

Inverse problems for stochastic partial differential equations

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Chapter 1

Introduction

Abstract This chapter introduces the motivation and main challenges of inverse problems for stochastic partial differential equations. The central questions addressed in this book are formulated and discussed.

1.1 Why should we study inverse problems for SPDEs?

Physical theories enable us to make predictions: given a complete description of a physical system, we can predict the outcomes of certain measurements. This process of predicting measurement results is commonly referred to as the “direct problem”. In contrast, the concept of an “inverse problem” involves deducing the values of the system’s parameters from actual measurements.

A general formulation of an inverse problem can be described as follows: Let U and V be two topological spaces, and $F : U \rightarrow V$ a given mapping. For a given input $u \in U$, computing $F(u)$ is called the direct problem. Conversely, for a given $v \in V$, finding an unknown input $u \in U$ such that $F(u) = v$ is called the inverse problem.

Before going to more details, let us recall the definition of the well-posedness by Hadamard:

Definition 1.1.1. Consider the equation

$$F(u) = v, \quad u \in U, \ v \in V. \quad (1.1.1)$$

The equation (1.1.1) is called well-posed in the sense of Hadamard if the following three conditions are satisfied:

- (i) For any $v \in V$, there exists a solution $u \in U$ such that $F(u) = v$.
- (ii) This solution is unique.
- (iii) The solution $u = F^{-1}(v)$ depends continuously on v in V .

If at least one of these conditions is failed, the problem is called ill-posed.

Remark 1.1.1. If (i) and (ii) are satisfied, we know there is a unique solution to the equation (1.1.1). However, for practical purposes, the necessity of condition (iii) becomes apparent. Indeed, in real-world scenarios, the exact value of v is often unattainable, and we must rely on an approximate value $\tilde{v}$ instead. Failure to meet condition (iii) can lead to a significant deviation between the solution $\tilde{u}$ of the equation $F(\tilde{u}) = \tilde{v}$ and the desired solution u . Therefore, the continuity of the inverse mapping F^{-1} plays a critical role in practical applications.

Inverse problems typically arise in situations where the parameters of a physical device or system cannot be measured directly. Instead, these parameters must be inferred from measurements of the effects they cause. A classic example is determining the electrical resistance of a material. This is done by measuring the voltage applied to the material and the current flowing through it (or the flux on its surface), and then applying Ohm's Law to deduce the resistance.

Over recent decades, the theory of inverse problems, particularly those involving partial differential equations (PDEs), has developed substantially, evolving into a major branch of applied mathematics (see, e.g., [19, 30, 35–37, 41, 43, 49, 65, 66, 71]). This growth has, in turn, stimulated significant advances in related areas such as partial differential equations, functional analysis, and numerical analysis.

However, many such inverse problems fail to satisfy Conditions (i)–(iii) in Definition 1.1.1. As a result, most inverse problems are inherently ill-posed in nature.

A variety of methods and techniques have been developed to address inverse problems for partial differential equations. Among these, the Carleman estimate stands out as one of the most powerful and widely used tools. Due to the extensive body of literature on this subject, we refrain from providing an exhaustive list here. For a comprehensive overview of the application of Carleman estimates to various inverse problems for PDEs, we refer the reader to [6, 7, 12, 26, 38, 71] and the numerous references therein.

Due to the complexity of real-world systems, many phenomena in fields such as chemistry, biology, microelectronics, pharmaceuticals, communications, and signal transmission require characterization through SPDEs for more accurate modeling (see, e.g., [11, 15, 33, 40, 62]). Consequently, the study of inverse problems for SPDEs arises naturally from practical needs. Moreover, such inverse problems are closely connected to other significant areas, including data assimilation and Kalman filtering for SPDEs (e.g., [39, 42]).

Next, we present an example of inverse problems for SPDEs.

Example 1.1.1. In interest rate modeling, the instantaneous forward rate $f(t, T)$ represents the rate agreed upon at time t for an infinitesimal loan at future time T . A widely used approach to describe the evolution of the forward rate curve is the Musiela parametrization, $r(t, x) = f(t, t + x)$, where $x = T - t$ denotes the time-to-maturity. This formulation allows us to treat the forward rate curve $x \mapsto r(t, x)$ as a single state variable that evolves over time.

To better capture the dynamics, the evolution of this curve can be further decomposed into deterministic the shape function and the random deviation. A specific model (e.g., [14]) for this is given by

$$r(t, x) = r(t, 0) + s(t)[u(x) + y(t, x)],$$

where $t \in [0, T]$ and $x \in [0, L]$. Here, L is the longest maturity available on the market. The curve's evolution is described relative to a short rate $r(t, 0)$ and scaled by a spread $s(t) = r(t, L) - r(t, 0)$. The shape of the curve is split into a deterministic average profile, $u(x)$, and a stochastic component, $y(t, x)$. This process $y(t, x)$ captures the random deviations of the term structure from its long-term average shape. We normalize $u(0) = 0$ and $u(L) = 1$ and impose $y(t, 0) = y(t, L) = 0$, consistently with the endpoint values of r .

The stochastic component y is modeled by the following stochastic parabolic equation (see [14]):

$$\begin{cases} dy = (\kappa y_{xx} + y_x)dt + \sigma(t, x)dW(t) & \text{in } (0, T) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{for } t \in (0, T), \\ y(0) = y_0 & \text{in } (0, L), \end{cases}$$

where κ is a positive constant and $\sigma(t, x)$ is the volatility function.

Model calibration seeks to use observed forward rates (measurement data) from the market to infer the volatility function $\sigma(t, x)$. This constitutes an inverse problem, and solving it is essential for effective risk management and hedging strategies. Studying of this inverse problem can be found in [Section 3.11](#).

In addition to their wide range of applications, inverse problems for SPDEs possess unique features that distinguish them from their deterministic counterparts. Some problems are inherently stochastic and different from the deterministic case.

To illustrate this, let us consider the following example.

Example 1.1.2. Let $T > 0$ and $G \subset \mathbb{R}^n$ be a given bounded domain with a C^2 boundary Γ . Put $Q = (0, T) \times G$ and $\Sigma = (0, T) \times \Gamma$. Consider the following inverse problem for the deterministic wave equation: Suppose y satisfies

$$\begin{cases} y_{tt} - \Delta y = f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_t(0) = 0 & \text{in } G. \end{cases} \quad (1.1.2)$$

Is it possible to uniquely determine f from the measurements $\frac{\partial y}{\partial \nu}|_{\Sigma}$ and $y(T)$? Here, ν denotes the outward normal vector on the boundary Γ . In fact, the answer is clearly negative.

If the answer were affirmative, then

$$\frac{\partial y}{\partial \nu}|_{\Sigma} = 0, \quad y(T) = 0 \implies f = 0 \text{ in } Q. \quad (1.1.3)$$

However, for any $y^* \in C_0^\infty(Q)$, let $f = y_{tt}^* - \Delta y^*$. Then y^* solves (1.1.2), and

$$\frac{\partial y^*}{\partial \nu} \Big|_{\Sigma} = 0, \quad y^*(T) = 0.$$

Thus, by (1.1.3), $f = 0$ in Q . But we can choose y^* so that f does not vanish in Q , leading to a contradiction.

