

Null Controllability for Fourth Order Stochastic Parabolic Equations*

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Abstract

We establish the null controllability for fourth order stochastic parabolic equations. Utilizing the duality argument, the null controllability is reduced to the observability for fourth order backward stochastic parabolic equations, and the desired observability estimate is obtained by a new global Carleman estimate. Our Carleman estimate is based on a new fundamental identity for a fourth order stochastic parabolic operator.

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1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Write $\mathbb{F}$ for the progressive σ -field with respect to $\mathbf{F}$. Let H be a Banach space. Denote by $L^2_{\mathcal{F}_t}(\Omega; H)$ the space of all $\mathcal{F}_t$ -measurable random variables ξ such that $\mathbb{E}|\xi|_H^2 < \infty$; by $L^2_{\mathbb{F}}(0, T; H)$ the space consisting of all H -valued $\mathbf{F}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{L^2(0, T; H)}^2) < \infty$; by $L^\infty_{\mathbb{F}}(0, T; H)$ the space consisting of all H -valued $\mathbf{F}$ -adapted bounded processes; and by $L^2_{\mathbb{F}}(\Omega; C[0, T]; H)$ the space consisting of all H -valued $\mathbf{F}$ -adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{C([0, T]; H)}^2) < \infty$. All these spaces are Banach spaces with the cononical norms (e.g., [20, Section 2.6]).

Let $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded domain with a C^∞ boundary ∂G . Set $Q = (0, T) \times G$ and $\Sigma = (0, T) \times \partial G$. Let $G_0 \subset G$ be a nonempty open subset. Set $Q_0 = (0, T) \times G_0$. As usual, χ_{G_0} denotes the characteristic function of G_0 .

Consider the following control system:

$$\begin{cases} dy + \Delta^2 y dt = (a_1 y + \chi_{G_0} u + a_3 v) dt + (a_2 y + v) dW(t) & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \quad (1.1)$$

where $y_0 \in L^2(G)$, $a_1, a_2, a_3 \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G))$ and $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$. In (1.1), y is the state and (u, v) are the controls. By the classical wellposedness result for stochastic

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evolution equations, we know that (1.1) admits a unique weak solution $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \times L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G))$ (e.g., [20, Theorem 3.24]).

Remark 1.1. The term a_3v reflects the effect of the control v in the diffusion term to the drift term. It is the side effect of the control v , i.e., once we put a control v in the diffusion term, a_3v appears passively. This will lead to some technical difficulties in the study of the null controllability of (1.1).

When $u = v = 0$, the equation (1.1) is widely used to describe some fast diffusion phenomenon under stochastic perturbations (e.g., [6, 12, 14]). Further, it is the linearized version of stochastic Cahn-Hilliard equation, which is used to model phase separation phenomena in a melted alloy that is quenched to a temperature at which just two distinctive phases can exist steadily (e.g., [1]). By acting control on the system, we aim to modify the diffusion of the material. It is an important goal to drive the state to the rest. This leads to the following definition.

Definition 1.1. The system (1.1) is said to be *null controllable* at time T if for any $y_0 \in L^2(G)$, there exists a pair $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$ such that the corresponding solution to (1.1) fulfills that $y(T) = 0$, $\mathbb{P}$ -a.s.

The main result of this paper is the following.

Theorem 1.1. *System (1.1) is null controllable at any time $T > 0$. Furthermore, there is a constant $C > 0$ such that for each given $y_0 \in L^2(G)$, one can choose controls (u, v) which steer the state of (1.1) to 0 and satisfy*

$$|(u, v)|_{L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))} \leq C|y_0|_{L^2(G)} e^{C(1+r_1^2)(T^{-1}+1)},$$

where $r_1 = |a_1|_{L^\infty(0, T; L^\infty(G))} + |a_2|_{L^\infty(0, T; L^\infty(G))} + |a_3|_{L^\infty(0, T; L^\infty(G))}$.

Here and in what follows, C stands for a generic positive constant depending only on G and G_0 whose value may vary from line to line.

Remark 1.2. Like in [21], due to the randomness, extra control is required in the diffusion term. Hence, the control of (1.1) is a pair of function (u, v) . It would be quite interesting to study the null controllability for system (1.1) with only one control in the drift. When a_1 is independent of (t, ω) and a_2 is independent of x , one can follow the idea in [18] to establish the null controllability of (1.1). However, we have no idea about handling the general case.

Remark 1.3. From the proof of Theorem 1.2, one can see that Theorem 1.1 still holds when the equation (1.1) contains ∇y and $\nabla^2 y$ in the drift terms. We do not introduce these two terms in the control system (1.1) for the simplicity of notations. On the other hand, we do not know how to handle the term of derivatives of the third order derivative of y now.

To prove the null controllability of (1.1), we introduce the adjoint equation:

$$\begin{cases} dz - \Delta^2 z dt = -(a_1 z + a_2 Z) dt + Z dW(t) & \text{in } Q, \\ z = \Delta z = 0 & \text{on } \Sigma, \\ z(T) = z_T & \text{in } G. \end{cases} \quad (1.2)$$

By the classical wellposedness result for backward stochastic evolution equations, we know that (1.2) admits a unique weak solution $(z, Z) \in (L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \times L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G))) \times L^2_{\mathbb{F}}(0, T; L^2(G))$ (e.g., [20, Theorem 4.11]).

By a standard duality argument, Theorem 1.1 is implied by the following observability estimate (e.g., [20, Theorem 7.17]).

Theorem 1.2. *There exists a constant $C > 0$ such that the solution (z, Z) to system (1.2) satisfies*

$$|z(0)|_{L^2(G)} \leq C e^{C(1+r_1^2)(T^{-1}+1)} (|z|_{L^2_{\mathbb{F}}(0,T;L^2(G_0))} + |a_3 z + Z|_{L^2_{\mathbb{F}}(0,T;L^2(G))}), \quad \forall z_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)).$$

Remark 1.4. In [21], the authors obtain observability estimates for second order stochastic parabolic equation. The constant in their observability inequality is $C e^{CT^{-4}}$ when T is small. In our case, it is $C e^{CT^{-1}}$.

Controllability problems for deterministic parabolic equations are extensively studied. Particularly, we refer readers to [2–5, 8, 10, 11, 22] and the rich references therein for controllability problems of fourth order parabolic equations. However, to the best of our knowledge, [9] is the only published work addressing null controllability of fourth order stochastic parabolic equations, in which the authors establish the null controllability of (1.1) when G is an interval.

We will use Carleman estimates to prove Theorem 1.2, which is deduced taking inspiration from the works [9, 10, 17, 21]. To state our Carleman estimate, we first introduce the weight functions:

$$\alpha(x, t) = \frac{e^{\lambda(|2\eta|_{C(\overline{G})} + \eta(x))} - e^{4\lambda|\eta|_{C(\overline{G})}}}{t^{1/2}(T-t)^{1/2}}, \quad \xi(x, t) = \frac{e^{\lambda(|2\eta|_{C(\overline{G})} + \eta(x))}}{t^{1/2}(T-t)^{1/2}}, \quad (1.3)$$

where η satisfies:

$$\eta \in C^\infty(\overline{G}), \quad \eta > 0 \text{ in } G, \quad \eta|_{\partial G} = 0, \quad |\nabla \eta| \geq C > 0 \text{ in } \overline{G} \setminus \overline{G_1} \quad (1.4)$$

with $G_1 \subset \overline{G_1} \subset G_0$ an open set. For the existence of η , see [7, Lemma 1.1]. We remark that our weight functions are the same as that in [10], except for a minus sign.

Let $\theta = e^\ell$, $\ell = s\alpha$. We have the following Carleman estimate for (1.2).

Theorem 1.3. *There exists a constant $C > 0$ such that for all $z_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$, the weak solution $(z(\cdot), Z(\cdot))$ to (1.2) satisfies that*

$$\begin{aligned} & \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |z|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla z|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta z|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 z|^2) dx dt \\ & \leq C \mathbb{E} \int_{Q_0} s^7 \lambda^8 \xi^7 \theta^2 |z|^2 dx dt + C e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |Z|^2 dx dt + C r_1^2 \mathbb{E} \int_Q \theta^2 (|z|^2 + |Z|^2) dx dt \end{aligned} \quad (1.5)$$

for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$.

Remark 1.5. Theorem 1.3 can be regarded as a generalization of [10, Theorem 1.2] for stochastic equations. Due to the appearance of the diffusion term, we cannot simply mimic the proof of [10, Theorem 1.2] to prove Theorem 1.3.

Remark 1.6. From the proof of Theorem 1.3, it is not difficult to see that Carleman inequality (1.5) is still valid when z satisfies the boundary condition

$$z = \frac{\partial z}{\partial \nu} = 0 \text{ on } \Sigma, \quad (1.6)$$

where $\nu(x) = (\nu^1(x), \dots, \nu^n(x))$ stands for the unit outward normal vector (of G) at $x \in \partial G$. However, the situation for

$$\frac{\partial z}{\partial \nu} = \frac{\partial \Delta z}{\partial \nu} = 0 \text{ on } \Sigma \quad (1.7)$$

is more difficult. As far as we know, it is still an open problem for the validity of Carleman estimates for deterministic fourth order parabolic operator under the boundary conditions (1.7) in the multi-dimensional case.

Remark 1.7. From Theorem 1.3, one can also get the approximate controllability of system (1.1). In fact, thanks to [20, Theorem 7.19], we only need to show the following unique continuation property holds:

for some $z_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$, the corresponding solution (z, Z) for equation (1.2) fulfills that $z = 0$ in Q_0 and $a_3 z + Z = 0$ in Q , $\mathbb{P}$ -a.s., then $z_T = 0$ in G , $\mathbb{P}$ -a.s.

By choosing $\lambda = C$ and $s = C(1 + r_1^2)(T^{1/2} + T)$ in (1.5), we can get

$$\mathbb{E} \int_Q s^6 \xi^6 \theta^2 |z|^2 dx dt \leq C \left[\mathbb{E} \int_{Q_0} s^7 \xi^7 \theta^2 |z|^2 dx dt + \mathbb{E} \int_Q s^4 \xi^4 \theta^2 |a_3 z + Z|^2 dx dt \right].$$

This implies the desired unique continuation property immediately.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. The Carleman estimate (1.5) is proved in Section 3. In Section 4, we prove Theorem 1.2. Finally, we finished our article with appendices in which we prove the weighted identity and lemmas that are important in deriving the Carleman estimate.

2 Some preliminary results

This section is devoted to presenting some preliminary results for the proof of Theorem 1.3.

Before we state our weighted identity, let's start with a simple example to understand its key idea, which is taken from [13, Section 8.1]. For constants $\lambda, K > 0$ such that $2\lambda K \geq 1$, we claim that

$$\int_{\mathbb{R}} |\varphi|^2 e^{\lambda t^2} dt \leq K \int_{\mathbb{R}} \left| \frac{d\varphi}{dt} \right|^2 e^{\lambda t^2} dt, \quad \forall \varphi \in C_0^\infty(\mathbb{R}). \quad (2.1)$$

Remark 2.1. Please note that (2.1) is not a special case of the famous Poincaré inequality since the integral is taken on $\mathbb{R}$.

Letting $\hat{\varphi} = e^{\lambda t^2/2} \varphi$, for $t \in \mathbb{R}$, we have

$$-2\lambda t \hat{\varphi} e^{\lambda t^2/2} \frac{d\varphi}{dt} = (2\lambda^2 t^2 + \lambda) \hat{\varphi}^2 - \frac{d}{dt} (\lambda t \hat{\varphi}^2). \quad (2.2)$$

Noting that $\hat{\varphi} \in C_0^\infty(\mathbb{R})$, integrating (2.2) on $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}} 2\lambda \hat{\varphi}^2 dt \leq \int_{\mathbb{R}} \left| \frac{d\varphi}{dt} \right|^2 e^{\lambda t^2} dt,$$

which completes the proof of (2.1).

The proof of (2.1) is very simple. But it reveals the two features in the deriving of a Carleman estimate. One is the introduction of the weight function $e^{\lambda t^2}$. The efficiency of Carleman estimate depends greatly on a clever choice of the weight function which is different for different types of equations. The other is the pointwise weighted identity (2.2). By multiplying the weighted multiplier $-2\lambda t \hat{\varphi} e^{\lambda t^2/2}$, the principal operator $\frac{d}{dt}$ can be rewritten as the sum of the “divergence” term $-\frac{d}{dt} (\lambda t \hat{\varphi}^2)$ and the “energy” term $(2\lambda^2 t^2 + \lambda) \hat{\varphi}^2$. Generally speaking, the divergence term will contribute the boundary term after applying the divergence theorem. Here it equals zero after integration on $\mathbb{R}$. The energy term leads to the appearance of the left hand side term in (2.1).

We have introduced the weight function by (1.3) and (1.4). Now, we give a pointwise weighted identity for the (formal) stochastic partial differential operator $-dh + \Delta^2 h dt$.

In what follows, for simplicity, we will use the notation $y_{x_i} \equiv y_{x_i}(x) = \partial y(x) / \partial x_i$, where x_i is the i -th coordinate of a generic point $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$.

