

A Geometric Inverse Source Problem for Stochastic Parabolic Equations

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Abstract

This paper addresses the geometric inverse problem of simultaneously recovering two unknown deterministic source supports in a stochastic parabolic equation, where one source appears in the drift term and the other in the diffusion term. We establish that partial boundary flux measurements alone uniquely determine both supports. Moreover, we prove the existence of minimizers for a perimeter-regularized objective functional. For smooth interfaces, we derive a Hadamard-type boundary representation of the shape derivative. To the best of our knowledge, this is the first such formula for the simultaneous recovery of a drift-source support and a diffusion-source support in an SPDE setting. The derivative exhibits a genuinely stochastic two-channel structure: the adjoint state governs the sensitivity of the drift-source interface, while the martingale component controls the sensitivity of the diffusion-source interface. Based on this formula, we develop a shape-gradient reconstruction method. Numerical experiments demonstrate its effectiveness and its capacity to distinguish between the two source channels under both full and partial boundary observations.

Keywords: Geometric inverse source problems, stochastic parabolic equations, shape calculus, uniqueness

1 Introduction

Geometric inverse problems seek to recover an unknown interface, inclusion, or source support from indirect measurements generated by a partial differential equation. When the unknown is a geometric set represented by its characteristic function, shape calculus provides a natural variational framework (see [1, 9, 18, 36, 39] and references therein). This approach is well established for deterministic equations. In the present paper, we extend it to stochastic partial differential equations and study the simultaneous recovery of two deterministic source supports in a stochastic parabolic equation from partial boundary flux measurements. One support enters the drift channel, and the other enters the diffusion channel.

Throughout, $T > 0$ is fixed, and $G \subset \mathbb{R}^n$, $n \geq 2$, is a bounded C^2 domain. The observation set Γ is a nonempty relatively open subset of ∂G , and $K \subset\subset G$ is a fixed compact set. We work on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ carrying a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$. The filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the augmented natural filtration. We denote by $\mathbb{F}$ the progressive σ -field with respect to $\mathbf{F}$ and by ν the unit outward normal on ∂G .

Let $\mathcal{H}$ be a Banach space. We use the following standard spaces of $\mathcal{H}$ -valued stochastic processes:

- $L_{\mathbb{F}}^2(0, T; \mathcal{H})$ denotes the space of all $\mathcal{H}$ -valued $\mathbf{F}$ -adapted processes $X(\cdot)$ with $\mathbb{E}(\|X(\cdot)\|_{L^2(0, T; \mathcal{H})}^2) < \infty$.
- $L_{\mathbb{F}}^\infty(0, T; \mathcal{H})$ denotes the space of all essentially bounded $\mathcal{H}$ -valued $\mathbf{F}$ -adapted processes.

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- $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathcal{H}))$ denotes the space of all $\mathcal{H}$ -valued $\mathbf{F}$ -adapted continuous processes $X(\cdot)$ satisfying $\mathbb{E}(\|X(\cdot)\|_{C([0, T]; \mathcal{H})}^2) < \infty$.

All these spaces are equipped with their usual norms; see [33, Section 2.6].

We consider the stochastic parabolic equation with two geometric source terms:

$$\begin{cases} du = (\Delta u + \chi_{D_0} + au) dt + (bu + \chi_{D_1}) dW(t) & \text{in } (0, T) \times G, \\ u = 0, & \text{on } (0, T) \times \partial G, \\ u(0, x) = u_0 & \text{in } G, \end{cases} \quad (1.1)$$

where $a \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G))$, $b \in L^\infty_{\mathbb{F}}(0, T; W^{1, \infty}(G))$, $u_0 \in H^1_0(G)$, and χ_{D_i} is the characteristic function of a measurable set $D_i \subset K$, $i = 0, 1$. The set D_0 supports the drift source and D_1 the diffusion source; the two sets may intersect, coincide, or one may be empty.

Problem 1 (Geometric inverse source problem). *Determine the source supports (D_0, D_1) from the partial boundary flux $\partial u / \partial \nu$ on $(0, T) \times \Gamma$.*

Remark 1.1. A concrete motivation for Problem 1 is thermal source localization under uncertain operating conditions. In electronic packages, inverse parabolic-conduction methods locate internal hot spots from surface measurements [24], and stochastic models account for non-stationary random thermal loads [27]. In a linearized stochastic model, D_0 represents the region of persistent mean parabolic release, while D_1 localizes the amplitude of random parabolic-release fluctuations.

Geometric reconstruction of discontinuous sources and inclusions has been extensively studied for deterministic parabolic equations. Hettlich and Rundell investigated the recovery of source supports and discontinuous sources in parabolic equations [20]. Their work treats a discontinuous source in the parabolic equation and can be seen as a direct deterministic precursor to our model. Shape-optimization approaches to parabolic interface identification were developed in [5, 16, 17, 19], boundary-data formulations in [7], and sparse boundary-flux reconstruction in [28, 29]. Recent advances include reduced-order reconstruction of geometric parabolic sources [22] and moving shape identification for parabolic sources and potentials [13]. These results confirm the effectiveness of geometry-aware formulations for unknowns with sharp interfaces, but all concern deterministic evolution equations.

A distinct direction is shape optimization under uncertainty, where deterministic geometry is paired with random loads or coefficients; see [6, 8]. The closest framework is the stochastic product-shape optimization of [14, 15]: the unknown (D_0, D_1) is a deterministic pair of shapes, the state is random, and the objective is an expected shape functional. The mathematical structures, however, differ in three key respects. First, their forward models rely on random-parameter PDEs that reduce to deterministic elliptic problems per realization, whereas ours is a genuinely stochastic parabolic equation. Second, their shapes are nonintersecting, while our D_0 and D_1 are additive source supports that may overlap. Third, their shape gradient uses a deterministic adjoint per realization, whereas the sensitivity of our diffusion support D_1 is governed by the martingale component q of a backward stochastic adjoint equation.

Inverse problems governed by genuinely stochastic PDEs are far less developed. As emphasized in the survey [34], the literature remains limited and many fundamental questions are open. Lü [31] proved a single-source uniqueness result for a stochastic parabolic equation via a global Carleman estimate. Simultaneous identification of function-valued sources in both the drift and diffusion channels has been obtained for stochastic wave equations [44], stochastic parabolic equations [45], and stochastic Grushin equations [43]. More recently, Wang, Yang, and Zhang [41] showed that partial boundary-flux data determine the temporal strength $|f(t)|$ of a separable random source $g(x)f(t)dW(t)$ in stochastic parabolic and wave equations. A distinct line of research concerns frequency-domain inverse scattering with random-field sources; for instance, [25] recovers the principal symbols of the covariance and relation operators of a generalized Gaussian random source from far-field patterns, rather than identifying sources in the drift and diffusion terms.

Moreover, available inverse source results for SPDEs rely on structural assumptions. For instance, the separated-source framework in [32, Chapters 3–4] requires a prescribed nonvanishing factor together with geometric, regularity, separability, and sometimes terminal-measurement conditions. These hypotheses do not accommodate a general characteristic function χ_D of an unknown support. In our setting, the unknowns are the characteristic functions of two independently varying sets that may overlap, and we use only partial boundary-flux data without terminal observation. Differentiating a characteristic source yields a distribution

supported on its interface, not an admissible L^2 or H^1 source, so existing Carleman arguments may not apply directly. Our uniqueness proof instead separates the two channels: expectation and a deterministic inverse-source argument for the drift source, and a stochastic transposition identity, Itô isometry, and approximate boundary reachability for the diffusion source.

We formulate the reconstruction as a perimeter-regularized shape optimization problem. Given noisy partial boundary data $g \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma))$, we seek

$$\min_{(D_0, D_1) \in \mathcal{A}_K} \mathcal{J}(D_0, D_1), \quad (1.2)$$

where

$$\mathcal{J}(D_0, D_1) := \frac{1}{2} \mathbb{E} \int_0^T \int_{\Gamma} \left(\frac{\partial u}{\partial \nu} - g \right)^2 d\sigma dt + \beta_0 \mathcal{P}_G(D_0) + \beta_1 \mathcal{P}_G(D_1),$$

with regularization parameters $\beta_0, \beta_1 > 0$ and admissible class

$$\mathcal{A}_K := \{(D_0, D_1) : D_0, D_1 \text{ are measurable subsets of } K\}. \quad (1.3)$$

The relative perimeter $\mathcal{P}_G(D)$ of a measurable set $D \subset G$ is

$$\mathcal{P}_G(D) := \sup \left\{ \int_D \operatorname{div} \phi dx : \phi \in C_c^1(G; \mathbb{R}^n), \|\phi\|_{L^\infty(G)} \leq 1 \right\}. \quad (1.4)$$

The boundary flux $\partial u / \partial \nu|_{\Gamma}$ is well defined because the sources are compactly supported in the interior of G ; see [Lemma 2.3](#).

Beyond uniqueness, the present formulation provides a perimeter-regularized existence theory, a shape-gradient reconstruction method, and a genuinely stochastic boundary shape derivative. The central novelty is the interplay between the two source channels and the stochastic adjoint structure. In the shape-optimization framework, the natural adjoint is a backward stochastic parabolic equation whose solution consists of two adapted components: the adjoint state p and the martingale component q . The shape derivative, obtained in boundary form in the spirit of deterministic boundary shape-gradient methods [\[21, 47\]](#), exhibits a genuinely stochastic two-channel structure: p determines the sensitivity of the drift interface ∂D_0 , while q determines the sensitivity of the diffusion interface ∂D_1 . This structure also creates a computational difficulty. Unlike a deterministic adjoint, the backward stochastic equation cannot be solved by direct time reversal; its backward recursion involves conditional expectations. One of these conditional expectations determines the martingale component, and its accurate approximation is a main computational challenge [\[10, 26, 35, 46\]](#).

The main contributions are as follows.

- (i) We formulate a two-support geometric inverse source problem for the stochastic parabolic equation [\(1.1\)](#) with partial boundary-flux data and two independently varying source supports.
- (ii) We prove that partial boundary-flux data alone uniquely determine both source supports, including that of the diffusion source; see [Theorem 3.1](#). This result is of independent interest. Unlike the simultaneous identification framework of [\[45\]](#), neither a terminal-time observation nor full boundary data is required: the flux is measured only on an arbitrary nonempty relatively open subset $\Gamma \subset \partial G$.
- (iii) We prove existence of minimizers for the perimeter-regularized functional [\(1.2\)](#) on the admissible class [\(1.3\)](#). The proof combines compactness of finite-perimeter sets with a local hidden boundary regularity estimate for interior H^{-1} sources.
- (iv) We derive the boundary shape derivative formula [\(6.6\)](#). The derivation requires a localization argument and interior regularity near the moving interfaces, exploiting the compact support of the sources to compensate for the limited regularity of stochastic parabolic equations.
- (v) We propose a shape-gradient descent method based on the boundary derivative formula [\(6.6\)](#). The method addresses the backward stochastic adjoint bottleneck through finite element discretization and least-squares Monte Carlo approximation of conditional expectations; numerical experiments demonstrate effective reconstruction of both source supports under full and partial boundary observations.

The remainder of the paper is organized as follows. [Section 2](#) establishes forward well-posedness and boundary regularity. [Section 3](#) proves uniqueness of the two source supports. [Section 4](#) proves existence of minimizers for the perimeter-regularized objective. [Section 5](#) recalls shape-calculus preliminaries. [Section 6](#) develops the adjoint equation, geometric-source linearization, interior regularity near moving interfaces, and the shape derivative formula. [Sections 7 and 8](#) describe the numerical method and present the experiments.

2 Well-posedness and boundary regularity for forward equations

This section records the well-posedness and boundary regularity results used in the uniqueness, existence, and shape sensitivity arguments.

For every pair (D_0, D_1) with $D_i \subset K$, the characteristic sources belong to $L^2(G)$. By the classical well-posedness result for stochastic evolution equations, (1.1) admits a unique weak solution $u \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^1_0(G))$; see, e.g., [33, Theorem 3.24].

For later use, consider the linear equation

$$\begin{cases} dy = (\Delta y + ay + \Phi) dt + (by + \Psi) dW(t) & \text{in } (0, T) \times G, \\ y = 0 & \text{on } (0, T) \times \partial G, \\ y(0, x) = y_0(x) & \text{in } G. \end{cases} \quad (2.1)$$

Definition 2.1. A process y is called a strong solution to (2.1) if $y \in L^2_{\mathbb{F}}(\Omega; L^2(0, T; H^2(G) \cap H^1_0(G))) \cap L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G)))$, and if, for all $t \in [0, T]$,

$$y(t) - \int_0^t (\Delta y(s) + a(s)y(s)) ds = y_0 + \int_0^t \Phi(s) ds + \int_0^t (b(s)y(s) + \Psi(s)) dW(s), \quad \mathbb{P}\text{-a.s.}$$

Theorem 2.1 ([40, Theorem 7.3]). Let $y_0 \in H^1_0(G)$, let $\Phi \in L^2_{\mathbb{F}}(0, T; L^2(G))$, and let $\Psi \in L^2_{\mathbb{F}}(0, T; H^1_0(G))$. If $a \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G))$ and $b \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G))$, then equation (2.1) admits a unique strong solution y . Furthermore, there exists a constant $C > 0$ such that

$$\|y\|_{L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G)) \cap L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G)))} \leq C(\|\Phi\|_{L^2_{\mathbb{F}}(0, T; L^2(G))} + \|\Psi\|_{L^2_{\mathbb{F}}(0, T; H^1_0(G))} + \|y_0\|_{H^1_0(G)}).$$

Here and in the sequel, C denotes a generic positive constant depending only on G, T , and the coefficient bounds $\|a\|_{L^\infty_{\mathbb{F}}(0, T; L^\infty(G))}$ and $\|b\|_{L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G))}$, and it may change from line to line. Constants denoted by C_M or C_K may also depend on the indicated interior support, in particular on its distance from ∂G .

For $f \in H^{-1}(G)$ and a closed set $K \subset G$, the notation $\text{supp } f \subset K$ means that $\langle f, \varphi \rangle_{H^{-1}(G), H^1_0(G)} = 0$ for all $\varphi \in C^\infty_c(G \setminus K)$. For a stochastic process, this condition is imposed for a.e. (t, ω) .

For $f_0, f_1 \in L^2_{\mathbb{F}}(0, T; H^{-1}(G))$ and $z_0 \in H^{-1}(G)$, consider the forward equation

$$\begin{cases} dz = (\Delta z + az + f_0) dt + (bz + f_1) dW(t) & \text{in } (0, T) \times G, \\ z = 0 & \text{on } (0, T) \times \partial G, \\ z(0) = z_0 & \text{in } G. \end{cases} \quad (2.2)$$

It follows from standard results for stochastic evolution equations (see, e.g., [33, Theorem 3.24]) that (2.2) admits a unique mild solution $z \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H^{-1}(G))) \cap L^2_{\mathbb{F}}(0, T; L^2(G))$ and

$$\|z\|_{L^2_{\mathbb{F}}(\Omega; C([0, T]; H^{-1}(G)))} + \|z\|_{L^2_{\mathbb{F}}(0, T; L^2(G))} \leq C(\|f_0\|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))} + \|f_1\|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))} + \|z_0\|_{H^{-1}(G)}). \quad (2.3)$$

Lemma 2.2. Let $M \subset\subset G$ and $U \subset\subset G \setminus \overline{M}$ be open sets. Let $f_0, f_1 \in L^2_{\mathbb{F}}(0, T; H^{-1}(G))$ satisfy $\text{supp } f_i(t, \omega) \subset \overline{M}$, $i = 0, 1$, for a.e. (t, ω) . Let $z_0 \in H^1_0(G)$, and let z be the mild solution of (2.2). Then

$$\|z\|_{L^2_{\mathbb{F}}(0, T; H^1(U))} \leq C_{M,U}(\|z_0\|_{H^1_0(G)} + \|f_0\|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))} + \|f_1\|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))}), \quad (2.4)$$

where $C_{M,U}$ depends on G, T , the coefficient bounds, and the distance between U and $\overline{M}$, but not on z_0, f_0, f_1 .