Next, let us consider the analogous inverse problem for the stochastic case: Suppose z satisfies

$$\begin{cases} dz_t - \Delta z dt = g dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_t(0) = 0 & \text{in } G. \end{cases} \quad (1.1.4)$$

Is it possible to uniquely determine g from the measurements $\frac{\partial z}{\partial \nu} \Big|_{\Sigma}$ and $z(T)$? In fact, the answer is affirmative: one can show that $\frac{\partial z}{\partial \nu} \Big|_{\Sigma} = 0$ and $z(T) = 0$ implies $g = 0$ in Q , $\mathbb{P}$ -a.s. Details can be found in [Section 4.6](#).

The above example illustrates us that the formulation of inverse problems for SPDEs can differ significantly from their deterministic counterparts. The key insight is that the stochastic noise, rather than being merely a complication, actually provides additional structural information through the quadratic variation terms, which makes certain inverse problems solvable in the stochastic setting that are impossible to solve deterministically.

Mathematically, inverse problems for SPDEs are relatively new and quite challenging. Compared to the extensive literature on deterministic inverse problems for PDEs, there is relatively less research on inverse problems for SPDEs. In our view, the main reasons are as follows:

- Compared to deterministic PDEs, much less is known about SPDEs, although many progresses have been made there in recent years.
- The formulation of inverse problems for SPDEs may differ fundamentally from that of PDEs, with some problems being genuinely stochastic and cannot be reduced to any deterministic ones. Some inverse problems may even be meaningful only within a stochastic setting (e.g. [Example 1.1.2](#)).
- Many powerful methods and tools applicable to solving inverse problems for deterministic PDEs cannot be directly applied to the study of inverse problems for SPDEs. For instance, due to the almost surely non-differentiability of sample paths of Brownian motion with respect to the time variable, solutions to SPDEs may not be continuously differentiable with respect to the time variable. Furthermore, many regularity estimates and extension theorems valid for deterministic PDEs no longer apply to SPDEs. Therefore, even for very simple SPDEs, it is required to present new mathematical techniques or even fundamentally new approaches.

1.2 Formulation of inverse problems for SPDEs

Similar to the formulation of inverse problems for deterministic PDEs, for inverse problems of SPDEs, we aim to determine unknown components within the system through some measurements derived from the solution of the equations. These inverse problems of SPDEs can be broadly categorized into the following four types based on the specific unknown components to be determined:

- (i) Inverse coefficient problems, i.e., to determine an unknown coefficient (usually a function) in an SPDE in such a way that the solution of the equation satisfies specific conditions or data. In other words, the aim is to infer the parameters or coefficients that lead to the observed phenomena. These problems have wide applications in science and engineering, such as in molecular biology for determining reaction constants [63], in astrophysics for reconstructing the internal structure of stars or planets [16], in finance for calibration of financial models [5], and in medical imaging for recovering tissue properties [45].
- (ii) Inverse initial value/state problems, which refer to the mathematical problems where the goal is to determine unknown initial value of the SPDEs based on observed data at a later time or in a time interval. Such problems are significant in many industrial and scientific applications where the initial state or the whole solution of an SPDE cannot be directly observed. Typical examples include ocean wave prediction [72] and state reconstruction in financial mathematics [5].
- (iii) Inverse source problems, which refer to the task of determining an unknown source or input function within an SPDE based on observed effects or responses. These types of problems are common in various scientific and engineering fields where understanding the hidden sources or parameters is crucial. These problems have wide-ranging applications, including optics and acoustics [18], environmental monitoring (such as groundwater pollution [13]), and the study of atmospheric systems and pollution [63], among many others.
- (iv) Inverse geometric shape of the domain problems, which refer to the task of determining the shape or geometry of a domain where the SPDE is involved from measured data or observations. Such problems arise in many practical scenarios, for example, determining the temperature distribution in a noisy medium, the dynamics of a chemical reaction in a noisy environment, or the evolution of the density of a bacteria population [15, 40, 48].

Due to space limitations, in this book we mainly focus on inverse state problems and inverse source problems.

Below, we further explain the aforementioned four inverse problems through a simple stochastic parabolic equation. In order to succinctly clarify the key points

of the issue, we will only consider a very basic equation. The formulation of these problems is similar for more complex equations.

Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a C^2 boundary Γ . Denote $Q = (0, T) \times G$ and $\Sigma = (0, T) \times \Gamma$.

Consider the following stochastic parabolic equation:

$$\begin{cases} dy - \Delta y dt = (ay + f)dt + (by + g)dW(t) & \text{in } Q, \\ \gamma_1 \frac{\partial y}{\partial \nu} + \gamma_2 y = \varphi & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \quad (1.2.1)$$

where a, b, f, g, γ_1 and γ_2 are suitable functions/stochastic processes.

The direct problem is to determine the solution y of system (1.2.1) by specifying $(a, b, f, g, \gamma_1, \gamma_2, \varphi, y_0)$ along with Q . This can be achieved using classical well-posedness results of stochastic evolution equations (Theorem 2.5.3). However, in many practical situation it is not possible to measure experimentally all these data. Instead, it is possible to measure certain information of the solution and use this information together with other data to recover the missing, unmeasurable data. More precisely, the inverse problem for (1.2.1) involves certain unknown components in $(a, b, f, g, \gamma_1, \gamma_2, \varphi, y_0)$ and Q , which need to be determined through measuring some information about the solution y :

$$\mathcal{M}(y)(t, x) = h(t, x), \quad (t, x) \in \mathcal{O} \subset [0, T] \times \overline{G}.$$

Here, $\mathcal{M}(\cdot)$ is a known operator, representing the method of obtaining measurement data, and $\mathcal{O}$ denotes the position of the measurement.

In general, there are three distinct types of measurements:

- Distributed measurement involves measuring the solution value on a subset of the domain.
- Terminal measurement entails measuring the solution value at the terminal state.
- Boundary measurement refers to observing Dirichlet or Neumann data on a portion of the boundary.

Determining unknown functions a and/or b are known as inverse coefficient problems. Reconstructing unknown components in f and/or g is termed as inverse source problems. When the goal is to reconstruct the initial condition y_0 , it is known as an inverse initial value problem. Furthermore, in cases where a section of the boundary Γ is both unknown or changing with time, deducing the shape or geometry of the domain from this boundary information can be seen as a type of inverse geometric shape of the domain problem.

For a given inverse problem, generally speaking, three main concerns are the following:

- (i) To what extent can measurement data uniquely determine the unknown parameters, and what information can be ascertained from it?
- (ii) Is it feasible to reconstruct unknown components from measurement data?
- (iii) Do there exist efficient algorithms for constructing approximate solutions?

The first issue pertains to the **uniqueness** of the inverse problem—that is, whether the mapping from the unknown parameters to the measurements is injective. This ensures that no two distinct parameter sets yield identical measurement data.

The second issue involves the **stability** of the inverse problem: specifically, whether the recovery of the unknowns from the measurements remains stable under perturbations or noise in the data. This typically requires the inverse mapping to be continuous, so that small errors in the measurements result only in small deviations in the reconstructed parameters.