Recall that $\theta = e^\ell$ and $\ell = s\alpha$. Set $w = \theta h$. Put

$$I = \Delta^2 w + \Psi_2 \Delta w + \sum_{i,j=1}^n \Psi_3^{ij} w_{x_i x_j} + \sum_{i=1}^n \Psi_4^i w_{x_i} + \sum_{i=1}^n \Psi_5^i w_{x_i} + \Psi_6 w, \quad I_1 = I dt, \quad (2.3)$$

where for $i, j = 1, \dots, n$,

$$\begin{aligned} \Psi_2 &= 2s^2 \lambda^2 \xi^2 |\nabla \eta|^2, \quad \Psi_3^{ij} = 4s^2 \lambda^2 \xi^2 \eta_{x_i} \eta_{x_j}, \quad \Psi_4^i = 12s^2 \lambda^3 \xi^2 |\nabla \eta|^2 \eta_{x_i} + 4s^2 \lambda^2 \xi^2 \Delta \eta \eta_{x_i}, \\ \Psi_5^i &= \sum_{j=1}^n 8s^2 \lambda^2 \xi^2 \eta_{x_i x_j} \eta_{x_j}, \quad \Psi_6 = s^4 \lambda^4 \xi^4 |\nabla \eta|^4. \end{aligned} \quad (2.4)$$

Let

$$I_2 = \sum_{i=1}^n \Phi_1^i \Delta w_{x_i} dt + \Phi_2 \Delta w dt + \sum_{i,j=1}^n \Phi_3^{ij} w_{x_i x_j} dt + \sum_{i=1}^n \Phi_4^i w_{x_i} dt + \Phi_5 w dt - dw, \quad (2.5)$$

where for $i, j = 1, \dots, n$,

$$\begin{aligned} \Phi_1^i &= -4s \lambda \xi \eta_{x_i}, \quad \Phi_2 = -2s \lambda^2 \xi |\nabla \eta|^2 - 2s \lambda \xi \Delta \eta, \\ \Phi_3^{ij} &= 4s \lambda \xi \eta_{x_i x_j} - 4s \lambda^2 \xi \eta_{x_i} \eta_{x_j}, \quad \Phi_4^i = -4s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \eta_{x_i}, \\ \Phi_5 &= -6s^3 \lambda^4 \xi^3 |\nabla \eta|^4 - 12s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) - 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta. \end{aligned} \quad (2.6)$$

Put

$$\begin{aligned} I_3 &= -8s \lambda \xi \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} dt - 4 \nabla \Delta \ell \cdot \nabla w dt + 4 (\nabla \ell \cdot \nabla \Delta \ell) w dt + 2 |\nabla^2 \ell|^2 w dt \\ &\quad - \Delta^2 \ell w dt + |\Delta \ell|^2 w dt + 8s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) w dt + s \alpha_t w dt. \end{aligned} \quad (2.7)$$

We have the following fundamental identity.

Theorem 2.1. *Let h be an $H^4(G)$ -valued Itô process. Set $\theta = e^\ell$, $\ell = s\alpha$ and $w = \theta h$, the function α, ξ , and η being as in (1.3) and (1.4). Then, for any $t \in [0, T]$ and a.e. $(x, \omega) \in G \times \Omega$,*

$$\begin{aligned} &2\theta I(-dh + \Delta^2 h dt) + 2 \sum_{i,j=1}^n (w_{x_i x_j} dw - w_{x_i x_j} dw_{x_i} + \Psi_2 w_{x_i} \delta_{ij} dw + \Psi_3^{ij} w_{x_i} dw)_{x_j} - 2 \operatorname{div}(V_1 + V_2) dt \\ &= 2I^2 dt + 2II_3 + 2M dt + 2 \sum_{i,j,k,p=1}^n \Lambda_1^{ijkp} w_{x_i x_j} w_{x_k x_p} dt + 2 \sum_{i,j=1}^n \Lambda_2^{ij} w_{x_i} w_{x_j} dt + 2\Lambda_3 w^2 dt \\ &\quad - 2d \left(\frac{1}{2} |\nabla^2 w|^2 - \frac{1}{2} \Psi_2 |\nabla w|^2 - 2s^2 \lambda^2 \xi^2 |\nabla \eta \nabla w|^2 + \frac{1}{2} \Psi_6 w^2 \right) - 4s^2 \lambda^2 \xi^2 \theta^2 |\nabla \eta d\nabla h + \nabla \eta \nabla \ell dh|^2 \\ &\quad + \theta^2 \sum_{i,j=1}^n [dh_{x_i x_j} + \ell_{x_i} dh_{x_j} + \ell_{x_j} dh_{x_i} + (\ell_{x_i x_j} + \ell_{x_i} \ell_{x_j}) dh]^2 - \theta^2 \Psi_2 |d\nabla h + \nabla \ell dh|^2 + \Psi_6 \theta^2 (dh)^2, \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} V_1 &= [V_1^1, \dots, V_1^j, \dots, V_1^n], \quad V_2 = [V_2^1, \dots, V_2^j, \dots, V_2^n], \\ V_1^j &= \sum_{i,k=1}^n \left[\sum_{p=1}^n \Phi_1^p w_{x_k x_k x_p} w_{x_i x_i x_j} - \frac{1}{2} \sum_{p=1}^n \Phi_1^j w_{x_k x_k x_p} w_{x_i x_i x_p} + \frac{1}{2} \Psi_2 \Phi_1^j w_{x_i x_i} w_{x_k x_k} \right. \\ &\quad \left. + \sum_{p=1}^n \Psi_3^{ik} \Phi_1^p w_{x_i x_k} w_{x_p x_j} - \frac{1}{2} \sum_{p=1}^n \Psi_3^{ij} \Phi_1^p w_{x_i x_k} w_{x_k x_p} + \Phi_4^k w_{x_i x_i x_j} w_{x_k} - \Phi_4^k w_{x_i x_j} w_{x_i x_k} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \Phi_4^j w_{x_i x_k}^2 + \left(\Psi_2 \Phi_4^k \delta_{ij} - \frac{1}{2} \Psi_2 \Phi_4^j \delta_{ik} - \frac{1}{2} \Psi_6 \Phi_1^j \delta_{ik} + \frac{1}{2} \Psi_3^{ij} \Phi_4^k \right) w_{x_i} w_{x_k} \Big], \\
V_2^j = & \sum_{i,k,p,r,q=1}^n \Theta_1^{ijkprq} w_{x_i x_k x_p} w_{x_r x_q} + \sum_{i,k,p,r=1}^n \Theta_2^{ijkpr} w_{x_i x_k} w_{x_p x_r} + \sum_{i,k,p=1}^n \Theta_3^{ijkp} w_{x_i x_k} w_{x_p} \\
& + \sum_{i,k=1}^n \Theta_4^{ijk} w_{x_i} w_{x_k} + \sum_{i=1}^n \Theta_5 w_{x_i x_i x_j} w + \sum_{i,k=1}^n \Theta_6^{ijk} w_{x_i x_k} w + \sum_{i=1}^n \Theta_7^{ij} w_{x_i} w + \Theta_8^j w^2,
\end{aligned}$$

$$\begin{aligned}
M = & 8s\lambda^2 \xi |\nabla \Delta w \cdot \nabla \eta|^2 + 32s^3 \lambda^4 \xi^3 |\nabla^2 w \nabla \eta \nabla \eta|^2 + 48s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 w \nabla \eta) (\nabla^2 w \nabla \eta) \\
& + 16s^3 \lambda^3 \xi^3 (\nabla^2 w \nabla \eta \nabla \eta) \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} - 16s^3 \lambda^4 \xi^3 |\nabla \eta|^2 |\nabla^2 w \nabla \eta|^2 \\
& + 6s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\nabla^2 w|^2 + 4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla^2 w|^2 + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta |\nabla^2 w|^2 \\
& + 2s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta w|^2 - 4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\Delta w|^2 - 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta |\Delta w|^2 \\
& + 40s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 + 64s^5 \lambda^5 \xi^5 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla w \cdot \nabla \eta|^2 \\
& - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla w|^2 + 8s^7 \lambda^8 \xi^7 |\nabla \eta|^8 w^2 - \sum_{i,k,j=1}^n (\Psi_5^i \Phi_1^k)_{x_j} w_{x_i} w_{x_k x_j},
\end{aligned}$$

$$\begin{aligned}
\Lambda_1^{ijkp} = & \sum_{r=1}^n \left(\frac{1}{2} \Phi_{2x_r x_r} \delta_{ij} \delta_{kp} - \Phi_{3x_r x_r}^{kp} \delta_{ij} + \sum_{q=1}^n \frac{1}{2} \Phi_{3x_r x_q}^{rq} \delta_{ij} \delta_{kp} - 2\Phi_{3x_j x_r}^{kr} \delta_{ip} + \Phi_{3x_r x_r}^{jk} \delta_{ip} \right) + 2\Phi_{3x_i x_j}^{kp}, \\
\Lambda_2^{ij} = & \sum_{k=1}^n \left[-\Phi_{4x_i x_k x_k}^j + \frac{1}{2} \Phi_{4x_i x_j x_k}^k + (\Psi_2 \Phi_3^{jk})_{x_i x_k} - \frac{1}{2} (\Psi_2 \Phi_3^{ij})_{x_k x_k} - \sum_{p=1}^n \frac{1}{2} (\Psi_2 \Phi_3^{kp} \delta_{ij})_{x_k x_p} \right. \\
& + (\Psi_3^{ik} \Phi_2)_{x_k x_j} - \sum_{p=1}^n \frac{1}{2} (\Psi_3^{kp} \Phi_2 \delta_{ij})_{x_k x_p} - \frac{1}{2} (\Psi_3^{ij} \Phi_2)_{x_k x_k} + \sum_{p=1}^n (\Psi_3^{ik} \Phi_3^{jp})_{x_k x_p} \\
& - \sum_{p=1}^n \frac{1}{2} (\Psi_3^{ij} \Phi_3^{kp})_{x_k x_p} - \sum_{p=1}^n \frac{1}{2} (\Psi_3^{kp} \Phi_3^{ij})_{x_k x_p} + \frac{1}{2} (\Psi_4^i \Phi_1^j)_{x_k x_k} + \frac{1}{2} (\Psi_4^k \Phi_2 \delta_{ij})_{x_k} \\
& - (\Psi_4^i \Phi_3^{jk})_{x_k} + \frac{1}{2} (\Psi_4^k \Phi_3^{ij})_{x_k} + \frac{1}{2} (\Psi_5^k \Phi_2 \delta_{ij})_{x_k} - (\Psi_5^i \Phi_3^{jk})_{x_k} + \frac{1}{2} (\Psi_5^k \Phi_3^{ij})_{x_k} \\
& \left. \right] - 2\Phi_{5x_i x_j} - (\Psi_4^i \Phi_2)_{x_j} - (\Psi_5^i \Phi_2)_{x_j} - \frac{1}{2} \Psi_{2t} \delta_{ij} - 4s^2 \lambda^2 \xi (\xi \eta_{x_j})_t \eta_{x_i}, \\
\Lambda_3 = & \frac{1}{2} \Delta^2 \Phi_5 + \frac{1}{2} \Delta (\Psi_2 \Phi_5) + \frac{1}{2} \sum_{i,j=1}^n (\Psi_3^{ij} \Phi_5)_{x_i x_j} - \frac{1}{2} \sum_{i=1}^n [(\Psi_4^i \Phi_5)_{x_i} + (\Psi_5^i \Phi_5)_{x_i}] \\
& - \frac{1}{2} \sum_{i,j=1}^n (\Psi_6 \Phi_1^i)_{x_i x_j x_j} + \frac{1}{2} \Delta (\Psi_6 \Phi_2) + \frac{1}{2} \sum_{i,j=1}^n (\Psi_6 \Phi_3^{ij})_{x_i x_j} + \frac{1}{2} \Psi_{6t}.
\end{aligned}$$

and

$$\begin{aligned}
\Theta_1^{ijkprq} = & \Phi_2 \delta_{ik} \delta_{pj} \delta_{rq} + \Phi_3^{rq} \delta_{ik} \delta_{pj} - \Phi_3^{ik} \delta_{rq} \delta_{pj} + \Phi_3^{ij} \delta_{kp} \delta_{rq}, \\
\Theta_2^{ijkpr} = & -\frac{1}{2} \Phi_{2x_j} \delta_{ik} \delta_{pr} + \Phi_{3x_j}^{pr} \delta_{ik} - \frac{1}{2} \sum_{q=1}^n \Phi_{3x_q}^{jq} \delta_{ik} \delta_{pr} - 2\Phi_{3x_i}^{pr} \delta_{kj} + 2\Phi_{3x_k}^{rj} \delta_{ip} - \Phi_{3x_j}^{kr} \delta_{ip},
\end{aligned}$$