Proof. Step 1. In this step, we prove the estimate first for smooth adapted sources, after choosing a cutoff away from the source support.

Choose an open set M_2 such that $\overline{M} \subset\subset M_2 \subset\subset G$ and $U \subset\subset G \setminus \overline{M_2}$. Choose $\eta \in C^\infty_c(G \setminus \overline{M_2})$ such that $\eta = 1$ on U . Assume that f_0, f_1 take values in $C^\infty_c(M_2)$. Then $f_0 \in L^2_{\mathbb{F}}(0, T; L^2(G))$ and $f_1 \in L^2_{\mathbb{F}}(0, T; H^1_0(G))$. Therefore Theorem 2.1, applied to (2.1) with $\Phi = f_0$ and $\Psi = f_1$, gives a strong solution satisfying $z \in L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G)) \cap L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G)))$.

Set $w := \eta z$. Since η is deterministic and independent of t , multiplying the strong equation by η gives the following identity in $L^2(G)$:

$$dw = (\eta \Delta z + aw) dt + bw dW(t).$$

Here $\eta f_0 = \eta f_1 = 0$, because $\text{supp } \eta \cap \overline{M_2} = \emptyset$ and $\text{supp } f_i \subset M_2$. Equivalently, using $\Delta(\eta z) = \eta \Delta z + 2\nabla \eta \cdot \nabla z + (\Delta \eta)z$, the localized equation becomes

$$dw = (\Delta w - 2\nabla \eta \cdot \nabla z - (\Delta \eta)z + aw) dt + bw dW(t). \quad (2.5)$$

All terms in (2.5) are $L^2(G)$ -valued in the present smooth-source setting: $w \in H^2(G) \cap H_0^1(G)$, while the commutator terms are in $L^2(G)$ because $z \in H_0^1(G)$ and η is smooth.

Applying Itô's formula to the $L^2(G)$ -valued semimartingale $w = \eta z$ and using the first form of the cutoff equation, we obtain

$$\begin{aligned} d\|w(t)\|_{L^2(G)}^2 &= 2(w(t), \eta \Delta z(t) + a(t)w(t))_{L^2(G)} dt + \|b(t)w(t)\|_{L^2(G)}^2 dt + 2(b(t)w(t), w(t))_{L^2(G)} dW(t) \\ &= 2(\eta^2 z(t), \Delta z(t))_{L^2(G)} dt + 2(a(t)w(t), w(t))_{L^2(G)} dt + \|b(t)w(t)\|_{L^2(G)}^2 dt \\ &\quad + 2(b(t)w(t), w(t))_{L^2(G)} dW(t). \end{aligned}$$

The integration by parts below is justified since $\eta^2 z \in H_0^1(G)$ and $\Delta z \in L^2(G)$:

$$\begin{aligned} (\eta^2 z, \Delta z)_{L^2(G)} &= - \int_G \nabla(\eta^2 z) \cdot \nabla z dx = - \int_G \eta^2 |\nabla z|^2 dx - 2 \int_G \eta z \nabla \eta \cdot \nabla z dx \\ &= - \|\nabla(\eta z)\|_{L^2(G)}^2 + \|\nabla \eta z\|_{L^2(G)}^2. \end{aligned}$$

Substituting this identity, and recalling that $w = \eta z$, gives

$$\begin{aligned} d\|\eta z(t)\|_{L^2(G)}^2 + 2\|\nabla(\eta z(t))\|_{L^2(G)}^2 dt \\ = \left[2\|\nabla \eta z(t)\|_{L^2(G)}^2 + 2(a(t)\eta z(t), \eta z(t))_{L^2(G)} + \|b(t)\eta z(t)\|_{L^2(G)}^2 \right] dt + 2(b(t)\eta z(t), \eta z(t))_{L^2(G)} dW(t). \end{aligned}$$

Taking expectations, using the boundedness of a and b , and dropping the nonnegative terminal term give, for every $t \in [0, T]$,

$$\mathbb{E} \int_0^t \|\nabla(\eta z(s))\|_{L^2(G)}^2 ds \leq C_\eta \left(\|z_0\|_{L^2(G)}^2 + \mathbb{E} \int_0^t \|z(s)\|_{L^2(G)}^2 ds \right).$$

Since $\eta = 1$ on U , this implies

$$\|z\|_{L_{\mathbb{F}}^2(0, T; H^1(U))}^2 \leq C_\eta (\|z_0\|_{L^2(G)}^2 + \|z\|_{L_{\mathbb{F}}^2(0, T; L^2(G))}^2).$$

Combining this with (2.3), and using $\|z_0\|_{H^{-1}(G)} + \|z_0\|_{L^2(G)} \leq C\|z_0\|_{H_0^1(G)}$, yields (2.4) for smooth adapted sources.

Step 2. In this step, we pass from smooth sources to rough sources.

For general $f_0, f_1 \in L_{\mathbb{F}}^2(0, T; H^{-1}(G))$ supported in $\overline{M}$, choose adapted smooth approximations $f_0^k, f_1^k \in L_{\mathbb{F}}^2(0, T; C_c^\infty(M_2))$ such that $f_i^k \rightarrow f_i$ in $L_{\mathbb{F}}^2(0, T; H^{-1}(G))$, $i = 0, 1$. This can be obtained by predictable simple-process approximation in time and spatial mollification inside M_2 . Let z^k be the corresponding solutions. By (2.3), $z^k \rightarrow z$ in $L_{\mathbb{F}}^2(0, T; L^2(G))$ and in $L_{\mathbb{F}}^2(\Omega; C([0, T]; H^{-1}(G)))$. The smooth-source estimate gives a uniform bound for $\{\eta z^k\}_{k=1}^\infty$ in $L_{\mathbb{F}}^2(0, T; H^1(G))$. Passing to a subsequence if necessary, $\{\eta z^k\}_{k=1}^\infty$ converges weakly in this space. Since $\eta z^k \rightarrow \eta z$ strongly in $L_{\mathbb{F}}^2(0, T; L^2(G))$, the weak limit of $\{\eta z^k\}_{k=1}^\infty$ is ηz . Lower semicontinuity gives the same estimate for z , and the proof is complete. $\square$

Lemma 2.3 (Local hidden boundary regularity for interior sources). *Let $M \subset\subset G$ be open, and let $f_0, f_1 \in L_{\mathbb{F}}^2(0, T; H^{-1}(G))$ satisfy $\text{supp } f_i(t, \omega) \subset \overline{M}$, $i = 0, 1$, for a.e. (t, ω) . Let $z_0 \in H_0^1(G)$, and let z be the solution of (2.2). Then $\partial z / \partial \nu \in L_{\mathbb{F}}^2(0, T; L^2(\partial G))$ and*

$$\left\| \frac{\partial z}{\partial \nu} \right\|_{L_{\mathbb{F}}^2(0, T; L^2(\partial G))} \leq C_M (\|f_0\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))} + \|f_1\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))} + \|z_0\|_{H_0^1(G)}), \quad (2.6)$$

where C_M is independent of f_0, f_1 , and z_0 .

Proof. Choose an open set M_2 such that $\overline{M} \subset\subset M_2 \subset\subset G$. Choose $\zeta \in C^\infty(\overline{G})$ such that $\zeta = 1$ in a collar neighborhood U_∂ of ∂G and $\text{supp } \zeta \cap \overline{M_2} = \emptyset$. Set $v := \zeta z$. Choose an open set M_1 such that $\text{supp } \nabla \zeta \cup \text{supp } \Delta \zeta \subset M_1 \subset\subset G \setminus \overline{M_2}$. By Lemma 2.2, applied with the present set M and $U = M_1$, we have

$$\|z\|_{L_{\mathbb{F}}^2(0, T; H^1(M_1))} \leq C_M (\|z_0\|_{H_0^1(G)} + \|f_0\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))} + \|f_1\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))}). \quad (2.7)$$

Consequently, $F_\zeta := -2\nabla \zeta \cdot \nabla z - (\Delta \zeta)z \in L_{\mathbb{F}}^2(0, T; L^2(G))$, and

$$\|F_\zeta\|_{L_{\mathbb{F}}^2(0, T; L^2(G))} \leq C_M (\|z_0\|_{H_0^1(G)} + \|f_0\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))} + \|f_1\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))}).$$

Since $\zeta = 0$ in a neighborhood of $\overline{M}$, we have $\zeta f_0 = \zeta f_1 = 0$ in $H^{-1}(G)$. Using the local H^1 regularity above, $v = \zeta z$ satisfies

$$dv = (\Delta v + av + F_\zeta) dt + bv dW(t), \quad v|_{\partial G} = 0, \quad v(0) = \zeta z_0. \quad (2.8)$$

Moreover, $\|\zeta z_0\|_{H_0^1(G)} \leq C_M \|z_0\|_{H_0^1(G)}$. By [Theorem 2.1](#), equation (2.8) has a strong solution with

$$\|v\|_{L_{\mathbb{F}}^2(0,T;H^2(G))} \leq C_M (\|z_0\|_{H_0^1(G)} + \|f_0\|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} + \|f_1\|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}). \quad (2.9)$$

Since $\zeta = 1$ on U_∂ , we have $v = z$ in a neighborhood of ∂G . Therefore the normal trace of z is defined by $\partial z / \partial \nu := \partial v / \partial \nu$ on $(0, T) \times \partial G$. The trace theorem for $H^2(G) \cap H_0^1(G)$ gives

$$\left\| \frac{\partial z}{\partial \nu} \right\|_{L_{\mathbb{F}}^2(0,T;L^2(\partial G))} = \left\| \frac{\partial v}{\partial \nu} \right\|_{L_{\mathbb{F}}^2(0,T;L^2(\partial G))} \leq C \|v\|_{L_{\mathbb{F}}^2(0,T;H^2(G))}.$$

Combining this estimate with (2.9) gives (2.6). $\square$

In particular, for the state equation (1.1) with sources (χ_{D_0}, χ_{D_1}) supported in K , [Lemma 2.3](#) gives

$$\frac{\partial u}{\partial \nu} \in L_{\mathbb{F}}^2(0, T; L^2(\partial G)), \quad (2.10)$$

so that $\partial u / \partial \nu|_\Gamma$ is well defined in $L_{\mathbb{F}}^2(0, T; L^2(\Gamma))$.

3 Uniqueness of the source supports

This section establishes one of the main theoretical results of the paper: both source supports are uniquely determined by the partial boundary flux alone.

Assumption 3.1. Assume $a(t, x, \omega) = a_0(t, \omega) + c(x)$, with $a_0 \in L_{\mathbb{F}}^\infty(0, T)$ and $c \in L^\infty(G)$, and $b \in L_{\mathbb{F}}^\infty(0, T; W^{1,\infty}(G))$.

Theorem 3.1. Assume [Assumption 3.1](#). Let $(D_0^1, D_1^1), (D_0^2, D_1^2) \in \mathcal{A}_K$, and let u^1, u^2 be the corresponding solutions, belonging to $L_{\mathbb{F}}^2(\Omega; C([0, T]; L^2(G))) \cap L_{\mathbb{F}}^2(0, T; H_0^1(G))$, with the initial datum $u_0 \in H_0^1(G)$ and the same coefficients a, b . If $\partial u^1 / \partial \nu = \partial u^2 / \partial \nu$ in $L_{\mathbb{F}}^2(0, T; L^2(\Gamma))$, then $\chi_{D_0^1} = \chi_{D_0^2}$ and $\chi_{D_1^1} = \chi_{D_1^2}$ a.e. in G .

Remark 3.1. The same argument gives a function-source version, showing that the uniqueness conclusion is not restricted to characteristic sources. If the drift sources in (1.1) are replaced by deterministic functions $f^j \in L^2(G)$ and the diffusion sources by $\Psi^j \in L_{\mathbb{F}}^2(0, T; H_0^1(G))$, $j = 1, 2$, then equality of the local boundary fluxes implies $f^1 = f^2$ in $L^2(G)$ and $\Psi^1 = \Psi^2$ in $L_{\mathbb{F}}^2(0, T; H_0^1(G))$. Thus the local boundary flux alone determines both a deterministic drift source and an adapted function-valued diffusion source, without any observation of the terminal state. This observation regime differs from that of [45], where simultaneous determination of two sources uses observations at the final time together with lateral-boundary data. We state [Theorem 3.1](#) in the characteristic-source form because the inverse problem considered in this paper is geometric and this notation keeps the presentation focused.

We now prepare the proof of [Theorem 3.1](#). The first ingredient is the boundary unique continuation property used to propagate zero Cauchy data from the observed boundary patch.

Lemma 3.2. Let $I = (t_0, t_1) \subset \mathbb{R}$, and let $c \in L^\infty(I \times G)$. Suppose that $v \in H^1(I; L^2(G)) \cap L^2(I; H^2(G))$ solves $v_t - \Delta v - cv = 0$ in $I \times G$. If $v = 0$ and $\partial v / \partial \nu = 0$ on $I \times \Gamma$, then $v = 0$ in $I \times G$. The same conclusion holds for the backward equation $-v_t - \Delta v - cv = 0$ after the time reversal $s = t_1 + t_0 - t$.

This is the homogeneous Cauchy-data consequence of [42, Theorem 5.1]; the backward assertion follows by time reversal.

The drift part of the uniqueness proof is reduced to a deterministic separated inverse source problem. For this purpose, define A on $L^2(G)$ by $D(A) = H^2(G) \cap H_0^1(G)$ and $Au = \Delta u + c(x)u$.

Lemma 3.3. Let $c \in L^\infty(G)$ be real-valued and $K \subset\subset G$. Assume $f \in L^2(G)$, $\text{supp } f \subset K$, and $r \in W^{1,\infty}(0, T)$ satisfying $r(0) \neq 0$. If the solution of

$$\begin{cases} y_t = \Delta y + c(x)y + r(t)f & \text{in } (0, T) \times G, \\ y = 0 & \text{on } (0, T) \times \partial G, \\ y(0) = 0 & \text{in } G, \end{cases} \quad (3.1)$$

satisfies $\frac{\partial y}{\partial \nu} = 0$ on $(0, T) \times \Gamma$, then $f = 0$.

Lemma 3.3 appears to be known, though a precise reference is not readily found in the literature. For completeness, we provide its proof here.

Proof. Let $S(t)$ be the Dirichlet semigroup generated by A . The solution admits the mild representation $y(t) = \int_0^t S(t-s)r(s)f ds$. The support condition $\text{supp } f \subset K \subset\subset G$ and standard parabolic smoothing away from K imply $g_f(t) := \frac{\partial S(t)f}{\partial \nu} \Big|_{\Gamma} \in L^2(0, T; L^2(\Gamma))$, and the boundary trace of the Duhamel formula is

$$\frac{\partial y(t)}{\partial \nu} \Big|_{\Gamma} = \int_0^t r(s)g_f(t-s) ds.$$

Thus $\frac{\partial y(t)}{\partial \nu} = 0$ on $(0, T) \times \Gamma$ gives, in $L^2(0, T; L^2(\Gamma))$, $\int_0^t r(t-s)g_f(s) ds = 0$. Since $r \in W^{1,\infty}(0, T)$, differentiating this gives $r(0)g_f(t) + \int_0^t r'(t-s)g_f(s) ds = 0$ for a.e. $t \in (0, T)$. Hence

$$\|g_f(t)\|_{L^2(\Gamma)} \leq \frac{\|r'\|_{L^\infty(0, T)}}{|r(0)|} \int_0^t \|g_f(s)\|_{L^2(\Gamma)} ds.$$

Gronwall's inequality yields $\frac{\partial S(t)f}{\partial \nu} \Big|_{\Gamma} = 0$ for a.e. $t \in (0, T)$.