The third issue concerns the **reconstruction** of the unknowns: namely, whether efficient algorithms exist to recover the unknown parameters from the observed measurement data.

With the growing importance of SPDEs, inverse problems governed by such equations arise naturally. Various types of inverse problems for SPDEs have attracted the interest of more and more researchers in many fields. In the past decade, some interesting progress has been made (e.g., [20, 24, 25, 28, 29, 31, 32, 44, 46, 47, 53–55, 57, 59, 61, 64, 73, 74] and the references therein). This book aims to provide a brief overview of recent advancements in the theory of inverse problems for stochastic partial differential equations. In order to keep the content concise, we will only discuss the inverse problems of two typical classes of stochastic partial differential equations: second-order stochastic parabolic equations and second-order stochastic hyperbolic equations.

The main tool for studying these inverse problem is Carleman estimate. We do not intend to pursue any general treatment of the Carleman estimates themselves and choose direct arguments based on basic stochastic calculus, rather than more general sophisticated methods.

As this field is still developing and many challenging issues remain, the purpose of this book is not to provide a comprehensive summary, but rather to inspire interest and encourage further exploration in this area. We prefer to present results that, from our perspective, include fresh and promising ideas. In cases where a complete mathematical theory is lacking, we only provide the available results. We do not intend for the current book to be encyclopedic in any sense, and the references are limited.

We aim to organize the material in such a way that no prerequisites except a moderate knowledge of partial differential equations and functional analysis are expected. During the preparation of this book we found that this goal was too ambitious. For example, it will cost lots of space for introducing the basic knowledge for SPDEs. Therefore, in [Chapter 2](#) we collect some definitions and

theorems concerning SPDEs needed in this book and provide references for further study.

From an expert's perspective, certain content in various chapters or sections may have slight overlaps, which are intentionally retained to aid the convenience of readers, especially beginners.

The rest of this book is organized as follows: In [Chapter 2](#), we recall the mathematical background material for SPDEs needed in this book. Chapters [3](#) and [4](#) are devoted to inverse problems for stochastic parabolic equations and stochastic hyperbolic equations, respectively.

To present the ideas and methods clearly, we do not aim for full generality in the equations, nor do we seek to establish the most general results. Readers interested in these broader aspects may refer to the discussions and references provided in [Sections 3.12](#) and [4.8](#).

A number of our colleagues, who are listed below in the alphabetical order, have helped us in our work on this book. Prof. Fangfang Dou and Dr. Peimin Lü have collaborated with the second author on the topic of reconstruction problem for stochastic parabolic equations ([\[22\]](#)). Prof. Zhongqi Yin has collaborated with the first author on the subject of inverse state problem for stochastic hyperbolic equations ([\[58\]](#)). He also read the draft of this book carefully and provided numerous useful suggestions. Prof. Xu Zhang has collaborated with the first author on the development of the idea of using Carleman estimate to study inverse source problem for stochastic hyperbolic equations ([\[59\]](#)). We sincerely appreciate the significant contributions of each of these individuals. This work is partially supported by the NSF of China under grants 12025105 and 12401589, by the New Cornerstone Science Foundation and the XPLOER PRIZE, and by the Fundamental Research Funds for the Central Universities 2682024CX013.

Chapter 2

Preliminaries in Stochastic Calculus

Abstract This chapter provides a concise introduction to stochastic calculus, supplying the essential background for the subsequent analysis. Key preliminaries relevant to inverse problems for SPDEs are collected, and a unified notational framework is established for use throughout the book.

Due to space limitations, we are unable to present the proofs of all results stated in this chapter. We provide standard and easily accessible references where readers can find the complete proofs. Readers already familiar with these topics may skip this chapter or use it as a quick reference.

2.1 Measure and integration

In this section, we present the definitions of measurability and integrability for Banach space-valued functions.

Definition 2.1.1. Let Ω be a nonempty set. A collection $\mathcal{F}$ of subsets of Ω is called a σ -field if

- (i) $\Omega \in \mathcal{F}$,
- (ii) $E \in \mathcal{F}$ implies $\Omega \setminus E \in \mathcal{F}$
- (iii) $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$ whenever each $E_i \in \mathcal{F}$.

We call $(\Omega, \mathcal{F})$ a measurable space if $\mathcal{F}$ is a σ -field on Ω . Any element $E \in \mathcal{F}$ is called a measurable set on $(\Omega, \mathcal{F})$.

In what follows, we fix a measurable space $(\Omega, \mathcal{F})$.

Definition 2.1.2. A measure on $(\Omega, \mathcal{F})$ is a function $\mu : \mathcal{F} \rightarrow [0, +\infty]$ such that

- (i) $\mu(\emptyset) = 0$,

- (ii) if $\{E_i\}_{i=1}^\infty$ is a sequence of disjoint sets in $\mathcal{F}$, i.e., $E_i \cap E_j = \emptyset$ for all $i, j \in \mathbb{N}$ with $i \neq j$, then $\mu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i)$.

If μ is a measure on $(\Omega, \mathcal{F})$, the triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Next, we fix a measure space $(\Omega, \mathcal{F}, \mu)$. A set $E \in \mathcal{F}$ with $\mu(E) = 0$ is called a μ -null set. If a proposition about points $x \in \Omega$ is true except for x in some μ -null set, it is usual to say that the proposition is true μ -a.e. (or simply a.e., if there is no ambiguity).

Definition 2.1.3. The measure space $(\Omega, \mathcal{F}, \mu)$ is called complete (and μ is said to be complete on $\mathcal{F}$), if $\mathcal{F}$ includes all subsets of null sets, i.e.,

$$\mathcal{N} = \{G \subset \Omega \mid G \subset E \text{ for some } \mu\text{-null set } E\} \subset \mathcal{F}. \quad (2.1.1)$$

Completeness can always be achieved by enlarging the domain of μ .

Definition 2.1.4. Let $\mathcal{X}$ be a topology space. The smallest σ -field containing all open sets of $\mathcal{X}$ is called the Borel σ -field of $\mathcal{X}$ and denoted by $\mathcal{B}(\mathcal{X})$. Any set $E \in \mathcal{B}(\mathcal{X})$ is called a Borel set.

Definition 2.1.5. Let $(\Omega, \mathcal{F})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}})$ be two measurable spaces. A mapping $f : \Omega \rightarrow \tilde{\Omega}$ is called $\mathcal{F}/\tilde{\mathcal{F}}$ -measurable or simply $\mathcal{F}$ -measurable or even *measurable* if $f^{-1}(\tilde{\mathcal{F}}) \subset \mathcal{F}$. In particular, if $(\tilde{\Omega}, \tilde{\mathcal{F}}) = (\mathcal{X}, \mathcal{B}(\mathcal{X}))$, then f is said to be an ($\mathcal{X}$ -valued) $\mathcal{F}$ -measurable (or measurable) function.