$$\begin{aligned}
\Theta_3^{ijkp} &= -\Phi_{4x_i}^p \delta_{kj} - \Phi_5 \delta_{ik} \delta_{pj} + \Psi_2 \Phi_3^{ik} \delta_{pj} - \Psi_2 \Phi_3^{ij} \delta_{kp} + \Psi_3^{pj} \Phi_2 \delta_{ik} - \Psi_3^{kp} \Phi_2 \delta_{ij} + \Psi_3^{ik} \Phi_3^{pj} - \Psi_3^{ij} \Phi_3^{kp} \\
&\quad + \Psi_4^p \Phi_1^k \delta_{ij} + \Psi_5^p \Phi_1^k \delta_{ij}, \\
\Theta_4^{ijk} &= \Phi_{4x_i x_j}^k - \frac{1}{2} \Phi_{4x_i x_k}^j + 2\Phi_{5x_i} \delta_{kj} - \Phi_{5x_j} \delta_{ik} - (\Psi_2 \Phi_3^{jk})_{x_i} + \frac{1}{2} (\Psi_2 \Phi_3^{ik})_{x_j} + \sum_{p=1}^n \frac{1}{2} (\Psi_2 \Phi_3^{jp})_{x_p} \delta_{ik} \\
&\quad - \sum_{p=1}^n (\Psi_3^{ip} \Phi_2)_{x_p} \delta_{kj} + \frac{1}{2} \sum_{p=1}^n (\Psi_3^{pj} \Phi_2)_{x_p} \delta_{ik} + \frac{1}{2} (\Psi_3^{ik} \Phi_2)_{x_j} - \sum_{p=1}^n (\Psi_3^{ij} \Phi_3^{kp})_{x_p} + \frac{1}{2} \sum_{p=1}^n (\Psi_3^{ik} \Phi_3^{jp})_{x_p} \\
&\quad + \frac{1}{2} \sum_{p=1}^n (\Psi_3^{pj} \Phi_3^{ik})_{x_p} - \frac{1}{2} (\Psi_4 \Phi_1^k)_{x_j} + \Psi_4^i \Phi_2 \delta_{kj} - \frac{1}{2} \Psi_4^j \Phi_2 \delta_{ik} + \Psi_4^i \Phi_3^{kj} - \frac{1}{2} \Psi_4^j \Phi_3^{ik} + \Psi_5^i \Phi_2 \delta_{kj} \\
&\quad - \frac{1}{2} \Psi_5^j \Phi_2 \delta_{ik} + \Psi_5^i \Phi_3^{kj} - \frac{1}{2} \Psi_5^j \Phi_3^{ik}, \\
\Theta_5 &= \Phi_5, \\
\Theta_6^{ijk} &= -\Phi_{5x_j} \delta_{ik} + \Psi_6 \Phi_1^k \delta_{ij}, \\
\Theta_7^{ij} &= \sum_{k=1}^n \Phi_{5x_k x_k} \delta_{ij} + \Psi_2 \Phi_5 \delta_{ij} + \Psi_3^{ij} \Phi_5 - (\Psi_6 \Phi_1^i)_{x_j} + \Psi_6 \Phi_2 \delta_{ij} + \Psi_6 \Phi_3^{ij}, \\
\Theta_8^j &= -\frac{1}{2} \sum_{i=1}^n \Phi_{5x_i x_i x_j} - \frac{1}{2} (\Psi_2 \Phi_5)_{x_j} - \frac{1}{2} \sum_{i=1}^n (\Psi_3^{ij} \Phi_5)_{x_i} + \frac{1}{2} \Psi_4^j \Phi_5 + \frac{1}{2} \Psi_5^j \Phi_5 + \frac{1}{2} \sum_{i=1}^n (\Psi_6 \Phi_1^j)_{x_i x_i} \\
&\quad - \frac{1}{2} (\Psi_6 \Phi_2)_{x_j} - \frac{1}{2} \sum_{i=1}^n (\Psi_6 \Phi_3^{ij})_{x_i} + \frac{1}{2} \Psi_6 \Phi_4^j.
\end{aligned}$$

Remark 2.2. The weighted identity (2.8) can be used to derive Carleman estimates for different boundary conditions, such as (1.6) and (1.7). Although the form of the identity is complex, many terms can merge or vanish under different boundary conditions.

Remark 2.3. In spite of its complicate form, the identity (2.8) enjoys the same features of (2.2). Namely, it represents $\theta I(-dh + \Delta^2 h dt)$ as divergence terms, positive energy terms and lower order term.

Proof of Theorem 2.1 is put in Appendix A.

In what follows, for a nonnegative integer m , we denote by $O(\lambda^m)$ a function of order λ^m for large λ (which is independent of s). Likewise, we use the notation $O(e^{\lambda|\eta|_{C(\bar{G})}})$ and so on. It is easy to check that, for $t \in (0, T)$ and $x \in G$,

$$|\xi_t| \leq \frac{T}{2} \xi^3, \quad |\alpha_t| \leq \frac{T}{2} \xi^3, \quad \xi^{-1} \leq \frac{T}{2}. \quad (2.9)$$

In fact, thanks to (1.3) and (1.4), we obtain

$$\begin{aligned}
|\xi_t| &= \left| \frac{(T-2t)e^{\lambda(|2\eta|_{C(\bar{G})} + \eta(x))}}{2t^{3/2}(T-t)^{3/2}} \right| \leq \left| \frac{T}{2} \frac{e^{\lambda(|2\eta|_{C(\bar{G})} + \eta(x))}}{t^{3/2}(T-t)^{3/2}} \right| \leq \frac{T}{2} \xi^3, \\
|\alpha_t| &\leq \frac{T}{2} \left| \frac{e^{4\lambda|\eta|_{C(\bar{G})}}}{t^{3/2}(T-t)^{3/2}} \right| \leq \frac{T}{2} \xi^3 e^{4\lambda|\eta|_{C(\bar{G})} - 3\lambda(|2\eta|_{C(\bar{G})} + \eta(x))} \leq \frac{T}{2} \xi^3,
\end{aligned}$$

and

$$|\xi^{-1}| \leq t^{1/2}(T-t)^{1/2} \leq \frac{T}{2}$$

for any $t \in (0, T)$ and $x \in G$.

Proposition 2.2. *When s and λ are large enough, it holds that*

$$\begin{cases} \sum_{i,j,k,p=1}^n \Lambda_1^{ijkp} \zeta_{ij} \zeta_{kp} \geq s \xi O(\lambda^4) |\zeta|^2, & \forall \zeta = (\zeta_{ij}) \in \mathbb{R}^{n \times n}, \\ \sum_{i,j=1}^n \Lambda_2^{ij} \zeta_i \zeta_j \geq [s^3 \xi^3 O(\lambda^6) + T s^2 \xi^4 O(\lambda^2)] |\zeta|^2, & \forall \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n, \\ \Lambda_3 \geq s^5 \xi^5 O(\lambda^8) + T s^4 \xi^6 O(\lambda^4). \end{cases} \quad (2.10)$$

for any $t \in (0, T)$ and $x \in G$.

Proof. Noting that $\nabla \xi = \lambda \xi \nabla \eta$, due to (1.4) and (2.6), for $i, j, p, q = 1, \dots, n$, we obtain

$$|\Phi_{2x_i x_j}| \leq s \xi O(\lambda^4), \quad |\Phi_{3x_i x_j}^{pq}| \leq s \xi O(\lambda^4).$$

This gives the first inequality of (2.10).

Thanks to (2.4) and (2.9), we have

$$|\Psi_{2t}| = 4s^2 \lambda^2 \xi |\xi_t| |\nabla \eta|^2 \leq T s^2 \xi^4 O(\lambda^2).$$

The rest of the proof runs as before. $\square$

The following is the Carleman inequality for the Laplace operator with homogeneous Dirichlet boundary conditions.

Lemma 2.3. [10, Lemma 2.4] *Let $q \in H^2(G) \cap H_0^1(G)$ and η be given by (1.4). Then there exists a constant $C > 0$ independent of s and λ , and parameters $\hat{\lambda} > 1$ and $\hat{s} > 1$ such that for all $\lambda \geq \hat{\lambda}$ and for all $\tilde{s} \geq \hat{s}$,*

$$\begin{aligned} & \tilde{s}^6 \lambda^8 \int_G e^{6\lambda\eta(x)} e^{2\tilde{s}e^{\lambda\eta(x)}} |q|^2 dx + \tilde{s}^4 \lambda^6 \int_G e^{4\lambda\eta(x)} e^{2\tilde{s}e^{\lambda\eta(x)}} |\nabla q|^2 dx \\ & \leq C \left(\tilde{s}^6 \lambda^8 \int_{G_1} e^{6\lambda\eta(x)} e^{2\tilde{s}e^{\lambda\eta(x)}} |q|^2 dx + \tilde{s}^3 \lambda^4 \int_G e^{3\lambda\eta(x)} e^{2\tilde{s}e^{\lambda\eta(x)}} |\Delta q|^2 dx \right). \end{aligned} \quad (2.11)$$

Remark 2.4. By taking $\tilde{s} = s \frac{\exp(\lambda|2\eta|_{C(\bar{G})})}{t^{1/2}(T-t)^{1/2}}$ in (2.11), multiplying (2.11) by $\exp\left(2s \frac{-\exp(4\lambda|\eta|_{C(\bar{G})})}{t^{1/2}(T-t)^{1/2}}\right)$, and integrating it in $(0, T)$, we obtain that for $\lambda \geq \hat{\lambda}$ and $s \geq \hat{s}T$,

$$\int_Q s^6 \lambda^8 \xi^6 \theta^2 |q|^2 dx dt + \int_Q s^4 \lambda^6 \xi^4 \theta^2 |\nabla q|^2 dx dt \leq C \left(\int_{(0,T) \times G_1} s^6 \lambda^8 \xi^6 \theta^2 |q|^2 dx dt + \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\Delta q|^2 dx dt \right).$$

Denote

$$\alpha_\star(t) = \frac{e^{\lambda|2\eta|_{C(\bar{G})}} - e^{4\lambda|\eta|_{C(\bar{G})}}}{t^{1/2}(T-t)^{1/2}}, \quad \xi_\star(t) = \frac{e^{\lambda|2\eta|_{C(\bar{G})}}}{t^{1/2}(T-t)^{1/2}}. \quad (2.12)$$

Lemma 2.4. *There exists a constant $C > 0$, such that for all $h \in L_{\mathbb{F}}^2(0, T; H^4(G))$, $f, g \in L_{\mathbb{F}}^2(0, T; L^2(G))$, satisfying*

$$\begin{cases} -dh + \Delta^2 h dt = f dt + g dW(t) & \text{in } Q, \\ h = \Delta h = 0 & \text{on } \Sigma, \end{cases}$$

we have

$$\begin{aligned} & \mathbb{E} \int_{\Sigma} \left[s^{\frac{9}{4}} \lambda^3 \xi^{\frac{9}{4}} \theta^2 |\nabla^2 h|^2 + s^{\frac{3}{4}} \lambda \xi^{\frac{3}{4}} \theta^2 |\nabla^3 h|^2 \right] d\sigma dt + \mathbb{E} \int_0^T e^{2s\alpha_*} |h|_{H^4(G)}^2 dt \\ & \leq C \mathbb{E} \int_Q s^6 \lambda^8 \xi^6 \theta^2 |h|^2 dx dt + C \mathbb{E} \int_Q \theta^2 |f|^2 dx dt, \quad \forall s \geq CT^{1/2}. \end{aligned}$$

Lemma 2.4 is a generalization of Lemma 2.3 in [10] to stochastic equations. Its proof is put in Appendix B.

3 Proof of the Carleman estimate for the backward equation

Proof of Theorem 1.3. Let $h \in L_{\mathbb{F}}^2(0, T; H^4(G))$, $f \in L_{\mathbb{F}}^2(0, T; L^2(G))$, $g \in L_{\mathbb{F}}^2(0, T; H^2(G))$ satisfying

$$\begin{cases} -dh + \Delta^2 h dt = f dt + g dW(t) & \text{in } Q, \\ h = \Delta h = 0 & \text{on } \Sigma. \end{cases} \quad (3.1)$$

In order to simplify the formulae used in the sequel, we define

$$\begin{aligned} \mathcal{A}_1 &= s^5 \mathbb{E} \int_Q \xi^5 [O(\lambda^8) + s \xi O(\lambda^7)] |w|^2 dx dt, & \mathcal{A}_2 &= s^3 \mathbb{E} \int_Q \xi^3 [O(\lambda^6) + s \xi O(\lambda^5)] |\nabla w|^2 dx dt, \\ \mathcal{A}_3 &= s \mathbb{E} \int_Q \xi [O(\lambda^4) + s \xi O(\lambda^3)] |\nabla^2 w|^2 dx dt, & \mathcal{B}_1 &= s^4 \mathbb{E} \int_{\Sigma} \xi^4 [O(\lambda^5) + s \xi O(\lambda^4)] \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt, \\ \mathcal{B}_2 &= s^2 \mathbb{E} \int_{\Sigma} \xi^2 O(\lambda^3) \theta^2 |\nabla^2 h|^2 d\sigma dt, & \mathcal{B}_3 &= s^{1/2} \mathbb{E} \int_{\Sigma} \xi^{1/2} O(\lambda) \theta^2 |\nabla^3 h|^2 d\sigma dt, \end{aligned} \quad (3.2)$$

and

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3, \quad \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3.$$

Integrating the equality (2.8) on Q , taking mathematical expectation in both sides, and noting (2.9), (2.10), we conclude that, for $s \geq CT^{1/2}$,

$$\begin{aligned} & 2 \mathbb{E} \int_Q \theta I(-dh + \Delta^2 h dt) dx + 2 \mathbb{E} \int_Q \sum_{i,j=1}^n [w_{x_i x_j} dw - w_{x_i x_j} dw_{x_i} + \Psi_2 w_{x_i} \delta_{ij} dw + \Psi_3^{ij} w_{x_i} dw]_{x_j} dx \\ & - 2 \mathbb{E} \int_Q \operatorname{div} V_1 dx dt - 2 \mathbb{E} \int_Q \operatorname{div} V_2 dx dt \\ & \geq 2 \mathbb{E} \int_Q I^2 dx dt + 2 \mathbb{E} \int_Q II_3 dx + 2 \mathbb{E} \int_Q M dx dt \\ & + \mathbb{E} \int_Q \theta^2 \sum_{i,j=1}^n [dh_{x_i x_j} + \ell_{x_i} dh_{x_j} + \ell_{x_j} dh_{x_i} + (\ell_{x_i x_j} + \ell_{x_i} \ell_{x_j}) dh]^2 dx \\ & - \mathbb{E} \int_Q \theta^2 \Psi_2 |d\nabla h + \nabla \ell dh|^2 dx - 4 \mathbb{E} \int_Q s^2 \lambda^2 \xi^2 \theta^2 |\nabla \eta d\nabla h + \nabla \eta \nabla \ell dh|^2 dx + \mathbb{E} \int_Q \Psi_6 \theta^2 (dh)^2 dx + \mathcal{A}. \end{aligned} \quad (3.3)$$

We will estimate the divergence terms, the interior terms, the stochastic terms, and the boundary terms in the following Steps 1–4.

