcal{M}} |z|^2 \Big|_{t=T} \right).$$

□

5 Proof of the controllability result

Proof of Theorem 1.1. For any $y_0 \in L_h^2(\mathcal{M})$, define

$$J_\varepsilon(z_T) = \frac{1}{2} \mathbb{E} \int_Q |Z|^2 dt + \frac{1}{2} \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} |z|^2 dt + \frac{\varepsilon}{2} \mathbb{E} |z_T|_{L_h^2(\mathcal{M})}^2 - \langle y_0, z(0) \rangle_{L_h^2(\mathcal{M})},$$

where $\varepsilon = e^{-\frac{C}{h}}$ and (z, Z) solves (1.5) with final datum z_T . It is clearly that $J_\varepsilon(\cdot)$ is continuous and convex. From Theorem 1.2 and Young's inequality, for $h \leq \min\{h_0, h_1\}$, we obtain

$$\begin{aligned} & J_\varepsilon(z_T) \\ & \geq \frac{1}{2} \mathbb{E} \int_Q |Z|^2 dt + \frac{1}{2} \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} |z|^2 dt + \frac{\varepsilon}{2} \mathbb{E} |z_T|_{L_h^2(\mathcal{M})}^2 - \frac{1}{4C_{obs}} \mathbb{E} |z(0)|_{L_h^2(\mathcal{M})}^2 - C_{obs} |y_0|_{L_h^2(\mathcal{M})}^2 \\ & \geq \frac{\varepsilon}{4} \mathbb{E} |z_T|_{L_h^2(\mathcal{M})}^2 - C_{obs} |y_0|_{L_h^2(\mathcal{M})}^2, \end{aligned}$$

which implies that $J_\varepsilon(\cdot)$ is coercive. Hence, $J_\varepsilon(\cdot)$ admits a unique minimizer z_T^* . Denote (z^*, Z^*) the solution to (1.5) with final datum z_T^* . Then we obtain

$$\mathbb{E} \int_Q Z Z^* dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} z z^* dt + \varepsilon \mathbb{E} \langle z_T, z_T^* \rangle_{L_h^2(\mathcal{M})} - \langle y_0, z(0) \rangle_{L_h^2(\mathcal{M})} = 0, \quad (5.1)$$

for any z_T with the associated solution (z, T) of (1.5). Choose $u = -z^*$ and $v = -Z^*$, by the duality of (1.3) with (1.5), we obtain

$$\mathbb{E} \langle y(T), z_T \rangle_{L_h^2(\mathcal{M})} - \langle y_0, z(0) \rangle_{L_h^2(\mathcal{M})} = \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} u z dt + \mathbb{E} \int_Q v Z dt.$$

Then $y(T) = \varepsilon z_T^*$. Noting that (5.1), we have

$$\mathbb{E} \int_Q |Z^*|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} |z^*|^2 dt + \varepsilon \mathbb{E} |z_T^*|_{L_h^2(\mathcal{M})}^2 \leq C_{obs} |y_0|_{L_h^2(\mathcal{M})}^2 + \frac{1}{4C_{obs}} |z^*(0)|_{L_h^2(\mathcal{M})}^2. \quad (5.2)$$

From Theorem 1.2, for $h \leq \min\{h_0, h_1\}$, we obtain

$$|z^*(0)|_{L_h^2(\mathcal{M})}^2 \leq C_{obs} \left(\mathbb{E} \int_Q |Z^*|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} |z^*|^2 dt + \varepsilon \mathbb{E} |z_T^*|_{L_h^2(\mathcal{M})}^2 \right) \quad (5.3)$$

Combining (5.2) and (5.3), for $h \leq \min\{h_0, h_1\}$, we have

$$\mathbb{E} \int_{\mathcal{M}} |y(T)|^2 = \varepsilon^2 \mathbb{E} |z_T^*|_{L_h^2(\mathcal{M})}^2 \leq C_{obs} e^{-\frac{C}{h}} \int_{\mathcal{M}} |y_0|^2,$$

and

$$\mathbb{E} \int_Q |v|^2 dt + \mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} |u|^2 dt = \mathbb{E} \int_Q |Z^*|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} |z^*|^2 dt \leq C_{obs} \int_{\mathcal{M}} |y_0|^2,$$

which implies that (u, v) satisfy our need. $\square$

6 Concluding remarks and future directions

In this paper, we established a global Carleman estimate for a backward stochastic fourth-order semi-discrete parabolic equation, with the large Carleman parameter explicitly linked to the mesh size. This yields a relaxed observability inequality and, by duality, a ϕ -null controllability result for the corresponding forward system.