Let $v(t) = S(t)f$. By parabolic regularization for analytic semigroups [37, Chapter 4, Section 4.1, Corollary 1.5], for every $\varepsilon \in (0, T)$, $v \in H^1(\varepsilon, T; L^2(G)) \cap L^2(\varepsilon, T; H^2(G) \cap H_0^1(G))$, solves $\partial_t v - \Delta v - cv = 0$ in $(\varepsilon, T) \times G$, and satisfies $\partial v / \partial \nu = 0$ on $(\varepsilon, T) \times \Gamma$. By Lemma 3.2, $v = 0$ in $(\varepsilon, T) \times G$. Since $\varepsilon > 0$ is arbitrary and $S(t)f \rightarrow f$ in $L^2(G)$ as $t \downarrow 0$, we obtain $f = 0$. $\square$

Let $\tau \in (0, T)$ and $c(x) \in L^\infty(G)$. Let $\mathcal{X} := H^1(0, T; L^2(\Gamma)) \cap L^2(0, T; H^{3/2}(\Gamma))$. Consider the backward parabolic equation

$$\begin{cases} -q_t = \Delta q + c(x)q & \text{in } (\tau, T) \times G, \\ q = \psi & \text{on } (\tau, T) \times \Gamma, \\ q = 0 & \text{on } (\tau, T) \times (\partial G \setminus \Gamma), \\ q(T) = 0 & \text{in } G. \end{cases} \quad (3.2)$$

Denote by $q(\cdot; \psi)$ the solution to (3.2) corresponding to the control ψ .

Lemma 3.4. (1) $\mathcal{R}(\tau) := \{q(\tau; \psi) : \psi \in C_c^\infty((\tau, T) \times \Gamma)\}$ is dense in $L^2(G)$.

(2) There exists a countable set $\{\psi_j\}_{j \geq 1} \subset C_c^\infty((0, T) \times \Gamma)$ such that

$$\text{span}\{q(t; \psi_j) : j \geq 1\} \text{ is dense in } L^2(G), \quad \forall t \in (0, T). \quad (3.3)$$

Remark 3.2. The approximate controllability of (3.2) with boundary controls in $L^\infty((0, T) \times \Gamma)$ was shown in [12], yielding the density of the reachable set in $L^2(G)$. Although the proof of part (1) of Lemma 3.4 is standard, it is not a direct consequence of that result, since $C_c^\infty((0, T) \times \Gamma)$ is not dense in $L^\infty((0, T) \times \Gamma)$. Part (2), on the other hand, appears to be new and is of independent interest for control theory.

Proof. Step 1. We prove that $\mathcal{R}(\tau)$ is dense in $L^2(G)$ by contradiction. Assume the contrary. Then there exists $\phi \in L^2(G)$ orthogonal to $\mathcal{R}(\tau)$. Let p solve

$$\begin{cases} p_t = \Delta p + c(x)p & \text{in } (\tau, T) \times G, \\ p = 0 & \text{on } (\tau, T) \times \partial G, \\ p(\tau) = \phi & \text{in } G. \end{cases} \quad (3.4)$$

By parabolic regularization for analytic semigroups, for every $\delta \in (0, T - \tau)$, $p \in H^1(\tau + \delta, T; L^2(G)) \cap L^2(\tau + \delta, T; H^2(G) \cap H_0^1(G))$. For $\psi \in C_c^\infty((\tau + \delta, T) \times \Gamma)$, multiplying the equation for $q(\cdot; \psi)$ by p and integrating by parts yields

$$0 = (\phi, q(\tau; \psi))_{L^2(G)} = - \int_{\tau + \delta}^T \int_{\Gamma} \psi \frac{\partial p}{\partial \nu} d\sigma dt. \quad (3.5)$$

Since $\delta > 0$ is arbitrary and $C_c^\infty((\tau + \delta, T) \times \Gamma)$ is dense in $L^2((\tau + \delta, T) \times \Gamma)$, we deduce from (3.5) that $\frac{\partial p}{\partial \nu} = 0$ on $(\tau, T) \times \Gamma$. Applying Lemma 3.2 on $(\tau + \delta, T)$ yields $p = 0$ in $(\tau + \delta, T) \times G$. Letting $\delta \downarrow 0$ and using $p \in C([\tau, T]; L^2(G))$, we obtain $\phi = p(\tau) = 0$. Hence $\mathcal{R}(\tau)$ is dense in $L^2(G)$.

Step 2. We construct a single countable family that works uniformly for all fixed times.

Define $\mathcal{X} := \{\psi \in H^1(0, T; L^2(\Gamma)) \cap L^2(0, T; H^{3/2}(\Gamma)) : \psi = 0 \text{ on } \partial\Gamma, \psi(0) = \psi(T) = 0\}$. Let $\mathcal{J} := \{(t_1, t_2) : 0 \leq t_1 < t_2 \leq T, t_1, t_2 \in \mathbb{Q}\}$. Since $\mathcal{X}$ is separable, for each $J \in \mathcal{J}$ we may select a countable

subset $\{\psi_j^J\}_{j \geq 1} \subset C_c^\infty((0, T) \times \Gamma)$ that is dense in $C_c^\infty(J \times \Gamma)$ with respect to the $\mathcal{X}$ -topology. As $\mathcal{J}$ is countable, the set $\{\psi_j\}_{j \geq 1} := \bigcup_{J \in \mathcal{J}} \{\psi_j^J\}_{j \geq 1}$ is countable.

Now fix $t \in (0, T)$. For any $\psi \in C_c^\infty((t, T) \times \Gamma)$, its support lies in $J \times \Gamma$ for some $J \in \mathcal{J}$. Consequently, there exists a subsequence $\{\psi_{j_k}\}_{k \geq 1} \subset \{\psi_j\}_{j \geq 1}$ such that

$$\lim_{k \rightarrow \infty} \|\psi_{j_k} - \psi\|_{H^1(0, T; L^2(\Gamma)) \cap L^2(0, T; H^{3/2}(\Gamma))} = 0.$$

By parabolic boundary regularity theory ([30, Chapter 4, Theorems 2.1 and 4.2]), obtained via a boundary lifting argument and the standard L^2 -regularity theory for the homogeneous Dirichlet problem, the map $\psi \mapsto q(t; \psi)$ is continuous from $\mathcal{X}$ to $L^2(G)$ for each fixed $t \in (0, T)$. Hence,

$$\lim_{k \rightarrow \infty} \|q(t; \psi_{j_k}) - q(t; \psi)\|_{L^2(G)} = 0.$$

Combined with assertion (1), this establishes (3.3). $\square$

We now combine the drift inverse-source lemma and the boundary reachability statement to prove the uniqueness theorem.

Proof of Theorem 3.1. Set $w = u^1 - u^2$, $f_0 = \chi_{D_0^1} - \chi_{D_0^2}$, and $f_1 = \chi_{D_1^1} - \chi_{D_1^2}$. Then $f_0, f_1 \in L^2(G)$ and $\text{supp } f_i \subset K$. By (1.1) and Lemma 2.3, w satisfies

$$\begin{cases} dw = (\Delta w + f_0 + aw) dt + (bw + f_1) dW(t) & \text{in } (0, T) \times G, \\ w = 0 & \text{on } (0, T) \times \partial G, \\ w(0, x) = 0 & \text{in } G, \end{cases} \quad (3.6)$$

and $\partial w / \partial \nu = 0$ on $(0, T) \times \Gamma$.

Step 1. Identification of the drift source. Define $\mu(t) = \exp(-\int_0^t a_0(s) ds)$ and $z = \mu w$. By Itô's formula, $dz = \mu dw + w d\mu = (Az + \mu f_0) dt + (bz + \mu f_1) dW(t)$, so that

$$\begin{cases} dz = (Az + \mu f_0) dt + (bz + \mu f_1) dW(t) & \text{in } (0, T) \times G, \\ z = 0 & \text{on } (0, T) \times \partial G, \\ z(0, x) = 0 & \text{in } G. \end{cases} \quad (3.7)$$

Since μ is independent of x , we have $\partial z / \partial \nu = \mu \partial w / \partial \nu = 0$ on $(0, T) \times \Gamma$.

Let $\zeta(t) = \mathbb{E}[z(t)]$ and $r(t) = \mathbb{E}[\mu(t)]$. Taking expectations in the variational formulation of (3.7) yields

$$\begin{cases} \partial_t \zeta = A\zeta + r(t)f_0 & \text{in } (0, T) \times G, \\ \zeta = 0 & \text{on } (0, T) \times \partial G, \\ \zeta(0, x) = 0 & \text{in } G. \end{cases} \quad (3.8)$$

Moreover, $\partial \zeta / \partial \nu = \mathbb{E}[\partial z / \partial \nu] = 0$ on $(0, T) \times \Gamma$. Since $\mu(0) = 1$ and $\mu'(t) = -a_0(t)\mu(t)$, we have $\mu \in L^\infty(\Omega; W^{1, \infty}(0, T))$, hence $r \in W^{1, \infty}(0, T)$ with $r(0) = 1$. Applying Lemma 3.3 gives $f_0 = 0$ in $L^2(G)$.

Step 2. Identification of the diffusion source. With $f_0 = 0$, set $F = bz + \mu f_1$. For any $\psi \in C_c^\infty((0, T) \times \Gamma)$, let $q(\cdot; \psi)$ solve

$$\begin{cases} -q_t = \Delta q + c(x)q & \text{in } (0, T) \times G, \\ q = \psi & \text{on } (0, T) \times \Gamma, \\ q = 0 & \text{on } (0, T) \times (\partial G \setminus \Gamma), \\ q(T) = 0 & \text{in } G. \end{cases}$$

Integration by parts gives

$$\int_0^T \int_G F(t) q(t; \psi) dx dW(t) = - \int_0^T \int_\Gamma \frac{\partial z}{\partial \nu} \psi d\sigma dt = 0.$$

Then Itô's isometry yields

$$\mathbb{E} \int_0^T \left| \int_G F(t) q(t; \psi) dx \right|^2 dt = 0. \quad (3.9)$$

Let $\{\psi_j\}_{j=1}^\infty$ be the countable family constructed in Lemma 3.4. For each j , (3.9) implies $\int_G F(t, \omega) q(t; \psi_j) dx = 0$ for a.e. (t, ω) . Removing the countable union of exceptional sets, we obtain a set $\mathcal{N} \subset \Omega \times (0, T)$ with $(\mathbb{P} \otimes dt)(\mathcal{N}) = 0$ such that, for all $(\omega, t) \notin \mathcal{N}$, $\int_G F(t, \omega) q(t; \psi_j) dx = 0$ for every $j \geq 1$. For any such fixed t , Lemma 3.4 asserts that $\text{span}\{q(t; \psi_j) : j \geq 1\}$ is dense in $L^2(G)$. Hence $F(t, \omega) = 0$ in $L^2(G)$ for a.e. (t, ω) .

The equation for z now reduces to $\partial_t z = Az$, $z|_{(0, T) \times \partial G} = 0$, $z(0) = 0$, hence $z \equiv 0$. Since μ is bounded away from zero, $F = bz + \mu f_1 = 0$ yields $f_1 = 0$. With $f_0 = 0$, the proof is complete. $\square$

4 Existence of the minimizer

This section returns to the general coefficient framework of (1.1); in particular, Assumption 3.1 is not required here. We prove the existence of a measurable minimizer $(D_0, D_1) \in \mathcal{A}_K$ for the shape optimization problem (1.2). The proof is based on the decomposition $\mathcal{J}(D_0, D_1) = \mathcal{J}_1(D_0, D_1) + \mathcal{J}_2(D_0, D_1)$, where

$$\mathcal{J}_1(D_0, D_1) = \frac{1}{2} \mathbb{E} \int_0^T \int_{\Gamma} \left| \frac{\partial u}{\partial \nu} - g \right|^2 d\sigma dt, \quad \mathcal{J}_2(D_0, D_1) = \beta_0 \mathcal{P}_G(D_0) + \beta_1 \mathcal{P}_G(D_1).$$

Consider the set of characteristic functions (see [39, p. 32]) $\text{Char}(G) = \{\chi \in L^2(G) \mid \chi(1 - \chi) = 0, \text{ a.e. in } G\}$. We also use the closed subset $\text{Char}_K(G) = \{\chi_D \in \text{Char}(G) : \chi_D = 0 \text{ a.e. in } G \setminus K\}$.

The perimeter term fits naturally into this framework. We therefore begin with the standard lower semicontinuity property of the perimeter functional.

Lemma 4.1 ([39, Lemma 2.6, p. 33]). *The functional $\chi_D \mapsto \mathcal{P}_G(D)$ is lower semicontinuous on $\text{Char}(G)$ with respect to the $L^2(G)$ -topology.*

We prove the continuity of the misfit term next.

Proposition 4.2. *The following functional is continuous on $\text{Char}_K(G) \times \text{Char}_K(G)$ with respect to the product $L^2(G) \times L^2(G)$ topology:*

$$(\chi_{D_0}, \chi_{D_1}) \mapsto \mathcal{J}_1(D_0, D_1) := \frac{1}{2} \mathbb{E} \int_0^T \int_{\Gamma} \left(\frac{\partial u}{\partial \nu} - g \right)^2 d\sigma dt,$$

where u solves (1.1) corresponding to (χ_{D_0}, χ_{D_1}) .

Proof of Proposition 4.2. Fix $(\chi_{D_0^1}, \chi_{D_1^1})$ and $(\chi_{D_0^2}, \chi_{D_1^2})$ in $\text{Char}_K(G) \times \text{Char}_K(G)$, and let u^1, u^2 be the corresponding solutions to (1.1). Set $w := u^1 - u^2$. Then w solves

$$\begin{cases} dw = (\Delta w + aw + \chi_{D_0^1} - \chi_{D_0^2}) dt + (bw + \chi_{D_1^1} - \chi_{D_1^2}) dW(t) & \text{in } (0, T) \times G, \\ w = 0 & \text{on } (0, T) \times \partial G, \\ w(0, x) = 0 & \text{in } G. \end{cases}$$

By Lemma 2.3, applied with an open set M satisfying $K \subset M \subset\subset G$, there exists a constant $C_K > 0$ such that

$$\left\| \frac{\partial w}{\partial \nu} \right\|_{L^2_{\mathbb{F}}(0, T; L^2(\partial G))} \leq C_K (\|\chi_{D_0^1} - \chi_{D_0^2}\|_{H^{-1}(G)} + \|\chi_{D_1^1} - \chi_{D_1^2}\|_{H^{-1}(G)}). \quad (4.1)$$

Since $L^2(G) \hookrightarrow H^{-1}(G)$ continuously, the right-hand side is bounded by the corresponding $L^2(G)$ norms. With the same set M , applying Lemma 2.3 directly to u^j , with initial datum u_0 and sources $(\chi_{D_0^j}, \chi_{D_1^j})$, and using $\|\chi_{D_i^j}\|_{L^2(G)} \leq |G|^{1/2}$ (the Lebesgue measure), yields the uniform bound

$$\left\| \frac{\partial u^j}{\partial \nu} \right\|_{L^2_{\mathbb{F}}(0, T; L^2(\partial G))} \leq C_K (|G|^{1/2} + \|u_0\|_{H_0^1(G)}), \quad j = 1, 2. \quad (4.2)$$

By the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |\mathcal{J}_1(D_0^1, D_1^1) - \mathcal{J}_1(D_0^2, D_1^2)| &= \frac{1}{2} \left| \mathbb{E} \int_0^T \int_{\Gamma} \left[\left(\frac{\partial u^1}{\partial \nu} - g \right)^2 - \left(\frac{\partial u^2}{\partial \nu} - g \right)^2 \right] d\sigma dt \right| \\ &\leq \frac{1}{2} \left\| \frac{\partial u^1}{\partial \nu} + \frac{\partial u^2}{\partial \nu} - 2g \right\|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))} \left\| \frac{\partial w}{\partial \nu} \right\|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}. \end{aligned}$$

Combining this estimate with (4.1) and (4.2), we obtain

$$|\mathcal{J}_1(D_0^1, D_1^1) - \mathcal{J}_1(D_0^2, D_1^2)| \leq C_K(u_0, g) \left(\|\chi_{D_0^1} - \chi_{D_0^2}\|_{L^2(G)} + \|\chi_{D_1^1} - \chi_{D_1^2}\|_{L^2(G)} \right).$$

Hence $\mathcal{J}_1(\cdot, \cdot)$ is continuous on $\text{Char}_K(G) \times \text{Char}_K(G)$ with respect to the product $L^2(G) \times L^2(G)$ topology. This completes the proof of Proposition 4.2. $\square$

Combining Lemma 4.1, Proposition 4.2, and (1.2) gives the following lower semicontinuity result.