For a measurable map $f : (\Omega, \mathcal{F}) \rightarrow (\tilde{\Omega}, \tilde{\mathcal{F}})$, denote by $f^{-1}(\tilde{\mathcal{F}})$ the σ -field generated by the set $\{f^{-1}(F) \mid F \in \tilde{\mathcal{F}}\}$. One can easily find that $f^{-1}(\tilde{\mathcal{F}})$ is a sub σ -field of $\mathcal{F}$ which is called the σ -field generated by f , and denoted by $\sigma(f)$. In addition, for a family of measurable maps $\{f_\lambda\}_{\lambda \in \Lambda}$, we denote by $\sigma(f_\lambda; \lambda \in \Lambda)$ the σ -field generated by $\bigcup_{\lambda \in \Lambda} \sigma(f_\lambda)$.

At last, we shall present the definition of measurable and integrable for Banach space-valued functions. We will fix a Banach space $\mathcal{X}$.

Definition 2.1.6. Let $f : \Omega \rightarrow \mathcal{X}$ be an $\mathcal{X}$ -valued function.

- (i) The function $f(\cdot)$ is called an $\mathcal{F}$ -simple function (or simple function when $\mathcal{F}$ is clear) if

$$f(\cdot) = \sum_{j=1}^k \chi_{E_j}(\cdot) u_j, \quad (2.1.2)$$

for some $k \in \mathbb{N}$, $u_j \in \mathcal{X}$, and $\{E_j\}_{j=1}^k \subset \mathcal{F}$. Here, $\chi_E(\cdot)$ is the characteristic function of E .

- (ii) The function $f(\cdot)$ is said to be strongly $\mathcal{F}$ -measurable with respect to μ (or simply strongly measurable) if it is measurable and there exist simple functions $\{f_k\}_{k=1}^\infty$ converging to f in $\mathcal{X}$, μ -a.e.

The following theorem comes from [76, Chapter V, 4].

Theorem 2.1.1. *If the Banach space $\mathcal{X}$ is separable and the measure space $(\Omega, \mathcal{F}, \mu)$ is σ -finite, then a function $f : \Omega \rightarrow \mathcal{X}$ is strongly measurable if and only if f is measurable.*

In what follows, we fix a σ -finite measure space $(\Omega, \mathcal{F}, \mu)$.

Definition 2.1.7. Write the simple function f in the form (2.1.2), where the E_j are disjoint and the u_j are its distinct nonzero values; the zero function is represented by the empty sum. We call $f(\cdot)$ Bochner integrable if $\mu(E_j) < \infty$ for any $j = 1, \dots, k$. For any $E \in \mathcal{F}$, we define the Bochner integral of $f(\cdot)$ over E as

$$\int_E f(s) d\mu = \sum_{j=1}^k \mu(E \cap E_j) u_j.$$

For general strongly measurable functions, we have the following definition.

Definition 2.1.8. We say the strongly measurable function $f(\cdot) : \Omega \rightarrow \mathcal{X}$ is Bochner integrable with respect to μ if there exists a sequence of Bochner integrable simple functions $\{f_i(\cdot)\}_{i=1}^\infty$ converging to $f(\cdot)$ in $\mathcal{X}$, μ -a.e., such that

$$\lim_{i,j \rightarrow \infty} \int_\Omega |f_i(s) - f_j(s)|_{\mathcal{X}} d\mu = 0.$$

For $E \in \mathcal{F}$, the Bochner integral of $f(\cdot)$ over E is defined by

$$\int_E f(s) d\mu = \lim_{i \rightarrow \infty} \int_\Omega \chi_E(s) f_i(s) d\mu(s) \quad \text{in } \mathcal{X}.$$

The following theorem (see [76, Chapter V, 5]) describes the relationship between the Bochner integral and the Lebesgue integral.

Theorem 2.1.2. *A strongly measurable function $f(\cdot) : \Omega \rightarrow \mathcal{X}$ is Bochner integrable if and only if the scalar function $|f(\cdot)|_{\mathcal{X}} : \Omega \rightarrow \mathbb{R}$ is integrable. In this case, for any $E \in \mathcal{F}$,*

$$\left| \int_E f(s) d\mu \right|_{\mathcal{X}} \leq \int_E |f(s)|_{\mathcal{X}} d\mu.$$

2.2 Probability, random variables and expectation

In this section, we present the definitions of sample space, Banach space-valued random variables, and expectation, as preparation for the subsequent introduction of stochastic processes.

A probability space is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(\Omega) = 1$, where $\mathbb{P}$ is called a probability measure. Set Ω is called a sample space; $\omega \in \Omega$ is called a sample point; any $A \in \mathcal{F}$ is called an event.

In the following, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Banach space $\mathcal{X}$.

Definition 2.2.1. The function $f : (\Omega, \mathcal{F}) \rightarrow (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is called an ($\mathcal{X}$ -valued) random variable if f is measurable and strongly measurable.

Definition 2.2.2. Let f be a random variable. If f is Bochner integrable with respect to the probability measure $\mathbb{P}$, then we denote $\mathbb{E}f = \int_{\Omega} f d\mathbb{P}$ and call it the mean or mathematical expectation of f .

Next, we recall the notion of independence, which is a fundamental feature of probability theory.

Definition 2.2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- (i) Let $A, B \in \mathcal{F}$. We say that A, B are independent w.r.t $\mathbb{P}$ if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- (ii) Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two subfamilies of $\mathcal{F}$. We say that $\mathcal{J}_1$ and $\mathcal{J}_2$ are independent if for all $(A, B) \in \mathcal{J}_1 \times \mathcal{J}_2$, it holds that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- (iii) Let $f, g : (\Omega, \mathcal{F}) \rightarrow (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ be two random variables. We say that f and g are independent if $\sigma(f)$ and $\sigma(g)$ are independent.
- (iv) Let $f : (\Omega, \mathcal{F}) \rightarrow (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ be a random variable and $\mathcal{J} \subset \mathcal{F}$. We say that f and $\mathcal{J}$ are independent if $\sigma(f)$ and $\mathcal{J}$ are independent.

Definition 2.2.4. Let $X = (X_1, \dots, X_n)$ be a $\mathbb{R}^n$ -valued random variable. The function

$$F(x) \equiv F(x_1, \dots, x_n) \triangleq \mathbb{P}\{X_1 \leq x_1, \dots, X_n \leq x_n\}$$

is called the distribution function of X . Furthermore, if there exists a nonnegative function $f(\cdot)$ such that

$$F(x) \equiv F(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(\xi_1, \dots, \xi_n) d\xi_1 \cdots d\xi_n,$$

then the function $f(\cdot)$ is called the density of X .

If random variable X has the following density:

$$f(x) = (2\pi)^{-\frac{n}{2}} |\det Q|^{-1/2} \exp \left\{ -\frac{1}{2} (x - \lambda)^\top Q^{-1} (x - \lambda) \right\},$$

where $\lambda \in \mathbb{R}^n$ and Q is a symmetric positive definite n -dimensional matrix, then X is called a normally distributed random variable (or X has a normal distribution) and denoted by $X \sim \mathcal{N}(\lambda, Q)$. Especially, we call X a standard normally distributed random variable if $\lambda = 0$ and Q is the n -dimensional identity matrix.