Step 1. In this step, we deal with the divergence term.

Using the boundary conditions satisfied by h , on the boundary Σ , we get that

$$w = 0, \quad \nabla w = \frac{\partial w}{\partial \nu} \nu, \quad \Delta w = 2s\lambda\xi \frac{\partial \eta}{\partial \nu} \frac{\partial w}{\partial \nu}, \quad \frac{\partial^2 w}{\partial \nu^2} = 2s\lambda\xi \frac{\partial \eta}{\partial \nu} \frac{\partial w}{\partial \nu} - \operatorname{div}(\nu) \frac{\partial w}{\partial \nu} \quad (3.4)$$

and

$$\begin{aligned} |\nabla^2 w|^2 &= s^2 \xi^2 O(\lambda^2) \left(\frac{\partial w}{\partial \nu} \right)^2 + O(1) \theta^2 |\nabla^2 h|^2, \\ |\nabla^3 w|^2 &= s^4 \xi^4 O(\lambda^4) \left(\frac{\partial w}{\partial \nu} \right)^2 + s^2 \xi^2 O(\lambda^2) \theta^2 |\nabla^2 h|^2 + O(1) \theta^2 |\nabla^3 h|^2, \end{aligned} \quad \text{for } s \geq CT. \quad (3.5)$$

Combining (2.9) and (3.4), for $s \geq CT$, we obtain that

$$\begin{aligned} & -2\mathbb{E} \int_Q \operatorname{div} V_1 dx dt \\ & \leq -2\mathbb{E} \int_{\Sigma} \left[-4s\lambda\xi \frac{\partial \eta}{\partial \nu} \left(\frac{\partial \Delta w}{\partial \nu} \right)^2 + 2s\lambda\xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta w|^2 - 90s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 + 12s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 w \cdot \nu|^2 \right. \\ & \quad \left. - 4s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \frac{\partial w}{\partial \nu} \frac{\partial \Delta w}{\partial \nu} - 2s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 w|^2 \right] d\sigma dt + \mathcal{B}_1. \end{aligned} \quad (3.6)$$

Combining (2.4), (2.6), (2.9), (3.4) and (3.5), for $s \geq CT$, we have

$$-2\mathbb{E} \int_Q \operatorname{div} V_2 dx dt \leq \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3. \quad (3.7)$$

Step 2. In this step, we simplify $2\mathbb{E} \int_Q M dx dt$ via integration by parts.

Thanks to (2.4), (2.6) and (3.2), we obtain

$$2\mathbb{E} \int_Q - \sum_{i,k,j} (\Psi_5^i \Phi_1^k)_{x_j} w_{x_i} w_{x_k x_j} dx dt \geq -(\mathcal{A}_2 + \mathcal{A}_3). \quad (3.8)$$

Combining (2.9), (3.2) and (3.4), using integration by parts, for $s \geq CT$, we have

$$\begin{aligned} & -\mathbb{E} \int_Q 32s^3 \lambda^4 \xi^3 |\nabla \eta|^2 |\nabla^2 w \nabla \eta|^2 dx dt \\ & \geq -8\mathbb{E} \int_Q s \lambda^2 \xi |\nabla \Delta w \nabla \eta|^2 dx dt - 32\mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \nabla \eta|^2 dx dt - (\mathcal{A}_2 + \mathcal{B}_1). \end{aligned} \quad (3.9)$$

Thanks to (1.4), we deduce that for $\lambda \geq C$,

$$\begin{aligned} & 2\mathbb{E} \int_Q 64s^5 \lambda^5 \xi^5 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla w \cdot \nabla \eta|^2 dx dt \\ & \geq -C\mathbb{E} \int_{(0,T) \times G_1} s^5 \lambda^5 \xi^5 |\nabla w|^2 dx dt - \mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 dx dt. \end{aligned} \quad (3.10)$$

From (2.9) and (3.2), we see that for $s \geq CT$,

$$2\mathbb{E} \int_Q (4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta) (|\nabla^2 w|^2 - |\Delta w|^2) \geq -(\mathcal{A}_2 + \mathcal{B}_1). \quad (3.11)$$

Using the argument similar to (3.9), we get that for $\lambda \geq C$ and $s \geq CT$,

$$\begin{aligned} & \mathbb{E} \int_Q 32s^3 \lambda^3 \xi^3 (\nabla^2 w \nabla \eta \nabla \eta) \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} dx dt \\ & \geq \mathbb{E} \int_Q 32s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 w \nabla \eta) (\nabla^2 w \nabla \eta) dx dt - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{B}_1 + \mathcal{B}_2) \end{aligned} \quad (3.12)$$

and, writing $F = s^3 \lambda^3 \xi^3$, integration by parts gives

$$\begin{aligned} & 2\mathbb{E} \int_Q 64F \nabla^2 \eta (\nabla^2 w \nabla \eta) (\nabla^2 w \nabla \eta) dx dt \\ & \geq -C\mathbb{E} \int_Q F \sum_{i,j,k} \eta_{x_i} \eta_{x_j} w_{x_i x_k} w_{x_j x_k} dx dt \\ & \geq C\mathbb{E} \int_Q F (\nabla \Delta w \cdot \nabla \eta) (\nabla w \cdot \nabla \eta) dx dt - (\mathcal{A}_2 + \mathcal{B}_1) \\ & \geq -\mathbb{E} \int_Q s \lambda^2 \xi |\nabla \Delta w \cdot \nabla \eta|^2 dx dt - \mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 dx dt \\ & \quad - C\mathbb{E} \int_{(0,T) \times G_1} s^5 \lambda^5 \xi^5 |\nabla w|^2 dx dt - (\mathcal{A}_2 + \mathcal{B}_1). \end{aligned} \quad (3.13)$$

The coefficient-derivative remainder in the integration by parts contains only $D^2(F \eta_{x_i} \eta_{x_j}) w_{x_i} w_{x_j}$ and boundary terms, which are bounded by $\mathcal{A}_2 + \mathcal{B}_1$. Young's inequality first leaves $Cs^5 \lambda^4 \xi^5 |\nabla w \cdot \nabla \eta|^2$. On $G \setminus G_1$ this is absorbed by the displayed $\lambda^6 |\nabla \eta|^4$ term for large λ ; on G_1 it is bounded by the added local gradient term. This local term is retained together with the one in (3.10).

Step 3. In this step, we estimate the stochastic terms.

Using Divergence Theorem and noting that $w = 0$ on Σ , we have

$$2\mathbb{E} \int_Q \sum_{i,j=1}^n (w_{x_i x_j} dw + \Psi_2 w_{x_i} \delta_{ij} dw + \Psi_3^{ij} w_{x_i} dw)_{x_j} dx = 0. \quad (3.14)$$

Thanks to (3.4), noting that $\lim_{t \rightarrow 0^+} \theta(t, \cdot) = \lim_{t \rightarrow T^-} \theta(t, \cdot) \equiv 0$, we deduce that for $s \geq CT$, Put $K = 2s\lambda\xi\partial_\nu\eta - \operatorname{div}\nu$ and $q = \partial_\nu w$. Then $K < 0$ for the chosen sufficiently large parameters, and the time endpoint terms vanish. The product rule gives

$$\begin{aligned} -2\mathbb{E} \int_Q \sum_{i,j} (w_{x_i x_j} dw_{x_i})_{x_j} dx &= -2\mathbb{E} \int_\Sigma K q dq d\sigma \\ &= -\mathbb{E} \int_\Sigma d(Kq^2) d\sigma + \mathbb{E} \int_\Sigma K_t q^2 d\sigma dt + \mathbb{E} \int_\Sigma K (dq)^2 d\sigma \\ &\leq \mathcal{B}_1. \end{aligned} \quad (3.15)$$

From (3.1), and noting that $\frac{1}{3}a^2 \leq (a+b+c)^2 + b^2 + c^2$ for any $a, b, c \in \mathbb{R}$, it follows that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 \sum_{i,j=1}^n [dh_{x_i x_j} + \ell_{x_i} dh_{x_j} + \ell_{x_j} dh_{x_i} + (\ell_{x_i x_j} + \ell_{x_i} \ell_{x_j}) dh]^2 dx \\ & \geq \frac{1}{3} \mathbb{E} \int_Q \theta^2 |\nabla^2 g|^2 dx dt - C\mathbb{E} \int_Q s^2 \lambda^2 \xi^2 \theta^2 |\nabla g|^2 dx dt - C\mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dx dt. \end{aligned} \quad (3.16)$$

From (3.1) again, we get that

$$\begin{aligned} & -\mathbb{E} \int_Q \theta^2 \Psi_2 |d\nabla h + \nabla \ell dh|^2 dx - 4\mathbb{E} \int_Q s^2 \lambda^2 \xi^2 \theta^2 |\nabla \eta d\nabla h + \nabla \eta \nabla \ell dh|^2 dx \\ & \geq -C\mathbb{E} \int_Q s^2 \lambda^2 \xi^2 \theta^2 |\nabla g|^2 dx dt - C\mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dx dt. \end{aligned} \quad (3.17)$$

For any $t \in (0, T)$ and $\varepsilon > 0$, we use the interpolation inequality (e.g., [15, Theorem 9.1]) and Young's inequality to obtain

$$\int_G |\nabla(\theta g)|^2 dx \leq \varepsilon \int_G |\nabla^2(\theta g)|^2 dx + \frac{C}{\varepsilon} \int_G \theta^2 |g|^2 dx.$$

Take

$$\varepsilon = \varepsilon_2 \left(\frac{s\lambda}{t^{1/2}(T-t)^{1/2}} \right)^{-2},$$

where ε_2 will be fixed later. Using

$$[t^{1/2}(T-t)^{1/2}]^{-1} \leq \xi \leq e^{3\lambda|\eta|_{C(\overline{G})}} [t^{1/2}(T-t)^{1/2}]^{-1},$$

we obtain

$$\begin{aligned} \int_G \theta^2 |\nabla g|^2 dx & \leq \varepsilon \int_G |\nabla^2(\theta g)|^2 dx + \frac{C}{\varepsilon} \int_G \theta^2 |g|^2 dx + 2 \int_G |\nabla \theta|^2 g^2 dx \\ & \leq \varepsilon_2 s^{-2} \lambda^{-2} \int_G t(T-t) |\nabla^2(\theta g)|^2 dx + C\varepsilon_2^{-1} s^2 \lambda^2 e^{C\lambda|\eta|_{C(\overline{G})}} \int_G t^{-1}(T-t)^{-1} \theta^2 g^2 dx, \end{aligned}$$

from which it follows that for $s \geq CT$,

$$\int_G \theta^2 |\nabla g|^2 dx \leq \varepsilon_2 s^{-2} \lambda^{-2} \int_G t(T-t) \theta^2 |\nabla^2 g|^2 dx + C\varepsilon_2^{-1} s^2 \lambda^2 e^{C\lambda|\eta|_{C(\overline{G})}} \int_G t^{-1}(T-t)^{-1} \theta^2 g^2 dx.$$

Hence,

$$\int_G s^2 \lambda^2 \xi^2 \theta^2 |\nabla g|^2 dx \leq \varepsilon_2 e^{C\lambda|\eta|_{C(\overline{G})}} \int_G \theta^2 |\nabla^2 g|^2 dx + C\varepsilon_2^{-1} s^4 \lambda^4 e^{C\lambda|\eta|_{C(\overline{G})}} \int_G \xi^4 \theta^2 g^2 dx. \quad (3.18)$$

From (3.16), (3.17), and (3.18), letting $\varepsilon_2 > 0$ sufficiently small, it follows that for $s \geq CT$,

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 \sum_{i,j=1}^n [dh_{x_i x_j} + \ell_{x_i} dh_{x_j} + \ell_{x_j} dh_{x_i} + (\ell_{x_i x_j} + \ell_{x_i} \ell_{x_j}) dh]^2 dx - \mathbb{E} \int_Q \theta^2 \Psi_2 |d\nabla h + \nabla \ell dh|^2 dx \\ & - 4\mathbb{E} \int_Q s^2 \lambda^2 \xi^2 \theta^2 |\nabla \eta d\nabla h + \nabla \eta \nabla \ell dh|^2 dx + \mathbb{E} \int_Q \Psi_6 \theta^2 (dh)^2 dx \\ & \geq -C e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dx dt. \end{aligned} \quad (3.19)$$

Combining (3.3), (3.6)–(3.15), and (3.19), for $s \geq CT$, we obtain that

$$2\mathbb{E} \int_Q \theta I(-dh + \Delta^2 h dt) dx$$

$$\begin{aligned}
&\geq 2\mathbb{E} \int_Q I^2 dx dt + 2\mathbb{E} \int_Q I I_3 dx - C\mathbb{E} \int_{(0,T) \times G_1} s^5 \lambda^5 \xi^5 |\nabla w|^2 dx dt - C e^{C\lambda|\eta|_{C(\overline{\mathcal{C}})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dx dt \\
&+ 2\mathbb{E} \int_Q [3s\lambda^2 \xi |\nabla \Delta w \cdot \nabla \eta|^2 + 32s^3 \lambda^4 \xi^3 |\nabla^2 w \nabla \eta \nabla \eta|^2 + 8s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta w|^2 \\
&\quad + 22s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla w|^2 + 8s^7 \lambda^8 \xi^7 |\nabla \eta|^8 w^2] dx dt \\
&+ 2\mathbb{E} \int_\Sigma \left[-4s\lambda \xi \frac{\partial \eta}{\partial \nu} \left(\frac{\partial \Delta w}{\partial \nu} \right)^2 + 2s\lambda \xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta w|^2 - 90s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 + 12s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 w \cdot \nu|^2 \right. \\
&\quad \left. - 4s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \frac{\partial w}{\partial \nu} \frac{\partial \Delta w}{\partial \nu} - 2s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 w|^2 \right] d\sigma dt - (\mathcal{A} + \mathcal{B}).
\end{aligned} \tag{3.20}$$

Step 4. In this step, we deal with the boundary terms.