Proposition 4.3. *The shape functional $\mathcal{J}(\cdot, \cdot)$ is lower semicontinuous on the set $\text{Char}_K(G) \times \text{Char}_K(G) \subset L^2(G) \times L^2(G)$.*

To pass from lower semicontinuity to existence, it remains to recover compactness for minimizing sequences. This is provided by the classical compactness theorem for characteristic functions with uniformly bounded perimeter.

Lemma 4.4 ([39, Lemma 2.8, p. 34]). *Let G be a bounded domain in $\mathbb{R}^n$. For any $R > 0$, the set $\text{Char}(G, R) := \{\chi_D \in \text{Char}(G) \mid \mathcal{P}_G(D) \leq R\}$ is compact in $L^2(G)$.*

The existence result is the following.

Theorem 4.5. *There exists a measurable pair $(\widehat{D}_0, \widehat{D}_1) \in \mathcal{A}_K$ such that $\mathcal{J}(\widehat{D}_0, \widehat{D}_1) \leq \mathcal{J}(D_0, D_1)$ for all $(D_0, D_1) \in \mathcal{A}_K$.*

Proof of Theorem 4.5. Let $j_0 = \mathcal{J}(\emptyset, \emptyset)$. By (2.10), $j_0 < +\infty$. Since $\mathcal{J} \geq 0$ on $\mathcal{A}_K$, the case $j_0 = 0$ trivially yields $(\emptyset, \emptyset)$ as a minimizer. We therefore assume $j_0 > 0$ in what follows.

Set $R_0 := j_0/\beta_0$ and $R_1 := j_0/\beta_1$, and introduce the truncated admissible class

$$\mathcal{K}_R := (\text{Char}_K(G) \cap \text{Char}(G, R_0)) \times (\text{Char}_K(G) \cap \text{Char}(G, R_1)).$$

As before, we identify each admissible set with its characteristic function. We claim that minimizing over $\mathcal{A}_K$ is equivalent to minimizing over $\mathcal{K}_R$, i.e.,

$$\inf_{(D_0, D_1) \in \mathcal{A}_K} \mathcal{J}(D_0, D_1) = \inf_{(\chi_{D_0}, \chi_{D_1}) \in \mathcal{K}_R} \mathcal{J}(D_0, D_1). \quad (4.3)$$

Since $\mathcal{K}_R \subset \text{Char}_K(G) \times \text{Char}_K(G)$, the inequality

$$\inf_{(D_0, D_1) \in \mathcal{A}_K} \mathcal{J}(D_0, D_1) \leq \inf_{(\chi_{D_0}, \chi_{D_1}) \in \mathcal{K}_R} \mathcal{J}(D_0, D_1)$$

is immediate. Suppose $(D_0, D_1) \in \mathcal{A}_K$ does not belong to $\mathcal{K}_R$. Then either $\mathcal{P}_G(D_0) > R_0$ or $\mathcal{P}_G(D_1) > R_1$. Because $\mathcal{J}_1(\cdot, \cdot) \geq 0$, the first case gives

$$\mathcal{J}(D_0, D_1) \geq \beta_0 \mathcal{P}_G(D_0) > j_0 = \mathcal{J}(\emptyset, \emptyset),$$

and the second case is analogous. Since $(\chi_\emptyset, \chi_\emptyset) \in \mathcal{K}_R$, no admissible pair outside $\mathcal{K}_R$ can lower the infimum below that over $\mathcal{K}_R$. This establishes (4.3).

It remains to minimize over $\mathcal{K}_R$. By Lemma 4.4, $\text{Char}(G, R_i)$ is compact in $L^2(G)$ for $i = 0, 1$. The subset $\text{Char}_K(G)$ is closed in $L^2(G)$, because the condition $\chi = 0$ a.e. on $G \setminus K$ is preserved under $L^2(G)$ -convergence. Consequently, each intersection $\text{Char}_K(G) \cap \text{Char}(G, R_i)$ is compact in $L^2(G)$, and hence $\mathcal{K}_R$ is compact in $L^2(G) \times L^2(G)$.

Let (D_0^k, D_1^k) be a minimizing sequence in $\mathcal{K}_R$. By compactness, passing to a subsequence if necessary, there exists $(\chi_{\widehat{D}_0}, \chi_{\widehat{D}_1}) \in \mathcal{K}_R$ such that $\chi_{D_0^k} \rightarrow \chi_{\widehat{D}_0}$ and $\chi_{D_1^k} \rightarrow \chi_{\widehat{D}_1}$ in $L^2(G)$. The lower semicontinuity established in Proposition 4.3 then yields

$$\mathcal{J}(\widehat{D}_0, \widehat{D}_1) \leq \liminf_{k \rightarrow \infty} \mathcal{J}(D_0^k, D_1^k) = \inf_{(\chi_{D_0}, \chi_{D_1}) \in \mathcal{K}_R} \mathcal{J}(D_0, D_1).$$

Combined with (4.3), this shows that $(\widehat{D}_0, \widehat{D}_1) \in \mathcal{A}_K$ is a minimizer over the original admissible class. This completes the proof of Theorem 4.5. $\square$

5 Preliminaries on shape calculus

This section recalls the notions from shape calculus used below. Details can be found in [39].

From this section on, let $D \subset\subset G$ be an open set with Lipschitz boundary. Let ν be the unit outward normal vector on ∂D . When the first variation of the perimeter is used later, we further assume that ∂D is of class C^2 .

Fix $V \in W_0^{1,\infty}(G; \mathbb{R}^n)$. For $|\varepsilon|$ sufficiently small, define the perturbation of identity by $F_\varepsilon(x) = x + \varepsilon V(x)$, $x \in G$. For the shape D , define the perturbed domain

$$D_\varepsilon = F_\varepsilon(D) \triangleq \{F_\varepsilon(x) \in \mathbb{R}^n \mid x \in D\}. \quad (5.1)$$

Since V is Lipschitz and $D \subset\subset G$, there exists $\varepsilon_0 > 0$ such that, for all $|\varepsilon| < \varepsilon_0$, the map F_ε is bi-Lipschitz on G , the perturbed set D_ε remains an open Lipschitz subdomain of G , and $F_\varepsilon(\partial D) = \partial D_\varepsilon$. This is the perturbation of identity method; see [9, 18].

Definition 5.1 (Material derivative and shape derivative). Let $X(D)$ be a Banach space of functions on D , and let $f_\varepsilon : D_\varepsilon \rightarrow \mathbb{R}$ be a family of functions such that $f = f_0$. The material derivative of f at D in the direction V is defined by $\dot{f} = \lim_{\varepsilon \rightarrow 0+} (f_\varepsilon \circ F_\varepsilon - f)/\varepsilon$, whenever the limit exists in $X(D)$. If, in addition, $f \in W_{\text{loc}}^{1,1}(D)$ and $\dot{f}$ exists, then the shape derivative of f at D in the direction V is defined by $f' = \dot{f} - \nabla f \cdot V$, whenever the right-hand side is well defined in the corresponding function space.

In the present problem, when a quantity is already defined on the fixed domain G , its derivative is understood directly in the corresponding function space over G .

Definition 5.2. The Eulerian derivative of a shape functional $J(D)$ at D in the direction V is defined as $dJ(D; V) = \lim_{\varepsilon \rightarrow 0+} (J(D_\varepsilon) - J(D))/\varepsilon$, if the limit exists.

Lemma 5.1 ([9, Theorem 4.1, p. 483]). Let $\tau > 0$. Assume that $\Phi \in C([0, \tau]; W_{\text{loc}}^{1,1}(G)) \cap C^1([0, \tau]; L_{\text{loc}}^1(G))$. Let $D \subset G$ be measurable, and define $J(\varepsilon) = \int_{D_\varepsilon} \Phi(\varepsilon, x) dx$. Then J admits a right derivative at $\varepsilon = 0$, and

$$J'(0+) = \int_D \left[\partial_\varepsilon \Phi(0, x) + \text{div}(\Phi(0, x)V(x)) \right] dx.$$

If, in addition, D is open with Lipschitz boundary, then

$$J'(0+) = \int_D \partial_\varepsilon \Phi(0, x) dx + \int_{\partial D} \Phi(0, x) V \cdot \nu d\sigma.$$

The following proposition is a direct consequence of Lemma 5.1.

Proposition 5.2. Assume that D is open with Lipschitz boundary. Then the shape derivative χ'_D of χ_D satisfies

$$\langle \chi'_D, \varphi \rangle_{H^{-1}(G), H_0^1(G)} = \int_{\partial D} \varphi V \cdot \nu d\sigma, \quad \forall \varphi \in H_0^1(G). \quad (5.2)$$

6 Shape sensitivity analysis

In this section, we derive a local Eulerian derivative formula for smooth interior shapes. This local sensitivity analysis is carried out on a smooth subclass of admissible shapes and is not claimed to be an Euler–Lagrange formula for arbitrary finite-perimeter minimizers obtained in Theorem 4.5. Throughout the section, unless otherwise stated, we assume that $D_0, D_1 \subset\subset G$ are open with C^2 boundaries and that $V_0, V_1 \in C_c^1(G; \mathbb{R}^n)$. Let ν_i be the unit outward normal vector on ∂D_i , and let $\rho_i = \text{div}_{\partial D_i} \nu_i$ be the scalar mean curvature, $i = 0, 1$. When this local formula is used for the constrained class $\mathcal{A}_K$, the perturbations are restricted so that $D_i^\varepsilon \subset K$ for all sufficiently small $\varepsilon > 0$.

For $i = 0, 1$ and $|\varepsilon|$ small, define $F_\varepsilon^i(x) = x + \varepsilon V_i(x)$ and $D_i^\varepsilon = F_\varepsilon^i(D_i)$. Since $D_i \subset\subset G$, we have $D_i^\varepsilon \subset\subset G$ for all sufficiently small $\varepsilon > 0$.

Let $u = u^{D_0, D_1}$ be the solution to (1.1) associated with (D_0, D_1) . From (2.10), we set

$$r := \frac{\partial u}{\partial \nu} \Big|_\Gamma - g \in L_{\mathbb{R}}^2(0, T; L^2(\Gamma)), \quad (6.1)$$

and $\tilde{r} = r(t, x)$ on Γ and $\tilde{r} = 0$ on $\partial G \setminus \Gamma$.

Consider the following adjoint equation:

$$\begin{cases} dp = -(\Delta p + ap + bq) dt + q dW(t) & \text{in } (0, T) \times G, \\ p = \tilde{r} & \text{on } (0, T) \times \partial G, \\ p(T) = 0 & \text{in } G. \end{cases} \quad (6.2)$$

In the following definition, the forward test state is the strong solution provided by Theorem 2.1.

Definition 6.1. A pair of processes $(p, q) \in L^2_{\mathbb{F}}(0, T; L^2(G)) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G))$ is called a transposition solution to (6.2) if, for every $\Phi \in L^2_{\mathbb{F}}(0, T; L^2(G))$ and every $\Psi \in L^2_{\mathbb{F}}(0, T; H^1_0(G))$, one has

$$\mathbb{E} \int_0^T \int_G p \Phi dx dt + \mathbb{E} \int_0^T \langle q, \Psi \rangle_{H^{-1}(G), H^1_0(G)} dt = -\mathbb{E} \int_0^T \int_{\Gamma} r \frac{\partial y^{\Phi, \Psi}}{\partial \nu} d\sigma dt, \quad (6.3)$$

where $y^{\Phi, \Psi}$ is the strong solution of (2.1) with zero initial datum and sources (Φ, Ψ) .

The following well-posedness result is an immediate corollary of the transposition solution theory for backward stochastic parabolic equations with nonhomogeneous boundary data [33, Theorem 7.14] and the regularity theory for stochastic parabolic equations [33, Theorem 4.11].

Proposition 6.1. For every $r \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma))$, there exists a unique transposition solution (p, q) to (6.2). Moreover, there exists a constant $C > 0$ such that

$$\|p\|_{L^2_{\mathbb{F}}(0, T; L^2(G))} + \|q\|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))} \leq C \|r\|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}. \quad (6.4)$$

Proof. Set $\mathfrak{X} := L^2_{\mathbb{F}}(0, T; L^2(G)) \times L^2_{\mathbb{F}}(0, T; H^1_0(G))$ equipped with the natural inner product

$$((\Phi_1, \Psi_1), (\Phi_2, \Psi_2))_{\mathfrak{X}} = \mathbb{E} \int_0^T \int_G \Phi_1 \Phi_2 dx dt + \mathbb{E} \int_0^T (\Psi_1, \Psi_2)_{H^1_0(G)} dt.$$

For $(\Phi, \Psi) \in \mathfrak{X}$, let $y^{\Phi, \Psi}$ be the strong solution of (2.1) with $y_0 = 0$. By Theorem 2.1 and the trace theorem,

$$\left\| \frac{\partial y^{\Phi, \Psi}}{\partial \nu} \right\|_{L^2_{\mathbb{F}}(0, T; L^2(\partial G))} \leq C (\|\Phi\|_{L^2_{\mathbb{F}}(0, T; L^2(G))} + \|\Psi\|_{L^2_{\mathbb{F}}(0, T; H^1_0(G))}).$$

Define a linear functional Λ_r on $\mathfrak{X}$ by

$$\Lambda_r(\Phi, \Psi) := -\mathbb{E} \int_0^T \int_{\Gamma} r \frac{\partial y^{\Phi, \Psi}}{\partial \nu} d\sigma dt.$$

The preceding estimate implies that Λ_r is bounded on $\mathfrak{X}$, with

$$\|\Lambda_r\|_{\mathfrak{X}'} \leq C \|r\|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}.$$

The Riesz type representation theorem ([33, Theorem 2.55]) yields unique elements $p \in L^2_{\mathbb{F}}(0, T; L^2(G))$ and $q \in L^2_{\mathbb{F}}(0, T; H^{-1}(G))$ such that

$$\Lambda_r(\Phi, \Psi) = \mathbb{E} \int_0^T \int_G p \Phi dx dt + \mathbb{E} \int_0^T \langle q, \Psi \rangle_{H^{-1}(G), H^1_0(G)} dt \quad \forall (\Phi, \Psi) \in \mathfrak{X}. \quad (6.5)$$

The estimate (6.4) follows directly from the operator norm bound of Λ_r and the isometric identification above. Uniqueness is immediate: if (p, q) is another solution, subtracting the corresponding transposition equations shows that $p = 0$ and $q = 0$ by testing against arbitrary $(\Phi, \Psi) \in \mathfrak{X}$. $\square$

Recall the definition of the Eulerian derivative of a shape functional in Definition 5.2. The main result of this section is the following explicit formula for the shape derivative of the cost functional $\mathcal{J}$, where

$$d\mathcal{J}(D_0, D_1; V_0, V_1) = \lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(D_0^\varepsilon, D_1^\varepsilon) - \mathcal{J}(D_0, D_1)}{\varepsilon}.$$

Theorem 6.2 (Shape derivative of the functional). For every $V_0, V_1 \in C^1_c(G; \mathbb{R}^n)$, the Eulerian derivative of $\mathcal{J}$ exists and is given by

$$\begin{aligned} d\mathcal{J}(D_0, D_1; V_0, V_1) = & -\mathbb{E} \int_0^T \int_{\partial D_0} p(t) V_0 \cdot \nu_0 d\sigma dt + \beta_0 \int_{\partial D_0} \rho_0 V_0 \cdot \nu_0 d\sigma \\ & - \mathbb{E} \int_0^T \int_{\partial D_1} q(t) V_1 \cdot \nu_1 d\sigma dt + \beta_1 \int_{\partial D_1} \rho_1 V_1 \cdot \nu_1 d\sigma, \end{aligned} \quad (6.6)$$

where (p, q) is the transposition solution of the adjoint equation (6.2).