2.3 Stochastic process

In this section, we present the definitions of Banach space-valued stochastic processes and Brownian motion. The coefficients and solutions of SPDEs are typically Banach space-valued stochastic processes. We also introduce the main function spaces used throughout this book.

Let $T > 0$. A stochastic process is a family of $\mathcal{X}$ -valued random variables $\{X(t)\}_{t \in [0, T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. If there is no ambiguity, we shall use $\{X(t)\}_{t \in [0, T]}$, $X(\cdot)$ or X to denote a stochastic process.

For every $\omega \in \Omega$ the function $t \mapsto X(t, \omega)$ is called a sample path (or path) of X . The stochastic process $X(\cdot)$ is said to be continuous if there exists a $\mathbb{P}$ -null set $N \in \mathcal{F}$ such that for any $\omega \in \Omega \setminus N$, the path $X(\cdot, \omega)$ is continuous in $\mathcal{X}$.

Definition 2.3.1. A family of sub- σ -field $\{\mathcal{F}_t\}_{t \in [0, T]}$ is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s, t \in [0, T]$ and $s \leq t$. We simply denote it by $\mathbf{F}$.

We put $\mathcal{F}_{0-} = \mathcal{F}_0$, $\mathcal{F}_{T+} = \mathcal{F}_T$, and

$$\mathcal{F}_{t+} \triangleq \bigcap_{s \in (t, T]} \mathcal{F}_s \quad (0 \leq t < T), \quad \mathcal{F}_{t-} \triangleq \sigma\left(\bigcup_{s \in [0, t)} \mathcal{F}_s\right) \quad (0 < t \leq T).$$

If $\mathcal{F}_{t+} = \mathcal{F}_t$ (resp. $\mathcal{F}_{t-} = \mathcal{F}_t$), then we call $\mathbf{F}$ is right (resp. left) continuous.

Definition 2.3.2. We call $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ a filtered probability space, and it satisfies the usual condition if $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets in $\mathcal{F}$, and $\mathbf{F}$ is right continuous.

In the sequel, we always assume that the filtered probability $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ satisfies the usual condition.

Definition 2.3.3. Let $X(\cdot)$ be an $\mathcal{X}$ -valued stochastic process.

- (i) We say that $X(\cdot)$ is measurable if the map $(t, \omega) \mapsto X(t, \omega)$ is strongly $(\mathcal{B}([0, T]) \times \mathcal{F})/\mathcal{B}(\mathcal{X})$ -measurable;
- (ii) We say that $X(\cdot)$ is $\mathbf{F}$ -adapted if it is measurable and for each $t \in [0, T]$, the map $\omega \mapsto X(t, \omega)$ is strongly $\mathcal{F}_t/\mathcal{B}(\mathcal{X})$ -measurable;
- (iii) We say that $X(\cdot)$ is $\mathbf{F}$ -progressively measurable if for each $t \in [0, T]$, the map $(s, \omega) \mapsto X(s, \omega)$ from $[0, t] \times \Omega$ is strongly $(\mathcal{B}([0, t]) \times \mathcal{F}_t)/\mathcal{B}(\mathcal{X})$ -measurable.

Definition 2.3.4. We call a set $A \subset [0, T] \times \Omega$ is progressively measurable with respect to $\mathbf{F}$ if the stochastic process $\chi_A(\cdot)$ is progressively measurable, where χ_A is the indicator function of A .

We denote the class of all progressively measurable sets by $\mathbb{F}$, which is a σ -field. It is clear that a process $\varphi : [0, T] \times \Omega \rightarrow \mathcal{X}$ is $\mathbf{F}$ -progressively measurable if and only if it is strongly $\mathbb{F}$ -measurable.

Clearly, if $X(\cdot)$ is $\mathbf{F}$ -progressively measurable, it must be $\mathbf{F}$ -adapted. Conversely, one can prove that, for any $\mathbf{F}$ -adapted process $X(\cdot)$, there exists an $\mathbf{F}$ -progressively measurable process $\bar{X}(\cdot)$, such that

$$\mathbb{P}(\{X(t) = \bar{X}(t)\}) = 1, \quad \forall t \in [0, T].$$

Hence, by saying that a process $X(\cdot)$ is $\mathbf{F}$ -adapted, we mean that it is $\mathbf{F}$ -progressively measurable.

Definition 2.3.5. The standard Brownian motion is a continuous, $\mathbb{R}$ -valued, $\mathbf{F}$ -adapted stochastic process $\{W(t)\}_{t \geq 0}$ such that

- (i) $W(0) = 0$, $\mathbb{P}$ -a.s.; and
- (ii) For any $s, t \in [0, \infty)$ with $s < t$, the random variable $W(t) - W(s)$ is independent of $\mathcal{F}_s$ and $W(t) - W(s) \sim \mathcal{N}(0, t - s)$.

We fix a standard Brownian motion $W(\cdot)$. For each $t \in [0, T]$, put $\mathcal{F}_t^W = \sigma(W(s); s \in [0, t]) \subset \mathcal{F}_t$. Let $\{\hat{\mathcal{F}}_t^W\}_{t \in [0, T]}$ be the augmentation of $\{\mathcal{F}_t^W\}_{t \in [0, T]}$ by adding all $\mathbb{P}$ -null sets. Then, $\{\hat{\mathcal{F}}_t^W\}_{t \in [0, T]}$ is continuous and $W(\cdot)$ is still a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\hat{\mathcal{F}}_t^W\}_{t \in [0, T]}, \mathbb{P})$. We call $\mathbf{F}$ the natural filtration generated by $W(\cdot)$, if $\mathbf{F}$ is $\{\hat{\mathcal{F}}_t^W\}_{t \in [0, T]}$ generated as described above.

The notations to be given below will be used in the rest of the book.

For any $p, q \in [1, \infty)$, denote

$$L_{\mathbb{F}}^p(\Omega; L^q(0, T; \mathcal{X})) \triangleq \left\{ \varphi : (0, T) \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbf{F}\text{-adapted and} \right. \\ \left. \mathbb{E} \left(\int_0^T |\varphi(t)|_{\mathcal{X}}^q dt \right)^{\frac{p}{q}} < \infty \right\},$$

and

$$L_{\mathbb{F}}^q(0, T; L^p(\Omega; \mathcal{X})) \triangleq \left\{ \varphi : (0, T) \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbf{F}\text{-adapted and} \right. \\ \left. \int_0^T (\mathbb{E} |\varphi(t)|_{\mathcal{X}}^p)^{\frac{q}{p}} dt < \infty \right\}.$$

Similarly, we may also define (for $1 \leq p, q < \infty$)

$$L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; \mathcal{X})), \quad L_{\mathbb{F}}^q(0, T; L^{\infty}(\Omega; \mathcal{X})), \quad L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(\Omega; \mathcal{X})).$$

All these spaces are Banach spaces with the canonical norms. In the sequel, we will simply denote $L_{\mathbb{F}}^p(0, T; L^p(\Omega; \mathcal{X}))$ by $L_{\mathbb{F}}^p(0, T; \mathcal{X})$; and denote $L_{\mathbb{F}}^p(0, T; \mathbb{R})$ by $L_{\mathbb{F}}^p(0, T)$.