Denote

$$\begin{aligned}
\overline{B} = &\mathbb{E} \int_\Sigma \left[-4s\lambda \xi \frac{\partial \eta}{\partial \nu} \left(\frac{\partial \Delta w}{\partial \nu} \right)^2 + 2s\lambda \xi \frac{\partial \eta}{\partial \nu} |\nabla \Delta w|^2 - 90s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 + 12s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 w \cdot \nu|^2 \right. \\
&\left. - 4s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \frac{\partial w}{\partial \nu} \frac{\partial \Delta w}{\partial \nu} - 2s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla^2 w|^2 \right] d\sigma dt.
\end{aligned} \tag{3.21}$$

On Σ , put $k = s\lambda\xi$, $r = \partial_\nu \eta < 0$, $q = \partial_\nu w$, and $H = \operatorname{div} \nu$. Let Π be the second fundamental form and ∇_Γ the full tangential gradient. Since $\eta = 0$ on Γ , k is constant along each boundary time slice. The boundary conditions imply

$$|\nabla^2 w \nu|^2 = (2kr - H)^2 q^2 + |\nabla_\Gamma q|^2, \quad |\nabla^2 w|^2 = (2kr - H)^2 q^2 + 2|\nabla_\Gamma q|^2 + |\Pi|^2 q^2.$$

Consequently,

$$\mathbb{E} \int_\Sigma 12k^3 r^3 |\nabla^2 w \nu|^2 d\sigma dt \geq \mathbb{E} \int_\Sigma (48k^5 r^5 q^2 + 12k^3 r^3 |\nabla_\Gamma q|^2) d\sigma dt - \mathcal{B}_1, \tag{3.22}$$

$$\mathbb{E} \int_\Sigma -2k^3 r^3 |\nabla^2 w|^2 d\sigma dt \geq \mathbb{E} \int_\Sigma (-8k^5 r^5 q^2 - 4k^3 r^3 |\nabla_\Gamma q|^2) d\sigma dt - \mathcal{B}_1. \tag{3.23}$$

Also $|\nabla \Delta w|^2 = |\partial_\nu \Delta w|^2 + 4k^2 |r \nabla_\Gamma q + q \nabla_\Gamma r|^2$. Use $|a + b|^2 \leq (1 + 1/8)|a|^2 + 9|b|^2$, remembering that $2kr < 0$, to obtain

$$\mathbb{E} \int_\Sigma 2kr |\nabla \Delta w|^2 d\sigma dt \geq \mathbb{E} \int_\Sigma (2kr |\partial_\nu \Delta w|^2 + 9k^3 r^3 |\nabla_\Gamma q|^2) d\sigma dt - \mathcal{B}_1. \tag{3.24}$$

Noting that $\frac{\partial \eta}{\partial \nu} < 0$ on Σ , we have

$$\mathbb{E} \int_\Sigma -4s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \frac{\partial w}{\partial \nu} \frac{\partial \Delta w}{\partial \nu} d\sigma dt \geq \mathbb{E} \int_\Sigma \left[16s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 + s\lambda \xi \frac{\partial \eta}{\partial \nu} \left(\frac{\partial \Delta w}{\partial \nu} \right)^2 \right] d\sigma dt. \tag{3.25}$$

Combining (3.21)–(3.25), for $s \geq CT$, we have

$$\overline{B} \geq \mathbb{E} \int_\Sigma \left[-s\lambda \xi \frac{\partial \eta}{\partial \nu} \left(\frac{\partial \Delta w}{\partial \nu} \right)^2 - 34s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 + 17s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla_\Gamma q|^2 \right] d\sigma dt - \mathcal{B}_1. \tag{3.26}$$

We now estimate the last term. Thanks to the interpolation inequality (e.g., [15, Theorem 9.1]),

(1.4), (2.12), Young's inequality, and noting that $\frac{\partial \eta}{\partial \nu} < 0$ on Σ , we find

$$\begin{aligned}
& \mathbb{E} \int_{\Sigma} 17s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\nabla_{\Gamma} q|^2 d\sigma dt \\
& \geq -C \mathbb{E} \int_0^T s^3 \lambda^3 \xi^3 \left| \frac{\partial w}{\partial \nu} \right|_{H^1(\partial G)}^2 dt \\
& \geq -C \mathbb{E} \int_0^T s^3 \lambda^3 \xi^3 \left| \frac{\partial w}{\partial \nu} \right|_{L^2(\partial G)}^{6/5} \left| \frac{\partial w}{\partial \nu} \right|_{H^{5/2}(\partial G)}^{4/5} dt \\
& \geq \mathbb{E} \int_{\Sigma} s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt - C \mathbb{E} \int_0^T e^{2s\alpha_*} |h|_{H^4(G)}^2 dt.
\end{aligned}$$

This, together with (3.26), implies that for $s \geq CT$,

$$\bar{B} \geq \mathbb{E} \int_{\Sigma} \left[-s \lambda \xi \frac{\partial \eta}{\partial \nu} \left(\frac{\partial \Delta w}{\partial \nu} \right)^2 - 33s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 \right] d\sigma dt - C \mathbb{E} \int_0^T e^{2s\alpha_*} |h|_{H^4(G)}^2 dt - \mathcal{B}_1. \quad (3.27)$$

By (1.4) and (2.9), for $\lambda \geq C$ and $s \geq CT$, we have

$$-\mathcal{B}_1 \geq \mathbb{E} \int_{\Sigma} s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt. \quad (3.28)$$

Combining (3.20), (3.21), (3.27), and (3.28), for $\lambda \geq C$ and $s \geq CT$, we get

$$\begin{aligned}
& 2\mathbb{E} \int_Q \theta I(-dh + \Delta^2 h dt) dx \\
& \geq 2\mathbb{E} \int_Q I^2 dx dt + 2\mathbb{E} \int_Q I I_3 dx - C e^{C\lambda|\eta|_{C(\bar{\mathcal{O}})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dx dt - C \mathbb{E} \int_0^T e^{2s\alpha_*} |h|_{H^4(G)}^2 dt \\
& \quad + 2\mathbb{E} \int_Q [3s \lambda^2 \xi |\nabla \Delta w \cdot \nabla \eta|^2 + 32s^3 \lambda^4 \xi^3 |\nabla^2 w \nabla \eta \nabla \eta|^2 + 8s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta w|^2 \\
& \quad + 22s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla w|^2 + 8s^7 \lambda^8 \xi^7 |\nabla \eta|^8 w^2] dx dt \\
& \quad - \mathbb{E} \int_{\Sigma} s^5 \lambda^5 \xi^5 \left(\frac{\partial \eta}{\partial \nu} \right)^5 \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt - (\mathcal{A} + \mathcal{B}_2 + \mathcal{B}_3) - C \mathbb{E} \int_{(0,T) \times G_1} s^5 \lambda^5 \xi^5 |\nabla w|^2 dx dt.
\end{aligned} \quad (3.29)$$

Step 5. In this step, we sort out the results of the previous steps and get the estimate of h . First, we have

$$\begin{aligned}
& 2\mathbb{E} \int_Q [8s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta w|^2 - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla w|^2 + 8s^7 \lambda^8 \xi^7 |\nabla \eta|^8 w^2] dx dt \\
& = 2\mathbb{E} \int_Q [8s^{-1} \xi^{-1} (s^4 \lambda^4 \xi^4 |\nabla \eta|^4 w + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \Delta w)^2 - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 w \Delta w \\
& \quad - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla w|^2] dx dt \\
& \geq 2\mathbb{E} \int_Q [8s^{-1} \xi^{-1} (s^4 \lambda^4 \xi^4 |\nabla \eta|^4 w + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \Delta w)^2] dx dt - \mathcal{A}_1.
\end{aligned} \quad (3.30)$$

Noting that $(a + b + c)^2 \leq 8a^2 + 2b^2 + 3c^2$ for $a, b, c \in \mathbb{R}$, we find

$$\begin{aligned}
& 2\mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\nabla \eta|^4 |\Delta h|^2 dxdt \\
&= 2\mathbb{E} \int_Q s^{-1} \xi^{-1} (s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \Delta w - 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \nabla \eta \nabla w + s^4 \lambda^4 \xi^4 |\nabla \eta|^4 w - s^3 \lambda^4 \xi^3 |\nabla \eta|^4 w \\
&\quad - s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta w)^2 dxdt \\
&\leq 2\mathbb{E} \int_Q [8s^{-1} \xi^{-1} (s^4 \lambda^4 \xi^4 |\nabla \eta|^4 w + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \Delta w)^2 + 8s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2] dxdt + \mathcal{A}_1.
\end{aligned} \tag{3.31}$$

Using (2.7) and (2.9), we obtain that for $s \geq C(T^{1/2} + T)$,

$$2\mathbb{E} \int_Q II_3 dx \geq -\mathbb{E} \int_Q I^2 dxdt - \mathcal{A}. \tag{3.32}$$

Thanks to (1.4), we get

$$\mathbb{E} \int_{(0,T) \times G_1} s^5 \lambda^5 \xi^5 |\nabla w|^2 dxdt \leq C\mathbb{E} \int_{(0,T) \times G_1} (s^7 \lambda^7 \xi^7 \theta^2 |h|^2 + s^5 \lambda^5 \xi^5 \theta^2 |\nabla h|^2) dxdt. \tag{3.33}$$

For the left hand side of (3.29), we have

$$2\mathbb{E} \int_Q \theta I(-dh + \Delta^2 h dt) dx \leq \mathbb{E} \int_Q I^2 dxdt + \mathbb{E} \int_Q \theta^2 f^2 dxdt. \tag{3.34}$$

From (3.29)–(3.34), for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$, we obtain that

$$\begin{aligned}
& \mathbb{E} \int_Q \theta^2 f^2 dxdt + C e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dxdt + C \left(\mathcal{A} + \mathcal{B}_2 + \mathcal{B}_3 + |e^{s\alpha_\star} h|_{L^2_{\mathbb{F}}(0,T;H^4(G))}^2 \right) \\
&+ C \left[\mathbb{E} \int_{(0,T) \times G_1} (s^7 \lambda^7 \xi^7 \theta^2 |h|^2 + s^5 \lambda^5 \xi^5 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2) dxdt \right] \\
&\geq s^3 \lambda^4 \mathbb{E} \int_Q \xi^3 \theta^2 |\Delta h|^2 dxdt.
\end{aligned} \tag{3.35}$$

Thanks to Remark 2.4, for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$, we have

$$\begin{aligned}
& s^4 \lambda^6 \mathbb{E} \int_Q \xi^4 \theta^2 (s^2 \lambda^2 \xi^2 |h|^2 + |\nabla h|^2) dxdt \\
&\leq C \left(\mathbb{E} \int_{(0,T) \times G_1} s^6 \lambda^8 \xi^6 \theta^2 |h|^2 dxdt + \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2 dxdt \right).
\end{aligned} \tag{3.36}$$

Hence, combining (2.9), (3.2), (3.35) and (3.36), for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$, we obtain

$$\begin{aligned}
& \mathbb{E} \int_Q \theta^2 f^2 dxdt + C e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dxdt + C \left(\mathcal{A}_3 + \mathcal{B}_2 + \mathcal{B}_3 + |e^{s\alpha_\star} h|_{L^2_{\mathbb{F}}(0,T;H^4(G))}^2 \right) \\
&+ C \left[\mathbb{E} \int_{(0,T) \times G_1} (s^7 \lambda^8 \xi^7 \theta^2 |h|^2 + s^5 \lambda^5 \xi^5 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2) dxdt \right]
\end{aligned} \tag{3.37}$$

$$\geq \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |h|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2) dx dt.$$

With the help of Lemma 2.4 and (2.9), we can absorb $\mathcal{B}_2, \mathcal{B}_3$, and $|e^{s\alpha^*} h|_{L^2_{\mathbb{F}}(0,T;H^4(G))}^2$.