The proof is organized according to the following strategy.

1. We identify the geometric source derivatives in $H^{-1}(G)$ and use Lemma 2.3 to control the corresponding boundary normal traces.
2. We prove the local H^1 regularity needed to give meaning to the traces of p and q on the moving interfaces.
3. We combine these two ingredients to prove Theorem 6.2.

6.1 Geometric source convergence

The following lemma is a consequence of the weak H^{-1} -differentiability of transported L^2 functions; see [39, Chapter 2, Proposition 2.39, pp. 72–74]. We include the proof for completeness.

Lemma 6.3 (Local convergence of a geometric source). *Let $D \subset\subset G$ be open with Lipschitz boundary, let $V \in C_c^1(G; \mathbb{R}^n)$, and let $\overline{D} \subset M \subset\subset G$, where M is an open bounded Lipschitz set. If $D_\varepsilon = (\text{Id} + \varepsilon V)(D)$ and $\xi_\varepsilon = (\chi_{D_\varepsilon} - \chi_D)/\varepsilon$, then there exists $\varepsilon_0 > 0$ such that $\text{supp } \xi_\varepsilon \subset\subset M$ for every $0 < \varepsilon < \varepsilon_0$. Moreover, when ξ_ε is regarded as an element of $H^{-1}(G)$ by*

$$\langle \xi_\varepsilon, \varphi \rangle_{H^{-1}(G), H_0^1(G)} := \int_G \xi_\varepsilon \varphi \, dx, \quad \varphi \in H_0^1(G),$$

one has

$$\|\xi_\varepsilon\|_{H^{-1}(G)} \leq C, \quad (6.7)$$

where C is independent of ε , and

$$\xi_\varepsilon \rightharpoonup \chi'_D \quad \text{weakly in } H^{-1}(G) \quad \text{as } \varepsilon \rightarrow 0+. \quad (6.8)$$

Proof of Lemma 6.3. Step 1. Support confinement. Let $\delta := \text{dist}(\overline{D}, \partial M) > 0$. Since $F_\varepsilon \rightarrow \text{Id}$ uniformly on $\overline{D}$, there exists $\varepsilon_0 > 0$ such that

$$\sup_{x \in \overline{D}} |F_\varepsilon(x) - x| < \frac{\delta}{2}, \quad 0 < \varepsilon < \varepsilon_0.$$

Hence $D_\varepsilon \subset\subset M$ for all $0 < \varepsilon < \varepsilon_0$, and therefore

$$\text{supp } \xi_\varepsilon \subset \overline{D_\varepsilon} \cup \overline{D} \subset\subset M.$$

Step 2. Uniform H^{-1} bound. For each fixed $\varepsilon \in (0, \varepsilon_0)$, $\xi_\varepsilon \in L^2(G)$ and consequently defines an element of $H^{-1}(G)$ via

$$\langle \xi_\varepsilon, \varphi \rangle_{H^{-1}(G), H_0^1(G)} := \int_G \xi_\varepsilon \varphi \, dx, \quad \varphi \in H_0^1(G).$$

Fix an arbitrary $\varphi \in H_0^1(G)$ and set $J_\varphi(\varepsilon) := \int_{D_\varepsilon} \varphi \, dx$. Since G is bounded, we have $H_0^1(G) \hookrightarrow W^{1,1}(G)$. Applying Lemma 5.1 with the integrand $\Phi(\varepsilon, x) = \varphi(x)$ (which is independent of ε) yields

$$\lim_{\varepsilon \rightarrow 0+} \langle \xi_\varepsilon, \varphi \rangle = J'_\varphi(0+) = \int_D \text{div}(\varphi V) \, dx. \quad (6.9)$$

Thus, for each $\varphi \in H_0^1(G)$, the numerical sequence $\{\langle \xi_\varepsilon, \varphi \rangle\}_{0 < \varepsilon < \varepsilon_0}$ is convergent and hence bounded. This establishes pointwise boundedness of the family $\{\xi_\varepsilon\}_{0 < \varepsilon < \varepsilon_0} \subset H^{-1}(G)$ on the Banach space $H_0^1(G)$. By the uniform boundedness principle,

$$\sup_{0 < \varepsilon < \varepsilon_0} \|\xi_\varepsilon\|_{H^{-1}(G)} < \infty, \quad (6.10)$$

which is precisely (6.7).

Step 3. Identification of the weak limit. It remains to prove (6.8). From (6.9) and Green's formula on the Lipschitz domain D , we obtain, for every $\varphi \in H_0^1(G)$,

$$\lim_{\varepsilon \rightarrow 0+} \langle \xi_\varepsilon, \varphi \rangle_{H^{-1}(G), H_0^1(G)} = \int_D \text{div}(\varphi V) \, dx = \int_{\partial D} \varphi V \cdot \nu \, d\sigma = \langle \chi'_D, \varphi \rangle_{H^{-1}(G), H_0^1(G)}.$$

The uniform bound (6.10) guarantees that every subsequence of $\{\xi_\varepsilon\}_{0 < \varepsilon < \varepsilon_0}$ contains a further subsequence converging weakly in $H^{-1}(G)$. By the uniqueness of the pointwise limit, every such weak limit must coincide with χ'_D . Consequently, the entire family converges weakly:

$$\xi_\varepsilon \rightharpoonup \chi'_D \quad \text{weakly in } H^{-1}(G) \quad \text{as } \varepsilon \rightarrow 0+.$$

This completes the proof of Lemma 6.3. $\square$

6.2 Interior regularity of the adjoint

Proposition 6.4. *Let $M \subset\subset G$ be an open set with Lipschitz boundary, and let (p, q) be the transposition solution of (6.2). Then $p|_M, q|_M \in L^2_{\mathbb{F}}(0, T; H^1(M))$ and*

$$\|p\|_{L^2_{\mathbb{F}}(0, T; H^1(M))} + \|q\|_{L^2_{\mathbb{F}}(0, T; H^1(M))} \leq C_M \|r\|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}. \quad (6.11)$$

Proof of Proposition 6.4. The $L^2(G)$ bound for p is already contained in (6.4). We divide the remaining proof into four steps.

Step 1. $L^2(M)$ -regularity of q . Let $\mathcal{D}_M$ denote the set of all finite linear combinations $\varphi(t, \omega, x) = \sum_{j=1}^N \alpha_j(t, \omega) \psi_j(x)$ ($N \in \mathbb{N}$), where each α_j is a bounded progressively measurable simple process and each $\psi_j \in C_c^\infty(M)$. The set $\mathcal{D}_M$ is dense in $L_{\mathbb{F}}^2(0, T; L^2(M))$. Taking $\Phi = 0$ and $\Psi = \varphi \in \mathcal{D}_M$ in (6.3) yields

$$\mathbb{E} \int_0^T \langle q, \varphi \rangle_{H^{-1}(G), H_0^1(G)} dt = -\mathbb{E} \int_0^T \int_{\Gamma} r \frac{\partial y^{0, \varphi}}{\partial \nu} d\sigma dt. \quad (6.12)$$

The strong solution $y^{0, \varphi}$ is also a mild solution. Applying Lemma 2.3 to this forward equation, we obtain

$$\left| \mathbb{E} \int_0^T \langle q, \varphi \rangle_{H^{-1}(G), H_0^1(G)} dt \right| \leq \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \left\| \frac{\partial y^{0, \varphi}}{\partial \nu} \right\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \leq C_M \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \|\varphi\|_{L_{\mathbb{F}}^2(0, T; L^2(M))}. \quad (6.13)$$

By density of $\mathcal{D}_M$ in $L_{\mathbb{F}}^2(0, T; L^2(M))$, the linear map $\varphi \mapsto \mathbb{E} \int_0^T \langle q, \varphi \rangle_{H^{-1}(G), H_0^1(G)} dt$ extends uniquely to a bounded linear functional on $L_{\mathbb{F}}^2(0, T; L^2(M))$. The Riesz representation theorem furnishes an element $\tilde{q} \in L_{\mathbb{F}}^2(0, T; L^2(M))$ such that

$$\mathbb{E} \int_0^T \langle q, \varphi \rangle_{H^{-1}(G), H_0^1(G)} dt = \mathbb{E} \int_0^T \int_M \tilde{q} \varphi dx dt \quad \forall \varphi \in \mathcal{D}_M. \quad (6.14)$$

By density, (6.14) holds for all $\varphi \in L_{\mathbb{F}}^2(0, T; L^2(M))$. Since the embedding $L^2(M) \hookrightarrow H^{-1}(M)$ is continuous and injective, the restriction $q|_M$ coincides with $\tilde{q}$ as an element of $L_{\mathbb{F}}^2(0, T; L^2(M))$. In the sequel we denote this $L^2(M)$ representative again by q , and the estimate

$$\|q\|_{L_{\mathbb{F}}^2(0, T; L^2(M))} \leq C_M \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \quad (6.15)$$

follows directly from (6.13) and the Riesz representation.

Step 2. Weak derivatives of p . Let $k \in \{1, \dots, n\}$ and let $\varphi \in \mathcal{D}_M$ be as above. Taking $\Phi = \partial_{x_k} \varphi$ and $\Psi = 0$ in (6.3) gives

$$\mathbb{E} \int_0^T \int_M p \partial_{x_k} \varphi dx dt = -\mathbb{E} \int_0^T \int_{\Gamma} r \frac{\partial y^{\Phi, 0}}{\partial \nu} d\sigma dt. \quad (6.16)$$

Since $\text{supp } \Phi \subset M$, applying Lemma 2.3 yields

$$\left| \mathbb{E} \int_0^T \int_M p \partial_{x_k} \varphi dx dt \right| \leq \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \left\| \frac{\partial y^{\Phi, 0}}{\partial \nu} \right\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \leq C_M \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \|\varphi\|_{L_{\mathbb{F}}^2(0, T; L^2(M))}. \quad (6.17)$$

Consequently, the linear functional $\varphi \mapsto -\mathbb{E} \int_0^T \int_M p \partial_{x_k} \varphi dx dt$ extends to a bounded linear functional on $L_{\mathbb{F}}^2(0, T; L^2(M))$. By the Riesz representation theorem, there exists $p_k \in L_{\mathbb{F}}^2(0, T; L^2(M))$ such that

$$-\mathbb{E} \int_0^T \int_M p \partial_{x_k} \varphi dx dt = \mathbb{E} \int_0^T \int_M p_k \varphi dx dt \quad \forall \varphi \in L_{\mathbb{F}}^2(0, T; L^2(M)). \quad (6.18)$$

We now verify that p_k is indeed the distributional derivative $\partial_{x_k} p$ on M . Fix $\psi \in C_c^\infty(M)$ and define the progressively measurable process

$$A_\psi(t, \omega) := \int_M p(t, \omega, x) \partial_{x_k} \psi(x) dx + \int_M p_k(t, \omega, x) \psi(x) dx, \quad (t, \omega) \in (0, T) \times \Omega.$$

Since $p, p_k \in L_{\mathbb{F}}^2(0, T; L^2(M))$, we have $A_\psi \in L^2(\Omega \times (0, T))$. Taking $\varphi(t, \omega, x) = \alpha(t, \omega) \psi(x)$ in (6.18), where α is an arbitrary bounded progressively measurable simple process, yields

$$\mathbb{E} \int_0^T \alpha(t, \omega) A_\psi(t, \omega) dt = 0.$$

Bounded progressively measurable simple processes are dense in $L_{\mathbb{F}}^2(0, T; \mathbb{R})$. Hence $A_\psi = 0$ for $(\mathbb{P} \otimes dt)$ -a.e. (ω, t) .

Now choose a countable subset $\{\psi_m\}_{m=1}^\infty \subset C_c^\infty(M)$ that is dense in $H_0^1(M)$. For each m , there exists a null set $\mathcal{N}_m \subset \Omega \times (0, T)$ such that $A_{\psi_m}(\omega, t) = 0$ for all $(\omega, t) \notin \mathcal{N}_m$. Set $\mathcal{N} := \bigcup_{m=1}^\infty \mathcal{N}_m$. Then $(\mathbb{P} \otimes dt)(\mathcal{N}) = 0$, and for every $(\omega, t) \notin \mathcal{N}$ we have

$$\int_M p(t, \omega, x) \partial_{x_k} \psi_m(x) dx = - \int_M p_k(t, \omega, x) \psi_m(x) dx \quad \forall m \geq 1.$$

For each such (ω, t) , both sides are continuous linear functionals of ψ with respect to the $H_0^1(M)$ -norm. By density of $\{\psi_m\}_{m=1}^\infty$ in $H_0^1(M)$, the identity extends to all $\psi \in H_0^1(M)$. In particular, $p_k = \partial_{x_k} p$ in the distributional sense on M , and

$$\|p_k\|_{L_{\mathbb{F}}^2(0, T; L^2(M))} \leq C_M \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))}. \quad (6.19)$$

Step 3. Weak derivatives of q . We repeat the argument of Step 2 for q . Take $\Phi = 0$ and $\Psi = \partial_{x_k} \varphi$ with $\varphi \in \mathcal{D}_M$ in (6.3). Using the $L^2(M)$ representative of q obtained in Step 1, we have

$$\mathbb{E} \int_0^T \int_M q \partial_{x_k} \varphi dx dt = - \mathbb{E} \int_0^T \int_\Gamma r \frac{\partial y^{0, \Psi}}{\partial \nu} d\sigma dt. \quad (6.20)$$

Applying Lemma 2.3 once more yields

$$\left| \mathbb{E} \int_0^T \int_M q \partial_{x_k} \varphi dx dt \right| \leq \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \left\| \frac{\partial y^{0, \Psi}}{\partial \nu} \right\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \leq C_M \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} \|\varphi\|_{L_{\mathbb{F}}^2(0, T; L^2(M))}. \quad (6.21)$$

By the same Riesz representation and density argument as in Step 2, there exists $q_k \in L_{\mathbb{F}}^2(0, T; L^2(M))$ such that $q_k = \partial_{x_k} q$ in the distributional sense on M , with

$$\|q_k\|_{L_{\mathbb{F}}^2(0, T; L^2(M))} \leq C_M \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))}.$$