For any $p \in [1, \infty)$, write

$$L_{\mathbb{F}}^p(\Omega; C([0, T]; \mathcal{X})) \triangleq \left\{ \varphi : [0, T] \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is continuous,} \right.$$

$$\mathbf{F}\text{-adapted and } \mathbb{E}(|\varphi(\cdot)|_{C([0,T];\mathcal{X})}^p) < \infty \}$$

It is clear that $L_{\mathbb{F}}^p(\Omega; C([0, T]; \mathcal{X}))$ is Banach spaces equipped with norms

$$|\varphi(\cdot)|_{L_{\mathbb{F}}^p(\Omega; C([0,T];\mathcal{X}))} = [\mathbb{E}|\varphi(\cdot)|_{C([0,T];\mathcal{X})}^p]^{1/p}.$$

2.4 Itô's integral

In this section, we introduce the Itô stochastic integral for Hilbert space-valued stochastic processes, summarize its fundamental properties, and present Itô's formula, a key result that will be frequently used throughout this book.

Let H be a Hilbert space. Let $W(\cdot)$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. We shall define the Itô integral $\int_0^T X(t) dW(t)$ of an H -valued, $\mathbf{F}$ -adapted stochastic process $X(\cdot)$ with respect to $W(t)$.

Let us start with the step process $f \in L_{\mathbb{F}}^2(0, T, H)$, which has the form

$$f(t, \omega) = \sum_{j=0}^n f_j(\omega) \chi_{[t_j, t_{j+1})}(t), \quad (t, \omega) \in [0, T] \times \Omega, \quad (2.4.1)$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_{n+1} = T$, f_j is $\mathcal{F}_{t_j}$ -measurable with

$$\sup \{ |f_j(\omega)|_H \mid j \in \{0, \dots, n\}, \omega \in \Omega \} < \infty.$$

The set of step processes will be denoted by $\mathcal{L}_0$ and is dense in $L_{\mathbb{F}}^2(0, T, H)$.

The stochastic integral of a step process $f \in \mathcal{L}_0$ of the form (2.4.1) is defined by

$$I(f)(t, \omega) = \sum_{j=0}^n f_j(\omega) [W(t \wedge t_{j+1}, \omega) - W(t \wedge t_j, \omega)].$$

Clearly, $I(f) \in L_{\mathcal{F}_t}^2(\Omega; H)$ and the following Itô isometry holds:

$$|I(f)|_{L_{\mathcal{F}_t}^2(\Omega; H)} = |f|_{L_{\mathbb{F}}^2(0, t; H)}.$$

Since $\mathcal{L}_0$ is dense in $L_{\mathbb{F}}^2(0, T; H)$, for $f \in L_{\mathbb{F}}^2(0, T; H)$, there exists a sequence of $\{f_k\}_{k=1}^{\infty} \subset \mathcal{L}_0$ such that

$$\lim_{k \rightarrow \infty} |f_k - f|_{L_{\mathbb{F}}^2(0, T; H)} = 0.$$

From Itô isometry, one can find that $\{I(f_k)\}_{k=1}^{\infty}$ is a Cauchy sequence in $L_{\mathcal{F}_t}^2(\Omega; H)$. Hence, it converges to a unique element in $L_{\mathcal{F}_t}^2(\Omega; H)$, which is independent of the choice of $\{f_k\}_{k=1}^{\infty}$. We call this element the Itô integral of f on $[0, t]$ and denote it by $\int_0^t f dW$.

For $0 \leq s < t \leq T$, we define the Itô integral of f on $[s, t]$ by

$$\int_s^t f dW = \int_s^t f(\tau) dW(\tau) \triangleq \int_0^t f dW - \int_0^s f dW.$$

For a progressively measurable f with $\int_0^T |f(t)|_H^2 dt < \infty$ a.s., the Itô integral is defined by localization. Stochastic integrals in the Itô formula below are understood in this sense when necessary.

One can prove the following properties of Itô integral (see [60, Theorem 2.124]).

Theorem 2.4.1. *Let $f, g \in L_{\mathbb{F}}^2(0, T; H)$ and $a, b \in L_{\mathcal{F}_s}^2(\Omega)$, $0 \leq s < t \leq T$. Then*

- (i) $\left(r \mapsto \int_s^r f dW\right) \in L_{\mathbb{F}}^2(\Omega; C([s, t]; H));$
- (ii) $\int_s^t (af + bg) dW = a \int_s^t f dW + b \int_s^t g dW, \text{ a.s.};$
- (iii) $\mathbb{E}\left(\int_s^t f dW\right) = 0;$
- (iv) $\mathbb{E}\left(\left\langle \int_s^t f dW, \int_s^t g dW \right\rangle_H\right) = \mathbb{E}\left(\int_s^t \langle f(\tau, \cdot), g(\tau, \cdot) \rangle_H d\tau\right).$

The integral on the left-hand side of (ii) is defined by localization.

For (ii), apply the bounded-multiplier identity on the events $\{|a| + |b| \leq N\} \in \mathcal{F}_s$ and let $N \rightarrow \infty$.

Definition 2.4.1. We call an H -valued, $\mathbf{F}$ -adapted, continuous stochastic process $X(\cdot)$ an Itô process, if there exist two H -valued process $\phi(\cdot) \in L_{\mathbb{F}}^1(0, T; H)$ and $\Phi(\cdot) \in L_{\mathbb{F}}^2(0, T; H)$ such that

$$X(t) = X(0) + \int_0^t \phi(s) ds + \int_0^t \Phi(s) dW(s), \quad \text{a.s.,} \quad \forall t \in [0, T]. \quad (2.4.2)$$

We write stochastic differential

$$dX(t) = \phi(t)dt + \Phi(t)dW(t),$$

or $dX = \phi dt + \Phi dW(t)$ to mean that $X(t)$ satisfies (2.4.2).

The following is a fundamental result called Itô formula (see [60, Theorem 2.139]).

Theorem 2.4.2. *Let $X(\cdot)$ be an Itô process given by (2.4.2) and $F : [0, T] \times H \rightarrow \mathbb{R}$ be a function such that its partial derivatives F_t, F_x, F_{xx} are uniformly continuous on any bounded subset of $[0, T] \times H$. It holds that*

$$\begin{aligned} & F(t, X(t)) - F(0, X(0)) \\ &= \int_0^t F_x(s, X(s)) \Phi(s) dW(s) + \int_0^t \left[F_t(s, X(s)) + F_x(s, X(s)) \phi(s) \right. \\ & \quad \left. + \frac{1}{2} \langle F_{xx}(s, X(s)) \Phi(s), \Phi(s) \rangle_H \right] ds, \quad \text{a.s.,} \quad \forall t \in [0, T]. \end{aligned} \quad (2.4.3)$$

Usually, we write (2.4.3) in the following differential form:

$$dF(t, X(t)) = F_x(t, X(t))\Phi(t)dW(t) + F_t(t, X(t))dt + F_x(t, X(t))\phi(t)dt \\ + \frac{1}{2} \langle F_{xx}(t, X(t))\Phi(t), \Phi(t) \rangle_H dt.$$