Thanks to (3.37), for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$, we have

$$\begin{aligned} & C\mathbb{E} \int_Q \theta^2 f^2 dx dt + C e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dx dt + C\mathcal{A}_3 \\ & + C \left[\mathbb{E} \int_{(0,T) \times G_1} (s^7 \lambda^8 \xi^7 \theta^2 |h|^2 + s^5 \lambda^5 \xi^5 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2) dx dt \right] \\ & \geq \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |h|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2) dx dt. \end{aligned} \quad (3.38)$$

Next, we estimate $\mathcal{A}_3$. Letting $\tilde{h} = s\lambda^2 \xi e^{s\alpha} h$, we have

$$\mathbb{E} \int_Q |\Delta \tilde{h}|^2 dx dt \leq C\mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |h|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla h|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\Delta h|^2) dx dt. \quad (3.39)$$

Since $\tilde{h} = 0$ on Σ , it follows that

$$|\tilde{h}|_{L^2_{\mathbb{F}}(0,T;H^2(G))}^2 \leq C |\Delta \tilde{h}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2. \quad (3.40)$$

Noting that

$$\mathbb{E} \int_Q s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 h|^2 dx dt \leq C |\tilde{h}|_{L^2_{\mathbb{F}}(0,T;H^2(G))}^2 + C\mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |h|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla h|^2) dx dt,$$

by (2.9), (3.2), (3.38)–(3.40), for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$, we obtain

$$\begin{aligned} & C\mathbb{E} \int_Q \theta^2 f^2 dx dt + C e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dx dt \\ & + C \left[\mathbb{E} \int_{(0,T) \times G_1} (s^7 \lambda^8 \xi^7 \theta^2 |h|^2 + s^5 \lambda^5 \xi^5 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2) dx dt \right] \\ & \geq \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |h|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 h|^2) dx dt. \end{aligned} \quad (3.41)$$

We are now ready to estimate the local terms on ∇h and Δh . Let us introduce a cut-off function ψ such that

$$\psi \in C_0^\infty(G_0), \quad \psi = 1 \text{ in } G_1, \quad 0 \leq \psi \leq 1. \quad (3.42)$$

Using Itô's formula and (3.1), we get

$$d(\psi^4 \xi^3 \theta^2 h^2) = (2s\psi^4 \xi^3 \alpha_t + 3\psi^4 \xi^2 \xi_t) \theta^2 h^2 dt + \psi^4 \xi^3 \theta^2 g^2 dt + 2\psi^4 \xi^3 \theta^2 h dh.$$

Hence,

$$\begin{aligned} 0 = & \mathbb{E} \int_{Q_0} (2s\psi^4 \xi^3 \alpha_t + 3\psi^4 \xi^2 \xi_t) \theta^2 h^2 dx dt + \mathbb{E} \int_{Q_0} \psi^4 \xi^3 \theta^2 g^2 dx dt + \mathbb{E} \int_{Q_0} 2\psi^4 \xi^3 \theta^2 h \Delta^2 h dx dt \\ & - \mathbb{E} \int_{Q_0} 2\psi^4 \xi^3 \theta^2 h f dx dt. \end{aligned} \quad (3.43)$$

From (3.42) and integration by parts, it follows that

$$\begin{aligned}
& \mathbb{E} \int_{Q_0} s^3 \lambda^4 \xi^3 \psi^4 \theta^2 |\Delta h|^2 dxdt \\
& \leq \mathbb{E} \int_{Q_0} s^3 \lambda^4 \xi^3 \psi^4 \theta^2 h \Delta^2 h dxdt + C \mathbb{E} \int_{Q_0} s^5 \lambda^6 \xi^5 \psi^4 \theta^2 |\nabla h|^2 dxdt \\
& \quad + C \mathbb{E} \int_{Q_0} s^7 \lambda^8 \xi^7 \theta^2 h^2 dxdt + C \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 (|h|^2 + |\nabla h|^2) dxdt.
\end{aligned} \tag{3.44}$$

For $s \geq CT$, we also have

$$\left| \mathbb{E} \int_{Q_0} 2s^3 \lambda^4 \xi^3 \psi^4 \theta^2 h f dxdt \right| \leq C \mathbb{E} \int_{Q_0} s^6 \lambda^8 \xi^6 \theta^2 h^2 dxdt + C \mathbb{E} \int_Q \theta^2 f^2 dxdt \tag{3.45}$$

and

$$\left| \mathbb{E} \int_{Q_0} s^3 \lambda^4 (2s\psi^4 \xi^3 \alpha_t + 3\psi^4 \xi^2 \xi_t) \theta^2 h^2 dxdt \right| \leq C \mathbb{E} \int_{Q_0} s^7 \lambda^8 \xi^7 \theta^2 h^2 dxdt. \tag{3.46}$$

Using integration by parts, we obtain

$$\mathbb{E} \int_{Q_0} s^5 \lambda^6 \xi^5 \psi^4 \theta^2 |\nabla h|^2 dxdt \leq \varepsilon \mathbb{E} \int_{Q_0} s^3 \lambda^4 \xi^3 \psi^4 \theta^2 |\Delta h|^2 dxdt + C\varepsilon^{-1} \mathbb{E} \int_{Q_0} s^7 \lambda^8 \xi^7 \theta^2 h^2 dxdt. \tag{3.47}$$

From (3.42)–(3.47), letting $\varepsilon > 0$ small enough, for $\lambda \geq C$ and $s \geq CT$, it follows that

$$\begin{aligned}
& \mathbb{E} \int_{(0,T) \times G_1} (s^5 \lambda^5 \xi^5 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2) dxdt \\
& \leq C \left[\mathbb{E} \int_Q \theta^2 f^2 dxdt + \mathbb{E} \int_{Q_0} s^7 \lambda^8 \xi^7 \theta^2 h^2 dxdt + \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 (|h|^2 + |\nabla h|^2) dxdt \right. \\
& \quad \left. + e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dxdt \right].
\end{aligned} \tag{3.48}$$

Combining (3.41) and (3.48), for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$, we deduce

$$\begin{aligned}
& C \left[\mathbb{E} \int_Q \theta^2 f^2 dxdt + e^{C\lambda|\eta|_{C(\overline{G})}} \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 \theta^2 |g|^2 dxdt + \mathbb{E} \int_{Q_0} s^7 \lambda^8 \xi^7 \theta^2 |h|^2 dxdt \right] \\
& \geq \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |h|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla h|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta h|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 h|^2) dxdt.
\end{aligned}$$

By a density argument, we obtain (1.5). □

4 Proof of Theorem 1.2

In this section, we use the Carleman estimate (1.5) to prove Theorem 1.2.

Proof of Theorem 1.2. We first consider the case $T \in (0, 1]$. Thanks to (1.5), and choosing $\lambda = C$ and $s = C(1 + r_1^2)$, we deduce that

$$\mathbb{E} \int_Q \xi^6 \theta^2 |z|^2 dxdt \leq C(1 + r_1^2) \left(\mathbb{E} \int_{Q_0} \xi^7 \theta^2 |z|^2 dxdt + \mathbb{E} \int_Q \xi^4 \theta^2 |a_3 z + Z|^2 dxdt \right). \tag{4.1}$$

Recalling (1.3), it follows from (4.1) that

$$\begin{aligned}
& \inf_{x \in G} (e^{2s\alpha(T/4, x)} \xi^6(T/2, x)) \mathbb{E} \int_{T/4}^{3T/4} \int_G z^2 dx dt \leq \mathbb{E} \int_Q e^{2s\alpha} \xi^6 z^2 dx dt \\
& \leq C(1 + r_1^2) \left[\mathbb{E} \int_{Q_0} e^{2s\alpha} \xi^7 z^2 dx dt + \mathbb{E} \int_Q e^{2s\alpha} \xi^4 (a_3 z + Z)^2 dx dt \right] \\
& \leq \sup_{(t, x) \in Q} [e^{2s\alpha(t, x)} (\xi^7(t, x) + \xi^4(t, x))] \cdot C(1 + r_1^2) \left[\mathbb{E} \int_{Q_0} z^2 dx dt + \mathbb{E} \int_Q (a_3 z + Z)^2 dx dt \right].
\end{aligned}$$

Therefore,

$$\mathbb{E} \int_{T/4}^{3T/4} \int_G z^2 dx dt \leq C e^{C(1+r_1^2)T^{-1}} \left[\mathbb{E} \int_{Q_0} z^2 dx dt + \mathbb{E} \int_Q (a_3 z + Z)^2 dx dt \right]. \quad (4.2)$$

Thanks to Itô's formula, we find that for $0 \leq t_1 \leq t_2 \leq T$,

$$\mathbb{E} \int_G z^2(t_2) dx - \mathbb{E} \int_G z^2(t_1) dx = \mathbb{E} \int_{t_1}^{t_2} \int_G d(z^2) dx \geq -C(1 + r_1^2) \mathbb{E} \int_{t_1}^{t_2} \int_G z^2 dx dt$$

Thus, in terms of the Gronwall inequality, for $0 \leq t_1 \leq t_2 \leq T$,

$$\mathbb{E} \int_G z^2(t_1) dx \leq C e^{CT(1+r_1^2)} \mathbb{E} \int_G z^2(t_2) dx. \quad (4.3)$$

From (4.2) and (4.3), we deduce

$$\mathbb{E} \int_G z^2(0) dx \leq C e^{C(1+r_1^2)T^{-1}} \left[\mathbb{E} \int_{Q_0} z^2 dx dt + \mathbb{E} \int_Q (a_3 z + Z)^2 dx dt \right].$$

Next, we deal with the case $T > 1$. By the result for $T = 1$ shown above, we have

$$\begin{aligned}
\mathbb{E} \int_G z^2(0) dx & \leq C e^{C(1+r_1^2)} (|z|_{L_{\mathbb{F}}^2(0,1;L^2(G_0))}^2 + |a_3 z + Z|_{L_{\mathbb{F}}^2(0,1;L^2(G))}^2) \\
& \leq C e^{C(1+r_1^2)(T^{-1}+1)} (|z|_{L_{\mathbb{F}}^2(0,T;L^2(G_0))}^2 + |a_3 z + Z|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2).
\end{aligned}$$

This completes the proof of Theorem 1.2. $\square$

A Proof of the weighted identity

Proof of Theorem 2.1. It is clear that

$$\theta dh = dw - s\alpha_t w dt, \quad (A.1)$$

and

$$\begin{aligned}
\theta \Delta^2 h & = \Delta^2 w - 4s\lambda \xi \nabla \eta \cdot \nabla \Delta w - 4s\lambda^2 \xi (\nabla^2 w \nabla \eta \nabla \eta) - 4s\lambda \xi \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} + 2s^2 \lambda^2 \xi^2 |\nabla \eta|^2 \Delta w \\
& \quad - 2s\lambda^2 \xi |\nabla \eta|^2 \Delta w - 2s\lambda \xi \Delta \eta \Delta w + 4s^2 \lambda^2 \xi^2 (\nabla^2 w \nabla \eta \nabla \eta) - 4\nabla \Delta \ell \cdot \nabla w \\
& \quad + 12s^2 \lambda^3 \xi^2 |\nabla \eta|^2 \nabla \eta \nabla w + 8s^2 \lambda^2 \xi^2 (\nabla^2 \eta \nabla \eta \nabla w) - 4s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \nabla \eta \nabla w \\
& \quad + 4s^2 \lambda^2 \xi^2 \Delta \eta \nabla \eta \nabla w + 4(\nabla \ell \cdot \nabla \Delta \ell) w + 2|\nabla^2 \ell|^2 w - \Delta^2 \ell w - 6s^3 \lambda^4 \xi^3 |\nabla \eta|^4 w
\end{aligned} \quad (A.2)$$

$$-4s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)w + s^4\lambda^4\xi^4|\nabla\eta|^4w - 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta w + |\Delta\ell|^2w.$$

From (2.3)–(2.7), (A.1), and (A.2), it follows that

$$2\theta I(-dh + \Delta^2 h dt) = 2I(I_1 + I_2 + I_3).$$

We will compute II_2 under the form $\sum_{i=1}^6 \sum_{j=1}^6 I_{ij}$, where I_{ij} is the product of the i -th term of I with the j -th term of I_2 .

By direct computations, we get the following equalities:

$$\begin{aligned} I_{11} &= \sum_{i,j,k,p=1}^n \left(\Phi_1^p w_{x_k x_k x_p} w_{x_i x_i x_j} - \frac{1}{2} \Phi_1^j w_{x_k x_k x_p} w_{x_i x_i x_p} \right)_{x_j} dt - \sum_{i,j,k,p=1}^n \Phi_{1x_j}^p w_{x_k x_k x_p} w_{x_i x_i x_j} dt \\ &\quad + \frac{1}{2} \sum_{i,j,k,p=1}^n \Phi_{1x_p}^p w_{x_k x_k x_j} w_{x_i x_i x_j} dt, \\ I_{12} &= \sum_{i,j,p=1}^n \left(\Phi_2 w_{x_i x_i x_j} w_{x_p x_p} - \frac{1}{2} \Phi_{2x_j} w_{x_p x_p} w_{x_i x_i} \right)_{x_j} dt - \Phi_2 |\nabla \Delta w|^2 dt + \frac{1}{2} \Delta \Phi_2 |\Delta w|^2 dt, \\ I_{13} &= \sum_{i,j,k,p=1}^n \left(\Phi_3^{kp} w_{x_k x_p} w_{x_i x_i x_j} - \Phi_3^{kp} w_{x_k x_p x_j} w_{x_i x_i} + \Phi_{3x_j}^{kp} w_{x_k x_p} w_{x_i x_i} + \Phi_3^{kj} w_{x_k x_p x_p} w_{x_i x_i} \right. \\ &\quad \left. - \frac{1}{2} \Phi_{3x_p}^{jp} w_{x_k x_k} w_{x_i x_i} - 2\Phi_{3x_i}^{kp} w_{x_k x_p} w_{x_i x_j} + 2\Phi_{3x_p}^{kj} w_{x_i x_k} w_{x_i x_p} - \Phi_{3x_j}^{kp} w_{x_i x_k} w_{x_i x_p} \right)_{x_j} dt \\ &\quad - \sum_{i,j,k,p=1}^n \Phi_3^{kp} w_{x_j x_j x_k} w_{x_i x_i x_p} dt - \sum_{i,j,k,p=1}^n \Phi_{3x_j x_j}^{kp} w_{x_k x_p} w_{x_i x_i} dt + \frac{1}{2} \sum_{k,p=1}^n \Phi_{3x_k x_p}^{kp} |\Delta w|^2 dt \\ &\quad + 2 \sum_{i,j,k,p=1}^n \Phi_{3x_i x_j}^{kp} w_{x_k x_p} w_{x_i x_j} dt - 2 \sum_{i,j,k,p=1}^n \Phi_{3x_j x_p}^{kp} w_{x_i x_j} w_{x_i x_k} dt + \sum_{i,j,k,p=1}^n \Phi_{3x_j x_j}^{kp} w_{x_i x_p} w_{x_i x_k} dt. \\ I_{14} &= \sum_{i,j,k=1}^n \left(\Phi_4^k w_{x_i x_i x_j} w_{x_k} - \Phi_{4x_i}^k w_{x_i x_j} w_{x_k} - \Phi_4^k w_{x_i x_j} w_{x_i x_k} + \Phi_{4x_i x_j}^k w_{x_i} w_{x_k} - \frac{1}{2} \Phi_{4x_i x_k}^j w_{x_i} w_{x_k} \right. \\ &\quad \left. + \frac{1}{2} \Phi_{4x_k}^j w_{x_i x_k}^2 \right)_{x_j} dt - \sum_{i,j,k=1}^n \Phi_{4x_i x_j x_j}^k w_{x_i} w_{x_k} dt + \sum_{i,j,k=1}^n \frac{1}{2} \Phi_{4x_i x_j x_k}^k w_{x_i} w_{x_j} dt \\ &\quad + 2 \sum_{i,j,k=1}^n \Phi_{4x_j}^k w_{x_i x_j} w_{x_i x_k} dt - \frac{1}{2} \sum_{i,j,k=1}^n \Phi_{4x_k}^k w_{x_i x_j}^2 dt, \\ I_{15} &= \sum_{i,j=1}^n \left(\Phi_5 w_{x_i x_i x_j} w - \Phi_{5x_j} w_{x_i x_i} w + \Phi_{5x_i x_i} w_{x_j} w - \frac{1}{2} \Phi_{5x_i x_i x_j} w^2 - \Phi_5 w_{x_i x_i} w_{x_j} + 2\Phi_{5x_i} w_{x_i} w_{x_j} \right. \\ &\quad \left. - \Phi_{5x_j} w_{x_i}^2 \right)_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n \Phi_{5x_i x_i x_j} w_{x_j}^2 dt - 2 \sum_{i,j=1}^n \Phi_{5x_i x_j} w_{x_i} w_{x_j} dt + \Phi_5 |\Delta w|^2 dt; \\ I_{21} &= \frac{1}{2} \sum_{i,j,k=1}^n (\Psi_2 \Phi_1^j w_{x_i x_i} w_{x_k x_k})_{x_j} dt - \frac{1}{2} \sum_{k=1}^n (\Psi_2 \Phi_1^k)_{x_k} |\Delta w|^2 dt, \quad I_{22} = \Phi_2 \Psi_2 |\Delta w|^2 dt, \end{aligned}$$