Step 4. Conclusion. Summing the estimates from Steps 1–3 over $k = 1, \dots, n$ and combining with the $L^2(G)$ bound for p from (6.4), we obtain

$$\|p\|_{L_{\mathbb{F}}^2(0, T; H^1(M))} + \|q\|_{L_{\mathbb{F}}^2(0, T; H^1(M))} \leq C_M \|r\|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))},$$

which completes the proof of Proposition 6.4. $\square$

6.3 Extension of the transposition identity to interior sources

Lemma 6.5 (Extension of the transposition identity to interior sources). *Let $M \subset\subset G$ be open with Lipschitz boundary, and let $f_0, f_1 \in L_{\mathbb{F}}^2(0, T; H^{-1}(G))$ satisfy $\text{supp } f_i(t, \omega) \subset \bar{M}$, $i = 0, 1$, for a.e. (t, ω) . Let z be the mild solution of (2.2) with $z_0 = 0$. If $\eta \in C_c^\infty(G)$ satisfies $\eta = 1$ in a neighborhood of $\bar{M}$, then*

$$\mathbb{E} \int_0^T \int_\Gamma r \frac{\partial z}{\partial \nu} d\sigma dt = - \mathbb{E} \int_0^T \langle f_0, \eta p \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^T \langle f_1, \eta q \rangle_{H^{-1}(G), H_0^1(G)} dt. \quad (6.22)$$

Proof. Choose an open set M_1 such that $\bar{M} \subset\subset M_1 \subset\subset G$ and $\eta = 1$ on M_1 . Choose adapted smooth approximations $f_0^m, f_1^m \in L_{\mathbb{F}}^2(0, T; C_c^\infty(M_1))$ such that $f_i^m \rightarrow f_i$ in $L_{\mathbb{F}}^2(0, T; H^{-1}(G))$, $i = 0, 1$. This can be obtained by predictable simple-process approximation in time and spatial mollification inside M_1 . Let z^m be the solution of (2.2) with sources (f_0^m, f_1^m) and zero initial datum. For each m , the sources are admissible in Definition 6.1. Hence

$$\begin{aligned} \mathbb{E} \int_0^T \int_\Gamma r \frac{\partial z^m}{\partial \nu} d\sigma dt &= - \mathbb{E} \int_0^T \int_G p f_0^m dx dt - \mathbb{E} \int_0^T \langle q, f_1^m \rangle_{H^{-1}(G), H_0^1(G)} dt \\ &= - \mathbb{E} \int_0^T \langle f_0^m, \eta p \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^T \langle f_1^m, \eta q \rangle_{H^{-1}(G), H_0^1(G)} dt. \end{aligned}$$

Here the last equality uses $\text{supp } f_i^m \subset M_1$ and $\eta = 1$ on M_1 , together with Proposition 6.4.

By (2.3), $z^m \rightarrow z$ in $L_{\mathbb{F}}^2(0, T; L^2(G))$ and in $L_{\mathbb{F}}^2(\Omega; C([0, T]; H^{-1}(G)))$. Applying Lemma 2.3 to $z^m - z$ gives $\partial z^m / \partial \nu \rightarrow \partial z / \partial \nu$ in $L_{\mathbb{F}}^2(0, T; L^2(\partial G))$. Since $\eta p, \eta q \in L_{\mathbb{F}}^2(0, T; H_0^1(G))$ by Proposition 6.4, the two duality pairings on the right also converge. Passing to the limit proves (6.22). $\square$

6.4 Proof of the shape derivative formula

Theorem 6.6 (Shape derivative of the misfit). *For every $V_0, V_1 \in C_c^1(G; \mathbb{R}^n)$,*

$$d\mathcal{J}_1(D_0, D_1; V_0, V_1) = -\mathbb{E} \int_0^T \int_{\partial D_0} p(t) V_0 \cdot \nu_0 d\sigma dt - \mathbb{E} \int_0^T \int_{\partial D_1} q(t) V_1 \cdot \nu_1 d\sigma dt. \quad (6.23)$$

Here (p, q) denotes the transposition solution of (6.2).

Proof of Theorem 6.6. Let $0 < \varepsilon \ll 1$ and define

$$\xi_\varepsilon^0 := \frac{\chi_{D_0^\varepsilon} - \chi_{D_0}}{\varepsilon}, \quad \xi_\varepsilon^1 := \frac{\chi_{D_1^\varepsilon} - \chi_{D_1}}{\varepsilon}.$$

Let u_ε be the solution to (1.1) with (D_0, D_1) replaced by $(D_0^\varepsilon, D_1^\varepsilon)$, and set $w_\varepsilon := (u_\varepsilon - u)/\varepsilon$. Then w_ε solves

$$\begin{cases} dw_\varepsilon = (\Delta w_\varepsilon + a w_\varepsilon + \xi_\varepsilon^0) dt + (b w_\varepsilon + \xi_\varepsilon^1) dW(t) & \text{in } (0, T) \times G, \\ w_\varepsilon = 0 & \text{on } (0, T) \times \partial G, \\ w_\varepsilon(0) = 0 & \text{in } G. \end{cases} \quad (6.24)$$

Step 1. Uniform estimate for the linearized state. Choose open bounded Lipschitz sets M_0 and M such that $\overline{D_0} \cup \overline{D_1} \subset M_0 \subset\subset M \subset\subset G$. By Lemma 6.3 applied to D_0 and D_1 with the enclosing set M_0 , there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$,

$$\text{supp } \xi_\varepsilon^i \subset\subset M_0, \quad \|\xi_\varepsilon^i\|_{H^{-1}(G)} \leq C_{M_0}, \quad i = 0, 1.$$

In particular,

$$\|\xi_\varepsilon^i\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))} \leq T^{1/2} C_{M_0}, \quad i = 0, 1. \quad (6.25)$$

Applying Lemma 2.3 to the equation for w_ε yields

$$\left\| \frac{\partial w_\varepsilon}{\partial \nu} \right\|_{L_{\mathbb{F}}^2(0, T; L^2(\partial G))} \leq C_{M_0} \left(\|\xi_\varepsilon^0\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))} + \|\xi_\varepsilon^1\|_{L_{\mathbb{F}}^2(0, T; H^{-1}(G))} \right) \leq C'_{M_0}. \quad (6.26)$$

Step 2. Expansion of the cost functional. Using the definition of $\mathcal{J}_1$ and the fact that $u_\varepsilon = u + \varepsilon w_\varepsilon$, we expand

$$\begin{aligned} \frac{\mathcal{J}_1(D_0^\varepsilon, D_1^\varepsilon) - \mathcal{J}_1(D_0, D_1)}{\varepsilon} &= \frac{1}{2\varepsilon} \mathbb{E} \int_0^T \int_\Gamma \left(\left| \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial w_\varepsilon}{\partial \nu} - g \right|^2 - \left| \frac{\partial u}{\partial \nu} - g \right|^2 \right) d\sigma dt \\ &= \mathbb{E} \int_0^T \int_\Gamma \left(\frac{\partial u}{\partial \nu} - g \right) \frac{\partial w_\varepsilon}{\partial \nu} d\sigma dt + \frac{\varepsilon}{2} \mathbb{E} \int_0^T \int_\Gamma \left| \frac{\partial w_\varepsilon}{\partial \nu} \right|^2 d\sigma dt. \end{aligned}$$

By (6.26), the second term is $O(\varepsilon)$. Denoting $r := \frac{\partial u}{\partial \nu} - g \in L_{\mathbb{F}}^2(0, T; L^2(\Gamma))$, we obtain

$$\frac{\mathcal{J}_1(D_0^\varepsilon, D_1^\varepsilon) - \mathcal{J}_1(D_0, D_1)}{\varepsilon} = \mathbb{E} \int_0^T \int_\Gamma r \frac{\partial w_\varepsilon}{\partial \nu} d\sigma dt + o(1) \quad (\varepsilon \rightarrow 0+). \quad (6.27)$$

Step 3. Duality with the adjoint state. Choose $\eta \in C_c^\infty(M)$ with $\eta \equiv 1$ on a neighborhood of $\overline{M_0}$. By Proposition 6.4, the transposition solution (p, q) of (6.2) satisfies $p, q \in L_{\mathbb{F}}^2(0, T; H^1(M))$. Since $\text{supp } \eta \subset M \subset\subset G$, extending ηp and ηq by zero outside M yields elements of $L_{\mathbb{F}}^2(0, T; H_0^1(G))$.

We now invoke Lemma 6.5. This lemma states that for any $f_0, f_1 \in L^2(G)$ supported in M_0 , the following identity holds with the truncated adjoint states:

$$\mathbb{E} \int_0^T \int_\Gamma r \frac{\partial y^{f_0, f_1}}{\partial \nu} d\sigma dt = -\mathbb{E} \int_0^T \langle f_0, \eta p(t) \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^T \langle f_1, \eta q(t) \rangle_{H^{-1}(G), H_0^1(G)} dt. \quad (6.28)$$

Applying (6.28) with $f_0 = \xi_\varepsilon^0$, $f_1 = \xi_\varepsilon^1$ and noting that $y^{\xi_\varepsilon^0, \xi_\varepsilon^1} = w_\varepsilon$, we obtain

$$\mathbb{E} \int_0^T \int_\Gamma r \frac{\partial w_\varepsilon}{\partial \nu} d\sigma dt = -\mathbb{E} \int_0^T \langle \xi_\varepsilon^0, \eta p(t) \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^T \langle \xi_\varepsilon^1, \eta q(t) \rangle_{H^{-1}(G), H_0^1(G)} dt. \quad (6.29)$$

Step 4. Passage to the limit. We pass to the limit $\varepsilon \rightarrow 0+$ in the two duality terms. The arguments for the two terms are completely symmetric; we present the details for the term involving p , the term involving q being obtained by replacing $(D_0, \xi_\varepsilon^0, p, V_0, \nu_0)$ with $(D_1, \xi_\varepsilon^1, q, V_1, \nu_1)$.

Pointwise convergence. By (6.8), $\xi_\varepsilon^0 \rightharpoonup \chi'_{D_0}$ weakly in $H^{-1}(G)$ as $\varepsilon \rightarrow 0+$. For each fixed (t, ω) , $\eta p(t, \omega) \in H_0^1(G)$, and therefore

$$\lim_{\varepsilon \rightarrow 0+} \langle \xi_\varepsilon^0, \eta p(t, \omega) \rangle_{H^{-1}(G), H_0^1(G)} = \langle \chi'_{D_0}, \eta p(t, \omega) \rangle_{H^{-1}(G), H_0^1(G)}. \quad (6.30)$$

Domination. By (6.7) and the properties of η , we have the pointwise estimate

$$|\langle \xi_\varepsilon^0, \eta p(t, \omega) \rangle_{H^{-1}(G), H_0^1(G)}| \leq \|\xi_\varepsilon^0\|_{H^{-1}(G)} \|\eta p(t, \omega)\|_{H_0^1(G)} \leq C_{M_0} \|\eta p(t, \omega)\|_{H_0^1(G)} \leq C_M \|p(t, \omega)\|_{H^1(M)}. \quad (6.31)$$

To verify the integrability of the dominating function, we compute

$$\mathbb{E} \int_0^T \|p(t, \omega)\|_{H^1(M)} dt \leq T^{1/2} (\mathbb{E} \int_0^T \|p(t, \omega)\|_{H^1(M)}^2 dt)^{1/2} = T^{1/2} \|p\|_{L^2_\mathbb{F}(0, T; H^1(M))} < \infty, \quad (6.32)$$

where the finiteness follows from Proposition 6.4.

Identification of the limit. By (6.30), (6.31), (6.32) and Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon \rightarrow 0+} \mathbb{E} \int_0^T \langle \xi_\varepsilon^0, \eta p(t) \rangle_{H^{-1}(G), H_0^1(G)} dt = \mathbb{E} \int_0^T \langle \chi'_{D_0}, \eta p(t) \rangle_{H^{-1}(G), H_0^1(G)} dt. \quad (6.33)$$

Since $\eta \equiv 1$ on a neighborhood of $\overline{M}_0 \supset \partial D_0$, (5.2) and the trace theorem on the Lipschitz boundary ∂D_0 give

$$\langle \chi'_{D_0}, \eta p(t, \omega) \rangle_{H^{-1}(G), H_0^1(G)} = \int_{\partial D_0} p(t, \omega) V_0 \cdot \nu_0 d\sigma. \quad (6.34)$$

Applying exactly the same reasoning to the term involving q , we obtain

$$\lim_{\varepsilon \rightarrow 0+} \mathbb{E} \int_0^T \langle \xi_\varepsilon^1, \eta q(t) \rangle_{H^{-1}(G), H_0^1(G)} dt = \mathbb{E} \int_0^T \int_{\partial D_1} q(t) V_1 \cdot \nu_1 d\sigma dt. \quad (6.35)$$

Step 5. Conclusion. Combining (6.27), (6.29), (6.33)–(6.35), we obtain

$$\begin{aligned} d\mathcal{J}_1(D_0, D_1; V_0, V_1) &= \lim_{\varepsilon \rightarrow 0+} \frac{\mathcal{J}_1(D_0^\varepsilon, D_1^\varepsilon) - \mathcal{J}_1(D_0, D_1)}{\varepsilon} \\ &= -\mathbb{E} \int_0^T \int_{\partial D_0} p(t) V_0 \cdot \nu_0 d\sigma dt - \mathbb{E} \int_0^T \int_{\partial D_1} q(t) V_1 \cdot \nu_1 d\sigma dt. \end{aligned}$$

This completes the proof of Theorem 6.6. $\square$

Lemma 6.7 ([9, Theorem 4.3, p. 486]). *Let $D \subset\subset G$ be open with C^2 boundary, and let $V \in C_c^1(G; \mathbb{R}^n)$. Then the Eulerian derivative of $\mathcal{P}_G$ at D in the direction V is given by*

$$d\mathcal{P}_G(D; V) = \int_{\partial D} \rho V \cdot \nu d\sigma,$$

where ν is the outward unit normal to D and $\rho = \operatorname{div}_{\partial D} \nu$.

Proof of Theorem 6.2. Combining the decomposition $\mathcal{J} = \mathcal{J}_1 + \beta_0 \mathcal{P}_G(D_0) + \beta_1 \mathcal{P}_G(D_1)$ with Theorem 6.6 and applying Lemma 6.7 to (D_i, V_i) , $i = 0, 1$, gives (6.6). $\square$

7 Numerical method

This section constructs a finite-dimensional approximation of the shape-gradient flow induced by the Eulerian derivative in (6.6). Let $h > 0$, let $\Delta t = T/N_t$, and let $M = N_{MC}$ be the number of Monte Carlo samples. For polygonal shape pairs (D_0^h, D_1^h) , define the sample-average functional

$$\mathcal{J}_{h,M}(D_0^h, D_1^h) := \frac{1}{2M} \sum_{m=1}^M \sum_{n=0}^{N_t-1} \Delta t \sum_{x_j \in \Gamma_h} w_j \left| \partial_n^h u_m^{n,h}(x_j) - g_{m,j}^{\delta,n} \right|^2 + \beta_0^h \mathcal{P}_h(D_0^h) + \beta_1^h \mathcal{P}_h(D_1^h), \quad (7.1)$$

where w_j are the boundary quadrature weights, $\partial_n^h u$ is the recovered discrete normal flux, and $\mathcal{P}_h(D_i^h)$ is the polygonal perimeter. The parameters β_i^h are the effective discrete perimeter weights.