Remark 2.4.1. If $H = \mathbb{R}^n$, $n \in \mathbb{N}^+$, and we write $X(t) = (X_1(t), \dots, X_n(t))^\top$ and similarly for $\phi(t)$ and $\Phi(t)$ (see (2.4.2)), then the Itô formula can be written as

$$dF(t, X(t)) = F_t(t, X(t))dt + \sum_{i=1}^n F_{x_i}(t, X(t))dX_i(t) \\ + \frac{1}{2} \sum_{i,j=1}^n F_{x_i x_j}(t, X(t))dX_i(t)dX_j(t).$$

Here, $\int_0^t dX_i(s)dX_j(s)$ represents the quadratic covariation of the processes X_i and X_j , which is defined by the following limit in probability:

$$\int_0^t dX_i(s)dX_j(s) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\lfloor tn \rfloor} \left[X_i\left(\frac{k}{n}\right) - X_i\left(\frac{k-1}{n}\right) \right] \left[X_j\left(\frac{k}{n}\right) - X_j\left(\frac{k-1}{n}\right) \right].$$

One can prove that (see [2, Section 10.2])

$$\int_0^t dX_i(s)dX_j(s) = \int_0^t \Phi_i(s)\Phi_j(s)ds, \quad \text{a.s.,} \quad \forall t \in [0, T].$$

2.5 Stochastic evolution equations

In this section, we present in a unified manner the well-posedness results for two classes of SPDEs considered in this book, namely stochastic parabolic equations and stochastic hyperbolic equations.

Let H be a separable Hilbert space, and A be an unbounded linear operator (with domain $D(A)$) on H . Assume that A is the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$.

We consider the following stochastic evolution equation:

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t)))dt + \tilde{F}(t, X(t))dW(t) & \text{in } (0, T], \\ X(0) = X_0, \end{cases} \quad (2.5.1)$$

where $X_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$ and $F(\cdot, \cdot)$, $\tilde{F}(\cdot, \cdot)$ are two functions from $[0, T] \times \Omega \times H$ to H .

Definition 2.5.1. We call an H -valued, $\mathbf{F}$ -adapted, continuous stochastic process $X(\cdot)$ a strong solution to (2.5.1) if

- (i) $X(t) \in D(A)$ for a.e. $(t, \omega) \in [0, T] \times \Omega$ and $AX(\cdot) \in L^1(0, T; H)$ a.s.;
- (ii) $F(\cdot, X(\cdot)) \in L^1(0, T; H)$ a.s., $\tilde{F}(\cdot, X(\cdot)) \in L^2_{\mathbb{F}}(0, T; H)$;
- (iii) For all $t \in [0, T]$,

$$X(t) = X_0 + \int_0^t (AX(s) + F(s, X(s)))ds + \int_0^t \tilde{F}(s, X(s))dW(s), \text{ a.s.}$$

In general, the conditions ensuring the existence of strong solutions to (2.5.1) are highly restrictive. Hence, we introduce the following definitions of the weak and the mild solution to (2.5.1), respectively.

Definition 2.5.2. We call an H -valued, $\mathbf{F}$ -adapted, continuous stochastic process $X(\cdot)$ a weak solution to (2.5.1) if $F(\cdot, X(\cdot)) \in L^1(0, T; H)$ a.s., $\tilde{F}(\cdot, X(\cdot)) \in L^2_{\mathbb{F}}(0, T; H)$, and for any $t \in [0, T]$ and $\xi \in D(A^*)$,

$$\begin{aligned} \langle X(t), \xi \rangle_H &= \langle X_0, \xi \rangle_H + \int_0^t (\langle X(s), A^*\xi \rangle_H + \langle F(s, X(s)), \xi \rangle_H) ds \\ &\quad + \int_0^t \langle \tilde{F}(s, X(s)), \xi \rangle_H dW(s), \quad \text{a.s.} \end{aligned}$$

Definition 2.5.3. We call an H -valued, $\mathbf{F}$ -adapted, continuous stochastic process $X(\cdot)$ a mild solution to (2.5.1) if $F(\cdot, X(\cdot)) \in L^1(0, T; H)$ a.s., $\tilde{F}(\cdot, X(\cdot)) \in L^2_{\mathbb{F}}(0, T; H)$, and for any $t \in [0, T]$,

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)\tilde{F}(s, X(s))dW(s), \text{ a.s.}$$

The following proposition states the equivalence between weak and mild solutions (see [60, Theorem 3.10]).

Proposition 2.5.1. *Any weak solution to (2.5.1) is also a mild solution to the same equation, and conversely.*

From Theorem 2.5.1, we do not distinguish the mild and the weak solution to (2.5.1).

It is obvious that a strong solution is also a weak and mild solution to the same equation. Conversely, the following is a sufficient condition (see [60, Lemma 3.11]) for a mild solution to be a strong solution to (2.5.1).

Proposition 2.5.2. *A mild solution $X(\cdot)$ is a strong solution to (2.5.1) if the following three conditions hold for all $x \in H$ and $0 \leq s \leq t \leq T$, a.s.,*

- (i) $X_0 \in D(A)$, $S(t-s)F(s, x) \in D(A)$ and $S(t-s)\tilde{F}(s, x) \in D(A)$;
- (ii) $|AS(t-s)F(s, x)|_H \leq \alpha(t-s)|x|_H$ for some real-valued stochastic process $\alpha(\cdot) \in L^1_{\mathbb{F}}(0, T)$ a.s.;

(iii) $|AS(t-s)\tilde{F}(s, x)|_H \leq \beta(t-s)|x|_H$ for some real-valued stochastic process $\beta(\cdot) \in L^2_{\mathbb{F}}(0, T)$ a.s.

In what follows, we assume that both $F(\cdot, x)$ and $\tilde{F}(\cdot, x)$ are $\mathbf{F}$ adapted for each $x \in H$, and there exist two nonnegative (real-valued) functions $L_1(\cdot), L_2(\cdot) \in L^2(0, T)$ such that for a.e. $t \in [0, T]$ and all $y, z \in H$,

$$\begin{cases} |F(t, y) - F(t, z)|_H \leq L_1(t)|y - z|_H, & \text{a.s.,} \\ |\tilde{F}(t, y) - \tilde{F}(t, z)|_H \leq L_2(t)|y - z|_H, & \text{a.s.,} \\ F(\cdot, 0) \in L^2_{\mathbb{F}}(0, T; H), \quad \tilde{F}(\cdot, 0) \in L^2_{\mathbb{F}}(0, T; H). \end{cases}$$

The following is the well-posedness of mild solutions for a class of stochastic evolution equations (see [60, Theorem 3.24]).