Compared with the deterministic semi-discrete fourth-order case, two features are specific to the present stochastic setting. First, after conjugation, the stochastic differential produces additional Itô terms involving dv , $D_h(dv)$, and $D_h^2(dv)$, which must be balanced with the diffusion variable. Second, the fourth-order discrete operator creates correction terms that cannot be handled independently and must be combined carefully inside the cross-product estimates.

Several interesting questions remain open. A first direction is to remove the exponentially small remainder term in the observability estimate and to study the uniform null controllability problem. A second direction is to consider more restrictive control configurations, for instance one-control problems or boundary controls. A third direction concerns inverse problems. The method developed in this paper may also be useful for studying inverse problems. Recent works already

address deterministic semi-discrete inverse problems and semi-discrete stochastic inverse problems for second-order operators in arbitrary dimension; see [32, 33]. It would be very interesting to extend these results to stochastic semi-discrete fourth-order operators.

Statements and Declarations

Funding. This work is partially supported by the NSF of China under grants 12401589, and by the Fundamental Research Funds for the Central Universities 2682024CX013.

Conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

References

- [1] D. ALLONSIUS, F. BOYER, AND M. MORANCEY, *Spectral analysis of discrete elliptic operators and applications in control theory*, Numer. Math., 140 (4) (2018), pp. 857–911.
- [2] F. BOYER, F. HUBERT, AND J. LE ROUSSEAU, *Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations*, J. Math. Pures Appl., 93 (2010), pp. 240–276.
- [3] F. BOYER, F. HUBERT, AND J. LE ROUSSEAU, *Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications*, SIAM J. Control Optim., 48 (8) (2010), pp. 5357–5397.
- [4] F. BOYER, *On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems*, ESAIM Proc., 41 (2013), pp. 15–58.
- [5] F. BOYER AND J. LE ROUSSEAU, *Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), pp. 1035–1078.
- [6] C. CARDON-WEBER, *Cahn-Hilliard stochastic equation: Existence of the solution and of its density*, Bernoulli, 7 (2001), pp. 777–816.
- [7] E. CERPA, R. LECAROS, T. N. T. NGUYEN, AND A. PÉREZ, *Carleman estimates and controllability for a semi-discrete fourth-order parabolic equation*, J. Math. Pures Appl., 164 (2022), pp. 93–130.
- [8] E. CERPA, A. MERCADO, *Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation*, J. Differ. Equ., 250 (4) (2011), pp. 2024–2044.
- [9] H. COOK, *Brownian motion in spinodal decomposition*, Acta Metallurgica, 18 (1970), pp. 297–306.
- [10] A. V. FURSIKOV AND O. Y. IMANUVILOV, *Controllability of Evolution Equations*, Lecture Notes Series, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [11] P. GAO, M. CHEN, AND Y. LI, *Observability estimates and null controllability for forward and backward linear stochastic Kuramoto–Sivashinsky equations*, SIAM J. Control Optim., 53 (1) (2015), pp. 475–500.
- [12] S. GUERRERO AND K. KASSAB, *Carleman estimate and null controllability of a fourth order parabolic equation in dimension $N \geq 2$* , J. Math. Pures Appl., 121 (9) (2019), pp. 135–161.