The discretization below specifies the state equation, the transposition adjoint equation, and the Riesz representation of the corresponding discrete shape derivative. It gives a finite-dimensional approximation

of the shape-gradient descent associated with (6.6). For a shape pair (D_0, D_1) , the residual r and its zero extension $\tilde{r}$ are defined by (6.1). The adjoint variable used in the descent method is the transposition solution (p, q) of (6.2) with boundary datum $\tilde{r}$. The local regularity result in Proposition 6.4 ensures that p and q admit H^1 -representatives in a neighborhood of $\partial D_0 \cup \partial D_1$. Hence the traces of p and q and the boundary integrals in (6.6) are well defined.

The finite element and least-squares Monte Carlo discretizations approximate the continuous state-adjoint optimality system on a common finite ensemble of Brownian paths. The same ensemble generates the synthetic boundary observations.

At the continuous level, a smooth critical shape pair is characterized by $d\mathcal{J}(D_0, D_1; V_0, V_1) = 0$ for all admissible (V_0, V_1) . For a descent method, the boundary form in (6.6) is represented through a Riesz map on a Hilbert space of vector fields on G . Set $\mathcal{H} := H_0^1(G; \mathbb{R}^n)$. Following the linear-elasticity metrics used in shape optimization [11, 38], the elasticity gradient flow is induced by the following bilinear form: for fixed Lamé parameters $\mu > 0$ and $\lambda \geq 0$,

$$a_{\text{el}}(V, W) := \int_G \left(2\mu \sum_{i,j=1}^n \varepsilon_{ij}(V) \varepsilon_{ij}(W) + \lambda \operatorname{div} V \operatorname{div} W \right) dx,$$

where $\varepsilon(V) := (\nabla V + \nabla V^\top)/2$.

The two independent velocity directions in the Eulerian derivative lead to the product metric on $\mathcal{H} \times \mathcal{H}$. The elasticity-smoothed descent direction $(\vartheta_0, \vartheta_1) \in \mathcal{H} \times \mathcal{H}$ is determined by the variational identity

$$a_{\text{el}}(\vartheta_0, W_0) + a_{\text{el}}(\vartheta_1, W_1) = -d\mathcal{J}(D_0, D_1; W_0, W_1), \quad \forall (W_0, W_1) \in \mathcal{H} \times \mathcal{H}. \quad (7.2)$$

The component ϑ_0 corresponds to variations of D_0 with D_1 fixed, and ϑ_1 corresponds to variations of D_1 with D_0 fixed. The product metric separates only the two components of the Riesz representation. However, the Eulerian derivative $d\mathcal{J}$ still couples the two supports through the state and adjoint variables.

By Korn's inequality, a_{el} is coercive on $\mathcal{H}$. Consequently, the product bilinear form on the left-hand side is coercive on $\mathcal{H} \times \mathcal{H}$, so the variational problem uniquely determines the pair $(\vartheta_0, \vartheta_1)$ associated with this elasticity metric. Substitution of $(W_0, W_1) = (\vartheta_0, \vartheta_1)$ into the variational identity yields

$$d\mathcal{J}(D_0, D_1; \vartheta_0, \vartheta_1) = -a_{\text{el}}(\vartheta_0, \vartheta_0) - a_{\text{el}}(\vartheta_1, \vartheta_1) \leq 0.$$

For a nonzero solution $(\vartheta_0, \vartheta_1)$, coercivity makes this quantity strictly negative, so $(\vartheta_0, \vartheta_1)$ is a first-order descent direction for the objective.

The shape-gradient descent framework is summarized in Algorithm 1, where $\mathcal{N}$ is the prescribed iteration count. Monte Carlo quadrature approximates the expectations in the misfit and the shape gradient by sample averages over N_{MC} Brownian paths. The LSMC approximation of the adjoint equation is based on the same paths.

Algorithm 1 Shape-gradient descent algorithm for reconstructing (D_0, D_1)

Require: Initial polygonal pair $(D_0^{h,0}, D_1^{h,0})$, observation boundary Γ , data g^δ , perimeter weights β_0, β_1 , curvature scaling c_κ , step parameters γ_0, η_γ , and iteration count $\mathcal{N}$

- 1: **for** $k = 0, 1, \dots, \mathcal{N} - 1$ **do**
 - 2: Solve the state equation (1.1).
 - 3: Solve the adjoint equation (6.2).
 - 4: Compute the shape derivative (6.6).
 - 5: Determine the elasticity descent direction (7.2).
 - 6: Update the polygonal interfaces by the deformation map $\text{Id} + \gamma_k \vartheta_i^h$ and regenerate the interface-fitted mesh for $(D_0^{h,k+1}, D_1^{h,k+1})$.
 - 7: **end for**
-

7.1 Discretization of the state equation

The state equation (1.1) is discretized in space by conforming $\mathbb{P}_1$ Lagrange finite elements. The finite element meshes form a family of interface-fitted triangular meshes $\{\mathcal{T}_h\}_{h>0}$. On these meshes, the polygonal interfaces ∂D_0 and ∂D_1 are represented by mesh edges. Let $V_h \subset H_0^1(G)$ be the conforming finite element space used for

variational solves with homogeneous Dirichlet condition. Let $\tilde{V}_h$ be the corresponding continuous piecewise linear space without the homogeneous boundary condition. This space provides Dirichlet lifts.

At each shape iteration the mesh is regenerated from the current polygonal interfaces. The discrete supports of the two characteristic sources are defined by the corresponding element classification. Elements in $D_0^h \cap D_1^h$ are included in both supports. The characteristic sources are discretized as elementwise constant functions associated with the polygonal sets D_i^h . Equivalently, their action on test functions $v_h \in V_h$ is represented through the $L^2(G)$ -duality pairing $(\chi_{D_i^h}, v_h)_{L^2(G)}$, $i = 0, 1$. For the prescribed target sources used to generate synthetic data, the characteristic functions are represented by nodal $\mathbb{P}_1$ interpolants in $\tilde{V}_h$.

The time interval is discretized by a uniform partition $0 = t_0 < \dots < t_{N_t} = T$, with $\Delta t = T/N_t$. The Brownian increments are defined by $\Delta W_n := W(t_{n+1}) - W(t_n)$, where $\Delta W_n \sim \mathcal{N}(0, \Delta t)$. The semi-implicit Euler–Maruyama discretization [23] reads

$$\begin{cases} u^{n+1} - \Delta t \Delta u^{n+1} = u^n + \Delta t (\chi_{D_0} + a^n u^n) + (b^n u^n + \chi_{D_1}) \Delta W_n & \text{in } G, \\ u^{n+1} = 0 & \text{on } \partial G. \end{cases} \quad (7.3)$$

Here $u^n \approx u(\cdot, t_n)$. On each Monte Carlo path, the coefficient fields are $a^n := a(t_n, \omega, \cdot)$ and $b^n := b(t_n, \omega, \cdot)$. The lower-order drift coefficient $a^n u^n$, the multiplicative-noise coefficient $b^n u^n$, and the stochastic source term χ_{D_1} are treated explicitly at the previous time level. The Laplacian is treated implicitly. The standard Galerkin projection of (7.3) onto V_h defines the fully discrete state equation. For the Monte Carlo approximation, the discrete state equation is solved for N_{MC} Brownian paths. The resulting state trajectories enter both the boundary residual and the terminal-to-initial adjoint discretization used below.

7.2 Discretization of the adjoint equation

The adjoint discretization uses the boundary residual associated with the discrete state. The boundary flux in the discrete cost is recovered from the variational time step. This residual-flux recovery avoids differentiating the piecewise linear state on boundary facets. Let Γ_h be the subset of boundary nodes lying on the observation boundary $\Gamma \subset \partial G$. For the m -th Brownian path, define the nodal boundary residual at time t_n by

$$r_{m,j}^n := \partial_n^h u_m^{n,h}(x_j) - g_{m,j}^{\delta,n}, \quad x_j \in \Gamma_h,$$

and denote its nodal zero extension to ∂G by $\tilde{r}_m^n$.

The adjoint discretization is based on the time grid of subsection 7.1 and the terminal condition $p^{N_t} = 0$. The time stepping follows standard BSDE and BSPDE schemes [26, 35, 46]. At each time level, adaptedness requires p^n and q^n to be $\mathcal{F}_{t_n}$ -measurable. The martingale integrand is determined through conditional expectations together with p^n . The Euler–Maruyama relation from t_{n+1} to t_n uses an implicit Laplacian and treats the lower-order terms explicitly:

$$p^{n+1} - p^n = -(\Delta p^n + a^n \bar{p}^{n+1} + b^n q^n) \Delta t + q^n \Delta W_n + \Delta M_{n+1} \quad \text{in } G, \quad (7.4)$$

where $\bar{p}^{n+1} := \mathbb{E}_n[p^{n+1}]$ and

$$\Delta M_{n+1} := p^{n+1} - \mathbb{E}_n[p^{n+1}] - q^n \Delta W_n, \quad \mathbb{E}_n[\Delta M_{n+1}] = \mathbb{E}_n[\Delta M_{n+1} \Delta W_n] = 0.$$

The remainder retains the martingale increment orthogonal to ΔW_n ; a general grid increment is not equal to $q^n \Delta W_n$ alone. The replacement $p^n \mapsto \bar{p}^{n+1}$ in the lower-order drift term is first-order consistent because $p^n = \mathbb{E}_n[p^{n+1}] + O(\Delta t)$. This discretization preserves the structure of the continuous adjoint equation $dp = -(\Delta p + ap + bq) dt + q dW(t)$. The implicit Laplacian guarantees unconditional stability for the terminal-to-initial time sweep.

The conditional-expectation framework for discrete-time BSDEs [3, 4] is then applied to (7.4). Multiplication by ΔW_n followed by conditioning on $\mathcal{F}_{t_n}$ extracts the martingale component. The resulting conditional-expectation system is

$$\begin{cases} q^n = \frac{1}{\Delta t} \mathbb{E}_n[p^{n+1} \Delta W_n] & \text{in } G, \\ p^n - \Delta t \Delta p^n = (1 + \Delta t a^n) \mathbb{E}_n[p^{n+1}] + \Delta t b^n q^n & \text{in } G, \\ p^n = \tilde{r}^n & \text{on } \partial G. \end{cases} \quad (7.5)$$

The Galerkin formulation of the second equation in (7.5) on V_h defines the fully discrete adjoint equation. The nonhomogeneous boundary datum is incorporated by a lift in $\tilde{V}_h$. The elliptic part has the same bilinear form as the state time step.

7.3 Least-squares Monte Carlo regression

The conditional expectations $\mathbb{E}_n[p^{n+1,h}]$ and $\mathbb{E}_n[p^{n+1,h}\Delta W_n]$ in (7.5) are approximated by least-squares projections onto a finite-dimensional regression space. For a pathwise quantity $X^{(m)}$, define

$$\widehat{\mathbb{E}}_n[X] := \operatorname{argmin}_{\phi \in \operatorname{span} \Phi_n} \frac{1}{M} \sum_{m=1}^M \left| X^{(m)} - \phi(\xi_n^{(m)}) \right|^2,$$

where $\xi_n^{(m)}$ is the vector of regression variables along the m -th Brownian path. The regression basis is

$$\Phi_n = (1, W(t_n), W(t_n)^2, \bar{u}_G^n, \bar{u}_{\partial D}^n)^\top \in \mathbb{R}^5, \quad (7.6)$$

where $\bar{u}_G^n$ is the finite element nodal average of the state over G . The interface average is

$$\bar{u}_{\partial D}^n := \frac{\int_{\partial D_0^h} u^{n,h} d\sigma_h + \int_{\partial D_1^h} u^{n,h} d\sigma_h}{|\partial D_0^h|_h + |\partial D_1^h|_h}.$$

Only nonempty interfaces are included in this average. The basis Φ_n depends on the current polygonal pair.

This regression basis is deliberately low-dimensional. The first three functions capture the leading dependence on the scalar Brownian history, while the two state averages summarize the forward trajectory in the bulk and near the moving interfaces. This choice is a variance-control compromise for the ensemble size used below ($M = N_{\text{MC}} = 200$), not an optimal one: increasing the basis dimension improves approximation capacity, but also makes the least-squares problems more ill-conditioned and increases sampling error. The numerical examples therefore use this fixed five-dimensional basis throughout.

The scaled variable $p^{n+1,h}\Delta W_n/\Delta t$ is the target of the second projection, so the approximation corresponds to the discrete martingale coefficient q^n . The LSMC approximation introduces a projection error from the chosen regression space and a sampling error from the finite ensemble. Under standard assumptions on the regression basis, the statistical error is of order K/M with $K = \dim \operatorname{span} \Phi_n$; the projection error is not estimated separately.

The adjoint time-stepping uses the same N_{MC} Brownian paths as the discrete state equation, with terminal condition $p^{N_t,h,(m)} = 0$ for $m = 1, \dots, N_{\text{MC}}$. These trajectories also generate the synthetic boundary observations; this common-random-numbers coupling reduces the Monte Carlo variance of the boundary residual. At each time level $n = N_t - 1, \dots, 0$, least-squares Monte Carlo regression with the basis Φ_n approximates the conditional expectations in (7.5). The resulting approximation of q^n enters the second equation of (7.5), and the finite element solution then determines $p^{n,h}$ on the sample ensemble.

7.4 Discrete shape gradient and mesh update

At each shape iteration the two interfaces are represented by closed polygonal curves ∂D_i^h with outward normals ν_i^h , $i = 0, 1$. The boundary densities in (6.6) are discretized by finite element traces of the adjoint variables and by the first variation of the polygonal perimeter, with effective discrete perimeter weights $\beta_i^h := c_\kappa \beta_i$. The scaling c_κ belongs to the discrete perimeter regularization and is not an additional term in the continuous shape derivative.