$$\begin{aligned}
I_{23} &= \sum_{j,k,p=1}^n \left[\Psi_2 \Phi_3^{kp} w_{x_j} w_{x_k x_p} - (\Psi_2 \Phi_3^{jp})_{x_k} w_{x_k} w_{x_p} + \frac{1}{2} (\Psi_2 \Phi_3^{kp})_{x_j} w_{x_k} w_{x_p} - \Psi_2 \Phi_3^{kj} w_{x_p} w_{x_k x_p} \right. \\
&\quad \left. + \frac{1}{2} (\Psi_2 \Phi_3^{jp})_{x_p} w_{x_k}^2 \right]_{x_j} dt + \sum_{j,k,p=1}^n (\Psi_2 \Phi_3^{kp})_{x_j x_k} w_{x_j} w_{x_p} dt - \frac{1}{2} \sum_{j,k,p=1}^n (\Psi_2 \Phi_3^{kp})_{x_j x_j} w_{x_k} w_{x_p} dt \\
&\quad - \frac{1}{2} \sum_{j,k,p=1}^n (\Psi_2 \Phi_3^{kp})_{x_k x_p} w_{x_j}^2 dt + \sum_{j,k,p=1}^n \Psi_2 \Phi_3^{kp} w_{x_j x_k} w_{x_j x_p} dt, \\
I_{24} &= \sum_{j,k=1}^n \left(\Psi_2 \Phi_4^k w_{x_j} w_{x_k} - \frac{1}{2} \Psi_2 \Phi_4^j w_{x_k}^2 \right)_{x_j} dt - \sum_{j,k=1}^n (\Psi_2 \Phi_4^k)_{x_j} w_{x_j} w_{x_k} dt + \frac{1}{2} \sum_{j,k=1}^n (\Psi_2 \Phi_4^k)_{x_k} w_{x_j}^2 dt, \\
I_{25} &= \sum_{j=1}^n \left[\Psi_2 \Phi_5 w_{x_j} w - \frac{1}{2} (\Psi_2 \Phi_5)_{x_j} w^2 \right]_{x_j} dt + \frac{1}{2} \sum_{j=1}^n (\Psi_2 \Phi_5)_{x_j x_j} w^2 dt - \Psi_2 \Phi_5 |\nabla w|^2 dt; \\
I_{31} &= \sum_{i,j,k,p=1}^n \left(\Psi_3^{ik} \Phi_1^p w_{x_i x_k} w_{x_p x_j} - \frac{1}{2} \Psi_3^{ij} \Phi_1^p w_{x_i x_k} w_{x_k x_p} \right)_{x_j} dt - \sum_{i,j,k,p=1}^n (\Psi_3^{ij} \Phi_1^p)_{x_k} w_{x_i x_j} w_{x_k x_p} dt \\
&\quad + \frac{1}{2} \sum_{i,j,k,p=1}^n (\Psi_3^{ij} \Phi_1^p)_{x_i} w_{x_j x_k} w_{x_k x_p} dt, \\
I_{32} &= \sum_{i,j,p=1}^n \left[\Psi_3^{ij} \Phi_2 w_{x_i} w_{x_p x_p} - (\Psi_3^{ip} \Phi_2)_{x_p} w_{x_i} w_{x_j} + \frac{1}{2} (\Psi_3^{ij} \Phi_2)_{x_i} w_{x_p}^2 - \Psi_3^{ip} \Phi_2 w_{x_i} w_{x_j x_p} \right. \\
&\quad \left. + \frac{1}{2} (\Psi_3^{ip} \Phi_2)_{x_j} w_{x_i} w_{x_p} \right]_{x_j} dt + \sum_{i,j,p=1}^n (\Psi_3^{ij} \Phi_2)_{x_j x_p} w_{x_i} w_{x_p} dt - \frac{1}{2} \sum_{i,j,p=1}^n (\Psi_3^{ij} \Phi_2)_{x_i x_j} w_{x_p}^2 dt \\
&\quad - \frac{1}{2} \sum_{i,j,p=1}^n (\Psi_3^{ij} \Phi_2)_{x_p x_p} w_{x_i} w_{x_j} dt + \sum_{i,j,p=1}^n \Psi_3^{ij} \Phi_2 w_{x_i x_p} w_{x_j x_p} dt, \\
I_{33} &= \sum_{i,j,k,p=1}^n \left[\Psi_3^{ip} \Phi_3^{kj} w_{x_i x_p} w_{x_k} - (\Psi_3^{ij} \Phi_3^{kp})_{x_p} w_{x_i} w_{x_k} + \frac{1}{2} (\Psi_3^{ik} \Phi_3^{jp})_{x_p} w_{x_i} w_{x_k} - \Psi_3^{ij} \Phi_3^{kp} w_{x_i x_p} w_{x_k} \right. \\
&\quad \left. + \frac{1}{2} (\Psi_3^{ij} \Phi_3^{kp})_{x_i} w_{x_p} w_{x_k} \right]_{x_j} dt + \sum_{i,j,k,p=1}^n (\Psi_3^{ij} \Phi_3^{kp})_{x_j x_p} w_{x_i} w_{x_k} dt - \frac{1}{2} \sum_{i,j,k,p=1}^n (\Psi_3^{ij} \Phi_3^{kp})_{x_k x_p} w_{x_i} w_{x_j} dt \\
&\quad - \frac{1}{2} \sum_{i,j,k,p=1}^n (\Psi_3^{ij} \Phi_3^{kp})_{x_i x_j} w_{x_p} w_{x_k} dt + \sum_{i,j,k,p=1}^n \Psi_3^{ij} \Phi_3^{kp} w_{x_i x_p} w_{x_k x_j} dt, \\
I_{34} &= \frac{1}{2} \sum_{i,j,p=1}^n (\Psi_3^{ij} \Phi_4^p w_{x_i} w_{x_p})_{x_j} dt - \frac{1}{2} \sum_{i,j,p=1}^n (\Psi_3^{ij} \Phi_4^p)_{x_j} w_{x_i} w_{x_p} dt, \\
I_{35} &= \sum_{i,j=1}^n \left[\Psi_3^{ij} \Phi_5 w_{x_i} w - \frac{1}{2} (\Psi_3^{ij} \Phi_5)_{x_i} w^2 \right]_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n (\Psi_3^{ij} \Phi_5)_{x_i x_j} w^2 dt - \sum_{i,j=1}^n \Psi_3^{ij} \Phi_5 w_{x_i} w_{x_j} dt; \\
I_{41} &= \sum_{i,k,j=1}^n \left[\Psi_4^i \Phi_1^k w_{x_i} w_{x_k x_j} - \frac{1}{2} (\Psi_4^i \Phi_1^k)_{x_j} w_{x_i} w_{x_k} \right]_{x_j} dt + \frac{1}{2} \sum_{i,k,j=1}^n (\Psi_4^i \Phi_1^k)_{x_j x_j} w_{x_i} w_{x_k} dt \\
&\quad - \sum_{i,k,j=1}^n \Psi_4^i \Phi_1^k w_{x_i x_j} w_{x_k x_j} dt,
\end{aligned}$$

$$\begin{aligned}
I_{42} &= \sum_{i,j=1}^n \left(\Psi_4^i \Phi_2 w_{x_i} w_{x_j} - \frac{1}{2} \Psi_4^j \Phi_2 w_{x_i}^2 \right)_{x_j} dt - \sum_{i,j=1}^n (\Psi_4^i \Phi_2)_{x_j} w_{x_i} w_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n (\Psi_4^j \Phi_2)_{x_j} w_{x_i}^2 dt, \\
I_{43} &= \sum_{i,j,k=1}^n \left(\Psi_4^i \Phi_3^{jk} w_{x_i} w_{x_k} - \frac{1}{2} \Psi_4^j \Phi_3^{ik} w_{x_i} w_{x_k} \right)_{x_j} dt - \sum_{i,j,k=1}^n (\Psi_4^i \Phi_3^{jk})_{x_k} w_{x_i} w_{x_j} dt \\
&\quad + \frac{1}{2} \sum_{i,j,k=1}^n (\Psi_4^i \Phi_3^{jk})_{x_i} w_{x_j} w_{x_k} dt, \\
I_{44} &= \sum_{i,j=1}^n \Psi_4^i \Phi_4^j w_{x_i} w_{x_j} dt, \quad I_{45} = \frac{1}{2} \sum_{j=1}^n (\Psi_4^j \Phi_5 w^2)_{x_j} dt - \frac{1}{2} \sum_{j=1}^n (\Psi_4^j \Phi_5)_{x_j} w^2 dt; \\
I_{51} &= \sum_{i,k,j=1}^n (\Psi_5^i \Phi_1^k w_{x_i} w_{x_k x_j})_{x_j} dt - \sum_{i,k,j=1}^n (\Psi_5^i \Phi_1^k)_{x_j} w_{x_i} w_{x_k x_j} dt - \sum_{i,k,j=1}^n \Psi_5^i \Phi_1^k w_{x_i x_j} w_{x_k x_j} dt, \\
I_{52} &= \sum_{i,j=1}^n \left(\Psi_5^i \Phi_2 w_{x_i} w_{x_j} - \frac{1}{2} \Psi_5^j \Phi_2 w_{x_i}^2 \right)_{x_j} dt - \sum_{i,j=1}^n (\Psi_5^i \Phi_2)_{x_j} w_{x_i} w_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n (\Psi_5^j \Phi_2)_{x_j} w_{x_i}^2 dt, \\
I_{53} &= \sum_{i,j,k=1}^n \left(\Psi_5^i \Phi_3^{jk} w_{x_i} w_{x_k} - \frac{1}{2} \Psi_5^j \Phi_3^{ik} w_{x_i} w_{x_k} \right)_{x_j} dt - \sum_{i,j,k=1}^n (\Psi_5^i \Phi_3^{jk})_{x_k} w_{x_i} w_{x_j} dt \\
&\quad + \frac{1}{2} \sum_{i,j,k=1}^n (\Psi_5^i \Phi_3^{jk})_{x_i} w_{x_j} w_{x_k} dt, \\
I_{54} &= \sum_{i,j=1}^n \Psi_5^i \Phi_4^j w_{x_i} w_{x_j} dt, \quad I_{55} = \frac{1}{2} \sum_{j=1}^n (\Psi_5^j \Phi_5 w^2)_{x_j} dt - \frac{1}{2} \sum_{j=1}^n (\Psi_5^j \Phi_5)_{x_j} w^2 dt;
\end{aligned}$$

and

$$\begin{aligned}
I_{61} &= \sum_{i,j=1}^n \left[\Psi_6 \Phi_1^i w_{x_i x_j} w - (\Psi_6 \Phi_1^i)_{x_j} w_{x_i} w + \frac{1}{2} (\Psi_6 \Phi_1^j)_{x_i x_i} w^2 - \frac{1}{2} \Psi_6 \Phi_1^j w_{x_i}^2 \right]_{x_j} dt \\
&\quad - \frac{1}{2} \sum_{i,j=1}^n (\Psi_6 \Phi_1^i)_{x_i x_j x_j} w^2 dt + \sum_{i,j=1}^n (\Psi_6 \Phi_1^i)_{x_j} w_{x_i} w_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n (\Psi_6 \Phi_1^i)_{x_i} w_{x_j}^2 dt \\
I_{62} &= \sum_{j=1}^n \left[\Psi_6 \Phi_2 w w_{x_j} - \frac{1}{2} (\Psi_6 \Phi_2)_{x_j} w^2 \right]_{x_j} dt + \frac{1}{2} \sum_{j=1}^n (\Psi_6 \Phi_2)_{x_j x_j} w^2 dt - \sum_{j=1}^n \Psi_6 \Phi_2 w_{x_j}^2 dt, \\
I_{63} &= \sum_{i,j=1}^n \left[\Psi_6 \Phi_3^{ij} w w_{x_i} - \frac{1}{2} (\Psi_6 \Phi_3^{ij})_{x_i} w^2 \right]_{x_j} dt + \frac{1}{2} \sum_{i,j=1}^n (\Psi_6 \Phi_3^{ij})_{x_i x_j} w^2 dt - \sum_{i,j=1}^n \Psi_6 \Phi_3^{ij} w_{x_i} w_{x_j} dt, \\
I_{64} &= \frac{1}{2} \sum_{j=1}^n (\Psi_6 \Phi_4^j w^2)_{x_j} dt - \frac{1}{2} \sum_{j=1}^n (\Psi_6 \Phi_4^j)_{x_j} w^2 dt, \quad I_{65} = \Psi_6 \Phi_5 w^2 dt.
\end{aligned}$$