- [13] P. GUZMÁN, *Local exact controllability to the trajectories of the Cahn-Hilliard equation*, Appl. Math. Optim., 82 (2020), pp. 279–306.
- [14] S. LABBÉ AND E. TRÉLAT, *Uniform controllability of semidiscrete approximations of parabolic control systems*, Systems Control Lett., 55 (7) (2006), pp. 597–609.
- [15] R. LECAROS, R. MORALES, A. PÉREZ, AND S. ZAMORANO, *Discrete Carleman estimates and application to controllability for a fully-discrete parabolic operator with dynamic boundary conditions*, J. Differential Equations, 365 (2023), pp. 832–881.
- [16] R. LECAROS, J. H. ORTEGA, AND A. PÉREZ, *Stability estimate for the semi-discrete linearized Benjamin–Bona–Mahony equation*, ESAIM Control Optim. Calc. Var., 27 (2021), pp. Paper No. 93, 30.
- [17] R. LECAROS, A. A. PÉREZ, AND M. F. PRADO, *Carleman estimate for semi-discrete stochastic parabolic operators in arbitrary dimension and applications to controllability*, Appl. Math. Optim., 93 (1) (2026), pp. Paper No. 12, 53.
- [18] A. LÓPEZ AND E. ZUAZUA, *Some new results related to the null controllability of the 1-d heat equation*, In Séminaire sur les Équations aux Dérivées Partielles, 1997–1998, pages Exp. No. VIII, 22. École Polytech., Palaiseau, 1998.
- [19] Q. LÜ AND X. ZHANG, *Mathematical Control Theory for Stochastic Partial Differential Equations*, in Probability Theory and Stochastic Modelling., Springer Nature Switzerland, 2021.
- [20] Q. LÜ AND Y. WANG, *Null controllability for fourth order stochastic parabolic equations*, SIAM J. Control Optim., 60 (3) (2022), pp. 1563–1590.
- [21] T. N. T. NGUYEN, *Carleman estimates for semi-discrete parabolic operators with a discontinuous diffusion coefficient and applications to controllability*, Math. Control Relat. Fields, 4 (2) (2014), pp. 203–259.
- [22] T. N. T. NGUYEN, *Uniform controllability of semidiscrete approximations for parabolic systems in Banach spaces*, Discrete Contin. Dyn. Syst. Ser. B, 20 (2) (2015), pp. 613–640.
- [23] A. A. PÉREZ, *Asymptotic behavior of Carleman weight functions and application to controllability*, arXiv:2412.19892.
- [24] S. TANG AND X. ZHANG, *Null controllability for forward and backward stochastic parabolic equations*, SIAM J. Control Optim., 48 (2009), pp. 2191–2216.
- [25] L. R. T. TÉBOU, *Uniform null controllability of the 1-D finite differences space, semi-discretization of the heat equation with locally distributed control*, J. Comput. Anal. Appl. 7 (2) (2005), 169–185.
- [26] Y. WANG, *Null controllability for stochastic coupled systems of fourth order parabolic equations*, J. Math. Anal. Appl., 538 (2) (2024), pp. 128426.
- [27] Y. WANG AND Q. ZHAO, *The ϕ -null controllability for semi-discrete stochastic semilinear parabolic equations*, ESAIM Control Optim. Calc. Var., 31 (2025), pp. Paper No. 98.
- [28] Q. ZHAO, *Null controllability for stochastic semidiscrete parabolic equations*, SIAM J. Control Optim., 63 (3) (2025), pp. 2007–2028.

- [29] X. ZHU, W. ZHOU, AND B. WU, *Carleman estimate for full-discrete approximations of the complex Ginzburg–Landau equation with dynamic boundary conditions and applications to controllability*, arXiv:2503.20628.
- [30] Z. ZHOU, *Observability estimate and null controllability for one-dimensional fourth order parabolic equation*, Taiwanese J. Math., 16 (2012), pp. 1991–2017.
- [31] E. ZUAZUA, *Control and numerical approximation of the wave and heat equations*, In International Congress of Mathematicians. Vol. III, pages 1389–1417. Eur. Math. Soc., Zürich, 2006.
- [32] R. LECAROS, J. LÓPEZ-RÍOS, AND A. A. PÉREZ, *Lipschitz stability and reconstruction in inverse problems for semi-discrete parabolic operators*, Inverse Problems, 41 (11) (2025), pp. 115012.
- [33] R. LECAROS, A. A. PÉREZ, AND M. F. PRADO, *Inverse problem for semidiscrete stochastic parabolic operators in arbitrary dimensions*, arXiv:2509.03760.
- [34] Y. LU AND L. LIU, *Null controllability for a cascade system of backward stochastic semi-discrete fourth- and second-order parabolic equations*, J. Math. Anal. Appl., 559 (2) (2026), pp. 130466.