The time integral and expectation in the p - and q -terms are approximated by time quadrature and Monte Carlo averaging. The resulting boundary densities are of the form

$$\mathcal{G}_0^h = -\frac{1}{M} \sum_{m=1}^M \sum_{n=0}^{N_t-1} \Delta t p_m^{n,h} \big|_{\partial D_0^h} + \beta_0^h \kappa_0^h,$$

and

$$\mathcal{G}_1^h = -\frac{1}{M} \sum_{m=1}^M \sum_{n=0}^{N_t-1} \Delta t q_m^{n,h} \big|_{\partial D_1^h} + \beta_1^h \kappa_1^h,$$

where κ_i^h is the discrete curvature of ∂D_i^h . The density $\mathcal{G}_0^h$ is associated with the adjoint state component p . The density $\mathcal{G}_1^h$ is associated with the martingale component q . These densities define an approximation to the continuous shape derivative, denoted by

$$\ell_h(V_0^h, V_1^h) := \int_{\partial D_0^h} \mathcal{G}_0^h V_0^h \cdot \nu_0^h d\sigma_h + \int_{\partial D_1^h} \mathcal{G}_1^h V_1^h \cdot \nu_1^h d\sigma_h. \quad (7.7)$$

The quantities $\mathcal{G}_i^h$ are scalar boundary densities of ℓ_h ; they are not themselves interface velocities. This optimize-then-discretize approximation is not identified with the exact derivative of the discrete objective

(7.1). The algorithm obtains an interface update field from the boundary functional using the finite element counterpart of the product-space elasticity metric in (7.2). The finite-dimensional deformation space is $\mathcal{H}_h := [V_h]^d \subset \mathcal{H}$. It is the vector-valued counterpart of the scalar finite element space used for the state and adjoint equations. The positive constants s_i are interpreted as weights in the product-space Riesz metric:

$$a_{\text{el},s}((V_0, V_1), (W_0, W_1)) := s_0^{-1} a_{\text{el}}(V_0, W_0) + s_1^{-1} a_{\text{el}}(V_1, W_1).$$

The choice $s_i = 1$ gives the unweighted product metric. Other positive values modify the metric representing the same functional ℓ_h . With full-boundary observation, the boundary density is not modified: $\tilde{\mathcal{G}}_i^h = \mathcal{G}_i^h$. The discrete update pair $(\vartheta_0^h, \vartheta_1^h) \in \mathcal{H}_h \times \mathcal{H}_h$ is defined by the following identity: for all $(W_0^h, W_1^h) \in \mathcal{H}_h \times \mathcal{H}_h$,

$$s_0^{-1} a_{\text{el}}(\vartheta_0^h, W_0^h) + s_1^{-1} a_{\text{el}}(\vartheta_1^h, W_1^h) = - \int_{\partial D_0^h} \tilde{\mathcal{G}}_0^h W_0^h \cdot \nu_0^h d\sigma_h - \int_{\partial D_1^h} \tilde{\mathcal{G}}_1^h W_1^h \cdot \nu_1^h d\sigma_h. \quad (7.8)$$

With partial-boundary observation, a positive spatial boundary weight is applied to the right-hand side: $\tilde{\mathcal{G}}_i^h = \omega_i^{(k)} \mathcal{G}_i^h$. The factor $\omega_i^{(k)}$ changes the discrete update field through the Riesz map. It is not part of the continuous Eulerian derivative in (6.6). The distance quantities are $d_\Gamma(x) := \text{dist}(x, \Gamma)$ and $\bar{d}_i^{(k)} := \max_{y \in \partial D_i^{h,k}} d_\Gamma(y)$. The boundary preconditioner is defined by

$$\omega_i^{(k)}(x) := \max \left\{ \omega_{\min}, \left(\frac{d_\Gamma(x)}{\bar{d}_i^{(k)}} \right)^{p_i} \right\}, \quad x \in \partial D_i^{h,k}. \quad (7.9)$$

This spatially weighted update is heuristic: positivity of $\omega_i^{(k)}$ does not imply $\ell_h(\vartheta_0^h, \vartheta_1^h) \leq 0$, since multiplication by $\omega_i^{(k)}$ need not commute with the inverse elasticity operator. With $\omega_i^{(k)} \equiv 1$, the Riesz identity gives $\ell_h(\vartheta_0^h, \vartheta_1^h) = - \sum_i s_i^{-1} a_{\text{el}}(\vartheta_i^h, \vartheta_i^h) \leq 0$. Neither identity alone guarantees decrease of (7.1), whose exact derivative differs from ℓ_h . Let $\gamma_k > 0$ be the accepted shape-update step length. The polygonal interfaces are updated by the deformation map

$$\partial D_0^{h,k+1} = (\text{Id} + \gamma_k \vartheta_0^h)(\partial D_0^{h,k}), \quad \partial D_1^{h,k+1} = (\text{Id} + \gamma_k \vartheta_1^h)(\partial D_1^{h,k}). \quad (7.10)$$

The sign convention is that of (7.2): the velocity ϑ_i^h already contains the negative sign from the Riesz representation and is used as the update field, subject to the qualifications above. Equivalently, one may solve the positive Riesz problem for $z_i^h = -\vartheta_i^h$ and update the interface nodes by $x \leftarrow x - \gamma_k z_i^h(x)$; this is only a change of notation. Only geometrically admissible candidate interfaces are retained.

The finite element mesh is then regenerated so that both polygonal interfaces are represented by mesh edges. This avoids relying on a global elastic mesh deformation for two independently moving internal interfaces, whose close approach or intersection may degrade element quality. The resulting remeshing cost is a limitation of the present discretization, especially for three-dimensional extensions. The accepted shape-update step length has the form $\gamma_k = \gamma_0 \eta_\gamma^k 2^{-m_k}$, where $\eta_\gamma \in (0, 1]$ is the decay factor and $m_k \geq 0$ is selected by the geometric admissibility condition.

8 Numerical experiments

This section presents numerical reconstructions using the discrete shape-gradient method of section 7. The examples illustrate how the drift and diffusion source roles in (6.6) separate under representative boundary observations.

Let $\Gamma_{\text{bottom}} := \partial G \cap \{x_2 = 0\}$ be the bottom side of G . The observation boundaries are $\Gamma^{\text{full}} := \partial G$ and $\Gamma^{\text{part}} := \partial G \setminus \Gamma_{\text{bottom}}$. The set Γ^{part} consists of the top, left, and right sides of the unit-square boundary. In the partial-data cases, the descent equation includes the boundary preconditioner defined in (7.9).

All reconstructions use $G = (0, 1)^2$, initial datum $u_0 = 0$, mesh parameter $h \approx 1/50$, final time $T = 1$ discretized into $N_t = 100$ uniform time steps, $M = N_{\text{MC}} = 200$ Monte Carlo samples, and iteration count $\mathcal{N} = 50$. The finite element discretization follows section 7 and is implemented in FEniCSx/DOLFINx [2]. The coefficients in (1.1) are prescribed by $a(t, \omega, x) = 0.1 \tanh(W(t, \omega)) \sin(\pi x_1)$ and $b \equiv 0.5$. For the synthetic-data experiments, the exact source supports used to generate the data are denoted by $(D_0^\dagger, D_1^\dagger)$. For each configuration, g_h denotes the noise-free discrete Neumann trace of the state associated with this target pair on the observation grid. The noisy data used in (7.1) are generated by adding a centered Gaussian perturbation,

$$g_h^{\delta_{\text{nom}}} = g_h + \delta_{\text{nom}} \|g_h\|_{Y_h} Z_h,$$

where the entries of Z_h are independent standard normal random variables and $\|\cdot\|_{Y_h}$ denotes the discrete version of the $L^2_{\mathbb{F}}(0, T; L^2(\Gamma))$ observation norm. The entries $g_{m,j}^{\delta,n}$ in (7.1) are the components of $g_h^{\delta_{\text{nom}}}$. Unless otherwise stated, the reconstructions below use $\delta_{\text{nom}} = 0.01$. The reconstruction configurations are summarized in Table 1, where $B_\rho(c)$ is the open disk with center c and radius ρ .

observation	configuration	target supports	initial geometries
Γ^{full}	single drift support	$D_0^\dagger = B_{0.1}(0.5, 0.5), D_1^\dagger = \emptyset$	$D_0^{h,0} = [0.25, 0.75]^2$
Γ^{full}	single diffusion support	$D_0^\dagger = \emptyset, D_1^\dagger = B_{0.1}(0.5, 0.5)$	$D_1^{h,0} = [0.25, 0.75]^2$
Γ^{part}	single drift support	$D_0^\dagger = B_{0.1}(0.5, 0.5), D_1^\dagger = \emptyset$	$D_0^{h,0} = [0.25, 0.75]^2$
Γ^{part}	single diffusion support	$D_0^\dagger = \emptyset, D_1^\dagger = B_{0.1}(0.5, 0.5)$	$D_1^{h,0} = [0.25, 0.75]^2$
Γ^{full}	two overlapping supports	$D_0^\dagger = B_{0.1}(0.45, 0.5),$ $D_1^\dagger = B_{0.1}(0.55, 0.5)$	$D_0^{h,0} = [0.220, 0.700] \times [0.260, 0.740],$ $D_1^{h,0} = [0.330, 0.810] \times [0.260, 0.740]$

Table 1: Reconstruction configurations and initial geometries.

The optimization parameters were selected empirically from preliminary computations and then fixed for the reconstructions reported below. These choices belong to the discrete descent scheme and are not part of the continuous shape derivative or the analytical results. The elasticity Riesz map and shape update use the parameters in Table 2. The values s_0 and s_1 are metric weights in the product-space Riesz representation. The parameter c_κ defines the effective discrete perimeter weights $\beta_i^h = c_\kappa \beta_i$. The boundary preconditioner $\omega_i^{(k)}$ appears only for partial boundary observations.

configuration	μ, λ	β_0, β_1	γ_0	η_γ	c_κ	additional parameters
single support, Γ^{full}	1, 5	0.005, 0.005	2.7	0.95	9.995	$s_0 = 2.0$ for drift, $s_1 = 0.84$ for diffusion
single support, Γ^{part}	1, 5	0.005, 0.005	2.4	0.953	10	$s_0 = 2.0$ for drift, $s_1 = 1.45$ for diffusion, $\omega_{\min} = 0.4, p_0 = 0, p_1 = 1.5$
overlapping supports, Γ^{full}	1, 5	0.005, 0.005	2.7	0.95	10	$s_0 = 1.2, s_1 = 0.85$

Table 2: Optimization and Riesz-metric parameters.

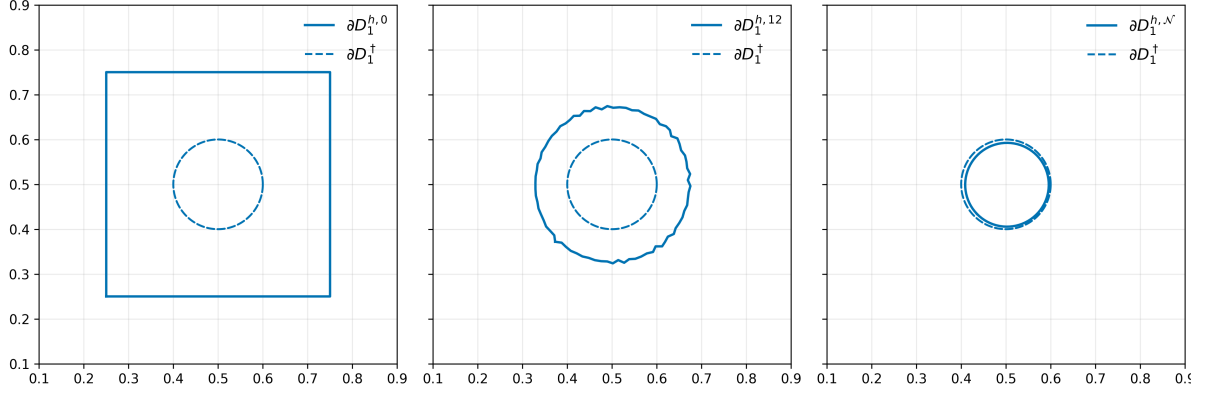
The reconstructions indicate that the two components of the discrete boundary derivative separate the two source channels. Figure 1 combines the geometric evolution and an archived cost diagnostic for the reconstruction of the diffusion support $D_1^\dagger$ with $\Gamma = \Gamma^{\text{full}}$. In Figure 1a, $\partial D_1^\dagger$ is the target interface used to generate $g_h^{\delta_{\text{nom}}}$, and $\partial D_1^{h,k}$ is the reconstructed polygonal interface after k iterations. The polygonal interface moves toward the prescribed support. The curve in Figure 1b cannot be identified with (7.1) using the reported parameters. Indeed, $\beta_1^h = 9.995 \times 0.005 = 0.049975$; at $k = 12$ the displayed support has horizontal width greater than 0.2, hence polygonal perimeter greater than 0.4. The nonnegative misfit then gives $\mathcal{J}_{h,M} > 0.01999$, whereas the plotted value is below 0.01. The normalization and penalty content of this archived curve are not documented in the available record, so its symbol J is retained only as a historical label. Its decrease is not evidence of decrease of the defined discrete objective.

Figure 2 compares the final polygonal reconstructions by observation regime. With $\Gamma = \Gamma^{\text{full}}$, the single drift-support case and the joint reconstruction recover the prescribed geometric scale up to the discretization scale; in the joint case, the two reconstructed interfaces remain distinguishable. This is consistent with the separate boundary terms involving p and q in (6.6). With $\Gamma = \Gamma^{\text{part}}$, removing the bottom side mainly affects localization in the direction associated with the unobserved boundary. The final supports nevertheless retain the correct qualitative scale for both the drift and diffusion source roles.

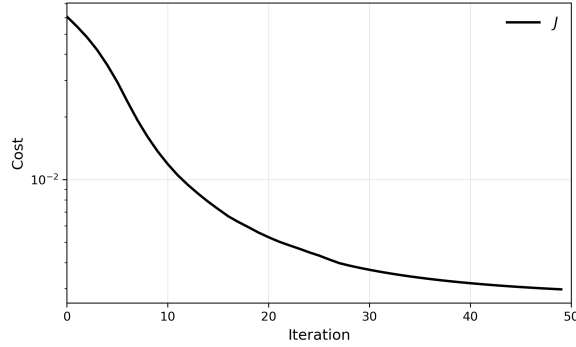
Together, the interface evolution and final reconstruction comparisons show that the discrete method can separate the two source channels under both full and partial boundary observations.

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(a) Interface evolution for $k = 0, 12, 50$.

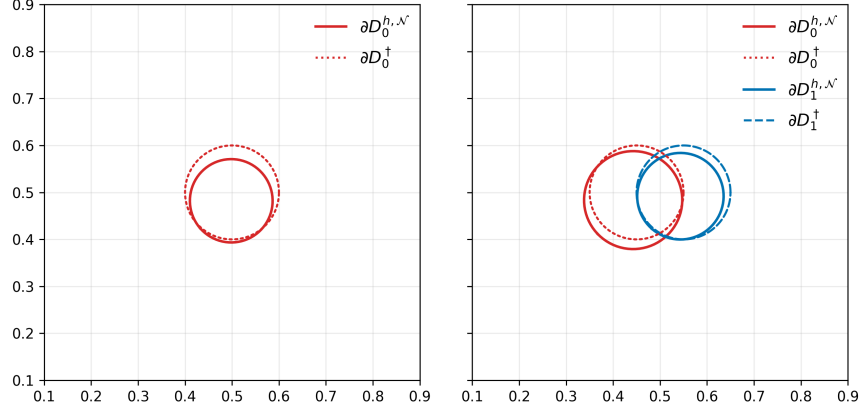


(b) Archived cost diagnostic; the plotted J is a historical label (see text).

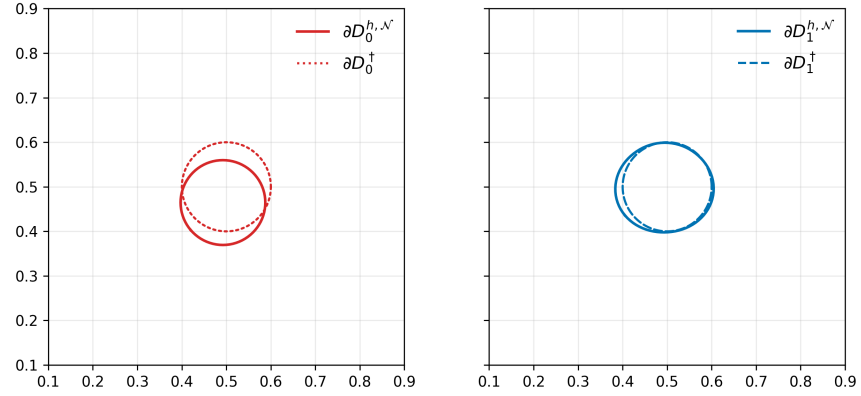
Figure 1: Full-boundary reconstruction of the diffusion support $D_1^\dagger$. The solid curves in (a) show the reconstructed interfaces $\partial D_1^{h,k}$, and the dashed curve shows the target interface $\partial D_1^\dagger$.

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(a) Full-boundary observation.



(b) Partial-boundary observation with the bottom side excluded.

Figure 2: Final polygonal reconstructions under the two observation regimes. Panel (a) includes the single drift-support reconstruction and the joint reconstruction of the target pair $(D_0^\dagger, D_1^\dagger)$; panel (b) shows the single drift- and diffusion-support reconstructions from partial boundary data.

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