Using Itô's formula, we conclude

$$I_{16} = - \sum_{i,j=1}^n (w_{x_i x_i x_j} dw - w_{x_i x_j} dw_{x_i})_{x_j} - \frac{1}{2} d(|\nabla^2 w|^2) + \frac{1}{2} |d\nabla^2 w|^2,$$

$$\begin{aligned}
I_{26} &= - \sum_{j=1}^n (\Psi_2 w_{x_j} dw)_{x_j} + \sum_{i=1}^n \Psi_{2x_i} w_{x_i} dw + \frac{1}{2} d(\Psi_2 |\nabla w|^2) - \frac{1}{2} \Psi_{2t} |\nabla w|^2 dt - \frac{1}{2} \Psi_2 |d\nabla w|^2, \\
I_{36} &= - \sum_{i,j=1}^n (\Psi_3^{ij} w_{x_i} dw)_{x_j} + \sum_{i,j=1}^n \Psi_{3x_j}^{ij} w_{x_i} dw + 2d(|s\lambda\xi\nabla\eta\nabla w|^2) - 4 \sum_{i,j=1}^n s^2 \lambda^2 \xi(\xi\eta_{x_j})_t \eta_{x_i} w_{x_i} w_{x_j} dt \\
&\quad - 2s^2 \lambda^2 \xi^2 |\nabla\eta d\nabla w|^2, \\
I_{46} &= - \sum_{i=1}^n \Psi_4^i w_{x_i} dw, \quad I_{56} = - \sum_{i=1}^n \Psi_5^i w_{x_i} dw, \quad I_{66} = - \frac{1}{2} d(\Psi_6 w^2) + \frac{1}{2} \Psi_{6t} w^2 dt + \frac{1}{2} \Psi_6 (dw)^2.
\end{aligned}$$

By summing all the I_{ij} , we get (2.8). $\square$

B Proof of Lemma 2.4

To prove Lemma 2.4, we need the following regularity estimate for the stochastic fourth order parabolic equation.

Lemma B.1. *There exists a constant $C > 0$, such that for all $(h, g) \in L^2_{\mathbb{F}}(0, T; H^4(G)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$, and $f \in L^2_{\mathbb{F}}(0, T; L^2(G))$ satisfying*

$$\begin{cases} dh = \Delta^2 h dt + f dt + g dW(t) & \text{in } Q, \\ h = \Delta h = 0 & \text{on } \Sigma, \\ h(T, \cdot) = 0 & \text{in } G, \end{cases}$$

we have

$$|h|_{L^2_{\mathbb{F}}(0, T; H^4(G))}^2 \leq C |f|_{L^2_{\mathbb{F}}(0, T; L^2(G))}^2.$$

Proof. The proof is standard. For the sake of completeness, we provide a sketch below. Define an unbounded operator A on $L^2(G)$ as follows:

$$\begin{cases} \mathcal{D}(A) = \{\varphi \in H^4(G) \mid \varphi, \Delta\varphi \in H_0^1(G)\}, \\ A\varphi = \Delta^2\varphi, \quad \forall \varphi \in \mathcal{D}(A). \end{cases}$$

Denote by $0 < \mu_1 < \mu_2 \leq \dots$ the eigenvalues of A and by $\{\phi_i\}_{i=1}^\infty \subset \mathcal{D}(A)$ the corresponding eigenfunctions such that $|\phi_i|_{L^2(G)} = 1$ ($i = 1, 2, 3, \dots$), which serves as an orthonormal basis of $L^2(G)$. Thanks to the regularity for solutions to the elliptic equations (see [15, Chapter 2, Theorem 5.1]), we have $\phi_i \in H^8(G)$, and $\Delta^3\phi_i = 0$ on ∂G for $i = 1, 2, 3, \dots$

For $i = 1, \dots, m$, consider the following backward stochastic differential equations

$$\begin{cases} dc_i^m = \mu_i c_i^m dt + (f, \phi_i)_{L^2(G)} dt + g_i^m dW(t), \\ c_i^m(T) = 0 \end{cases} \quad (\text{B.1})$$

We have $(f, \phi_i)_{L^2(G)} \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$. According to the classical theory of backward stochastic differential equations (see [20, Theorem 4.2]), we know that there is a unique solution $(c_i^m(\cdot), g_i^m(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R})) \times L^2_{\mathbb{F}}(0, T; \mathbb{R})$ for $i = 1, \dots, m$.

Denote

$$h^m = \sum_{i=1}^m c_i^m(t) \phi_i. \quad (\text{B.2})$$

From (B.1) and (B.2), we get

$$dh^m = \Delta^2 h^m dt + f^m dt + g^m dW(t),$$

where $f^m = \sum_{i=1}^m (f, \phi_i)_{L^2(G)} \phi_i$, $g^m = \sum_{i=1}^m g_i^m \phi_i$ and $h^m(T) \equiv 0$.

Using Itô's formula, we get that

$$\begin{aligned} d(h^m \Delta^2 h^m) &= (\Delta^2 h^m)^2 dt + f^m \Delta^2 h^m dt + h^m \Delta^4 h^m dt + h^m \Delta^2 f^m dt + g^m \Delta^2 g^m dt \\ &\quad + g^m \Delta^2 h^m dW(t) + h^m \Delta^2 g^m dW(t). \end{aligned} \quad (\text{B.3})$$

Integrating the equality (B.3) on Q , taking expectation in both sides, we obtain

$$-\int_G h^m(0) \Delta^2 h^m(0) dx = \mathbb{E} \int_Q [(\Delta^2 h^m)^2 + f^m \Delta^2 h^m + h^m \Delta^4 h^m + h^m \Delta^2 f^m + g^m \Delta^2 g^m] dx dt. \quad (\text{B.4})$$

Thanks to (B.2), we get

$$-\int_G h^m(0) \Delta^2 h^m(0) dx = -\sum_{i=1}^m \mu_i (c_i^m(0))^2 \leq 0. \quad (\text{B.5})$$

Utilizing integration by parts, noting $h^m = \Delta h^m = \Delta^2 h^m = \Delta^3 h^m = f^m = \Delta f^m = g^m = \Delta g^m = 0$ on Σ , we have

$$\mathbb{E} \int_Q [h^m \Delta^4 h^m + h^m \Delta^2 f^m + g^m \Delta^2 g^m] dx dt = \mathbb{E} \int_Q [(\Delta^2 h^m)^2 + f^m \Delta^2 h^m + (\Delta g^m)^2] dx dt. \quad (\text{B.6})$$

Combining (B.4)–(B.6), using Cauchy's inequality, we find

$$\mathbb{E} \int_Q |\Delta^2 h^m|^2 dx dt \leq C \mathbb{E} \int_Q |f^m|^2 dx dt. \quad (\text{B.7})$$

Thanks to elliptic regularity theory (see [16, Chapter 4, Theorem 4.1]), we get

$$|h^m|_{L_{\mathbb{F}}^2(0,T;H^4(G))}^2 \leq C |\Delta^2 h^m|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2. \quad (\text{B.8})$$

Combining (B.7) and (B.8), we have

$$|h^m|_{L_{\mathbb{F}}^2(0,T;H^4(G))}^2 \leq C |f^m|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2.$$

Therefore $\{h^m\}_{m=1}^\infty$ is bounded in $L_{\mathbb{F}}^2(0,T;H^4(G))$, and there exists a subsequence, which is still denoted by $\{h^m\}_{m=1}^\infty$, such that h^m weakly convergence to $\tilde{h} \in L_{\mathbb{F}}^2(0,T;H^4(G))$ as m tends to ∞ . On the other hand, thanks to the well-posedness of backward stochastic evolution equations (see [20, Theorem 4.11]), we know that $\lim_{m \rightarrow \infty} |h^m - h|_{L_{\mathbb{F}}^2(0,T;L^2(G))} = 0$. Hence, $\tilde{h} = h$ for a.e. $(t, x, \omega) \in [0, T] \times G \times \Omega$ and

$$|h|_{L_{\mathbb{F}}^2(0,T;H^4(G))} \leq \liminf_{m \rightarrow \infty} |h^m|_{L_{\mathbb{F}}^2(0,T;H^4(G))} \leq C |f|_{L_{\mathbb{F}}^2(0,T;L^2(G))}.$$

□

Proof of Lemma 2.4 . Using the trace inequality (e.g., [15, Theorem 9.4]) and interpolation in-

equality between Sobolev spaces (e.g., [15, Theorem 9.1]), we have

$$\begin{aligned}\|\nabla^2 h\|_{L^2(\partial G)}^2 &\leq C\|h\|_{H^2(G)}\|h\|_{H^3(G)}, \\ \|\nabla^3 h\|_{L^2(\partial G)}^2 &\leq C\|h\|_{H^3(G)}\|h\|_{H^4(G)}.\end{aligned}\tag{B.9}$$

By (B.9), (2.12), and interpolation results between the Hilbert spaces $L_{\mathbb{F}}^2(0, T; H^r(G))$ ($r \in [0, 4]$) (see [15, Chapter 1, Remark 14.4]), we have

$$\begin{aligned}\mathbb{E} \int_{\Sigma} s^{\frac{9}{4}} \lambda^3 \xi^{\frac{9}{4}} \theta^2 |\nabla^2 h|^2 d\sigma dt &= \mathbb{E} \int_0^T s^{\frac{9}{4}} \lambda^3 \xi_{\star}^{\frac{9}{4}} e^{2s\alpha_{\star}} |\nabla^2 h|_{L^2(\partial G)}^2 dt \\ &\leq C \mathbb{E} \int_0^T s^{\frac{9}{4}} \lambda^3 \xi_{\star}^{\frac{9}{4}} e^{2s\alpha_{\star}} |h|_{H^2(G)} |h|_{H^3(G)} dt \\ &\leq C \mathbb{E} \int_0^T s^{\frac{9}{4}} \lambda^3 \xi_{\star}^{\frac{9}{4}} e^{2s\alpha_{\star}} |h|_{L^2(G)}^{\frac{3}{4}} |h|_{H^4(G)}^{\frac{5}{4}} dt \\ &\leq C \left(\mathbb{E} \int_Q s^6 \lambda^8 \xi^6 \theta^2 h^2 dx dt + \mathbb{E} \int_0^T e^{2s\alpha_{\star}} |h|_{H^4(G)}^2 dt \right),\end{aligned}\tag{B.10}$$

where we use $\alpha \geq \alpha_{\star}$ and $\xi \geq \xi_{\star}$ in $\overline{Q}$.

Similarly, we can obtain that

$$\begin{aligned}\mathbb{E} \int_{\Sigma} s^{\frac{3}{4}} \lambda \xi^{\frac{3}{4}} \theta^2 |\nabla^3 h|^2 d\sigma dt &\leq C \mathbb{E} \int_0^T s^{\frac{3}{4}} \lambda \xi_{\star}^{\frac{3}{4}} e^{2s\alpha_{\star}} |h|_{L^2(G)}^{\frac{1}{4}} |h|_{H^4(G)}^{\frac{7}{4}} dt \\ &\leq C \left(\mathbb{E} \int_Q s^6 \lambda^8 \xi^6 \theta^2 h^2 dx dt + \mathbb{E} \int_0^T e^{2s\alpha_{\star}} |h|_{H^4(G)}^2 dt \right).\end{aligned}\tag{B.11}$$

To estimate $\mathbb{E} \int_0^T e^{2s\alpha_{\star}} |h|_{H^4(G)}^2 dt$, we set $h_{\star} = e^{s\alpha_{\star}} h$. This function satisfies

$$\begin{cases} -dh_{\star} + \Delta^2 h_{\star} dt = [e^{s\alpha_{\star}} f - (e^{s\alpha_{\star}})_t h] dt + e^{s\alpha_{\star}} g dW(t) & \text{in } Q, \\ h_{\star} = \Delta h_{\star} = 0 & \text{on } \Sigma, \\ h_{\star}(T, \cdot) = 0 & \text{in } G, \end{cases}$$

where we use $\lim_{t \rightarrow T^-} e^{s\alpha_{\star}} = 0$.

Thanks to Lemma B.1 and (2.9), for $s \geq CT^{1/2}$, we have

$$\begin{aligned}\mathbb{E} \int_0^T e^{2s\alpha_{\star}} |h|_{H^4(G)}^2 dt &= |h_{\star}|_{L_{\mathbb{F}}^2(0, T; H^4(G))}^2 \leq C |e^{s\alpha_{\star}} f - (e^{s\alpha_{\star}})_t h|_{L_{\mathbb{F}}^2(0, T; L^2(G))}^2 \\ &\leq C \left(\mathbb{E} \int_Q \theta^2 |f|^2 dx dt + \mathbb{E} \int_Q s^6 \xi^6 \theta^2 h^2 dx dt \right).\end{aligned}\tag{B.12}$$

Combining (B.10)–(B.12), the proof is complete. $\square$

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