Theorem 2.5.3. *Assume that A is a self-adjoint, negative definite linear operator on H . Then, the equation (2.5.1) admits a unique mild solution $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H)) \cap L^2_{\mathbb{F}}(0, T; D((-A)^{\frac{1}{2}}))$. Moreover,*

$$\begin{aligned} & |X(\cdot)|_{L^2_{\mathbb{F}}(\Omega; C([0, T]; H))} + |X(\cdot)|_{L^2_{\mathbb{F}}(0, T; D((-A)^{\frac{1}{2}}))} \\ & \leq \mathcal{C}(|X_0|_{L^2_{\mathcal{F}_0}(\Omega; H)} + |F(\cdot, 0)|_{L^2_{\mathbb{F}}(0, T; H)} + |\tilde{F}(\cdot, 0)|_{L^2_{\mathbb{F}}(0, T; H)}). \end{aligned}$$

Generally speaking, the stochastic convolution $\int_0^t S(t-s)\tilde{F}(s, X(s))dW(s)$ is no longer a martingale. Hence, one cannot apply Itô's formula directly to mild solutions to (2.5.1). For example, when establishing the pointwise identities (for Carleman estimates) for stochastic partial differential equations of second order, we need the function to be second differentiable in the sense of weak derivative w.r.t the space variables. Nevertheless, this sort of problems can be solved by the following strategy:

1. Introduce suitable approximating equations with strong solutions such that the limit of these strong solutions is the mild/weak solution to the original equation;
2. Obtain the desired properties for the above strong solutions;
3. Utilize the density argument to establish the desired properties for the mild/weak solutions.

There are many methods to implement the above three steps in the setting of deterministic partial differential equations. Roughly speaking, any of these methods, which do not destroy the adaptedness of solutions, can be applied to stochastic partial differential equations. Here we present one. To this end, we introduce the following approximating equations of (2.5.1):

$$\begin{cases} dX_{\lambda}(t) = (AX_{\lambda}(t) + R(\lambda)F(t, X_{\lambda}(t)))dt \\ \quad + R(\lambda)\tilde{F}(t, X_{\lambda}(t))dW(t) & \text{in } (0, T], \\ X_{\lambda}(0) = R(\lambda)X_0. \end{cases} \quad (2.5.2)$$

Here, λ belongs to $\rho(A)$, the resolvent set of A , and $R(\lambda) \triangleq \lambda(\lambda I - A)^{-1}$.

We have the following result (see [60, Theorem 3.22]).

Theorem 2.5.4. *Assume that A generates a contraction semigroup, $F(\cdot, 0) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; H))$. Then, for each $X_0 \in L^2_{\mathcal{F}_0}(\Omega; H)$ and $\lambda \in \rho(A)$, the equation (2.5.2) admits a unique strong solution $X_\lambda(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; D(A)))$. Moreover, as $\lambda \rightarrow \infty$, the solution $X_\lambda(\cdot)$ converges to $X(\cdot)$ in $L^2_{\mathbb{F}}(\Omega; C([0, T]; H))$, where $X(\cdot)$ solves (2.5.1) in the sense of the mild solution.*

Chapter 3

Inverse problems for stochastic parabolic equations

Abstract This chapter focuses on inverse problems for stochastic parabolic equations. We study the determination of unknown initial data and source terms from partial observations, including internal, terminal, and boundary measurements. We analyze uniqueness, stability, and reconstruction results. The main results are derived via Carleman estimates based on a fundamental weighted identity, and several related open problems are briefly discussed.

Throughout this chapter, let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$. Write $\mathbb{F}$ for the progressive σ -field with respect to $\mathbf{F}$. Readers who are not familiar with the notions aforementioned are referred to [Chapter 2](#).

Let $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a C^2 boundary Γ and $G_0 \subset G$ be a nonempty open subset. Denote by $\mathbf{L}(G)$ all Lebesgue measurable subsets of G . Put

$$Q = (0, T) \times G, \quad \Sigma = (0, T) \times \Gamma, \quad Q_0 = (0, T) \times G_0.$$

Also, unless other stated, we denote by $\nu(x) = (\nu^1(x), \nu^2(x), \dots, \nu^n(x))$ the unit outward normal vector of Γ at $x \in \Gamma$. Let $(b^{jk})_{1 \leq j, k \leq n} \in C^1(\overline{G}; \mathbb{R}^{n \times n})$ satisfying that $b^{jk} = b^{kj}$ ($j, k = 1, 2, \dots, n$), and for some constant $s_0 > 0$,

$$\sum_{j,k=1}^n b^{jk} \xi_j \xi_k \geq s_0 |\xi|^2, \quad \forall (x, \xi) \triangleq (x, \xi_1, \dots, \xi_n) \in G \times \mathbb{R}^n. \quad (3.0.1)$$

In what follows, we use $\mathcal{C}$ to denote a generic positive constant, which may vary from line to line.

3.1 Formulation of the problems

Let $F : [0, T] \times \Omega \times G \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ and $K : [0, T] \times \Omega \times G \times \mathbb{R} \rightarrow \mathbb{R}$ be two functions such that, for each $(\varrho, \zeta) \in \mathbb{R}^n \times \mathbb{R}$, the functions $F(\cdot, \cdot, \cdot, \cdot, \varrho, \zeta) : [0, T] \times \Omega \times G \rightarrow \mathbb{R}$ and $K(\cdot, \cdot, \cdot, \zeta) : [0, T] \times \Omega \times G \rightarrow \mathbb{R}$ are $\mathbb{F} \times \mathbf{L}(G)$ -measurable, where $\mathbf{L}(G)$ denotes all Lebesgue measurable sets; and for a.e. $(t, \omega, x) \in [0, T] \times \Omega \times G$ and any $(\varrho_i, \zeta_i) \in \mathbb{R}^n \times \mathbb{R}$ ($i = 1, 2$),

$$\begin{cases} |F(t, x, \varrho_1, \zeta_1) - F(t, x, \varrho_2, \zeta_2)| \leq L(|\varrho_1 - \varrho_2|_{\mathbb{R}^n} + |\zeta_1 - \zeta_2|), \\ |K(t, x, \zeta_1) - K(t, x, \zeta_2)| \leq L|\zeta_1 - \zeta_2|, \\ |F(\cdot, \cdot, \cdot, 0, 0)| \in L^2_{\mathbb{F}}(0, T; L^2(G)), \quad |K(\cdot, \cdot, \cdot, 0)| \in L^2_{\mathbb{F}}(0, T; H^1_0(G)) \end{cases} \quad (3.1.1)$$

for some constant $L > 0$.

Let us consider the following semilinear stochastic parabolic equation:

$$\begin{cases} dy - \sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt = F(\nabla y, y) dt + K(y) dW(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \quad (3.1.2)$$

where $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G))$ is the initial data.

By the standard variational well-posedness result, using uniform ellipticity and (3.1.1), (3.1.2) admits a unique weak solution $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^1_0(G))$.

In order to simplify notation, we will suppress the time variable t ($\in [0, T]$), the sample point ω ($\in \Omega$), and/or the space variable x ($\in G$) in the functions if there is no risk of confusion.

The inverse problems studied in this chapter, roughly speaking, concern determining unknown components of system (3.1.2) based on measurements of part of its solution. Depending on which unknowns to be recovered, we consider two main classes of inverse problems: inverse state problems and inverse source problems. The difficulty, results, and proof techniques of these problems can vary greatly according to the region where the solution is observed. Due to space limitations and the fact that many aspects in this area remain open, we only present a selection of inverse problems.

3.1.1 Inverse state problems

The most classical problem is the inverse state problem, which concerns determining the unknown initial value of the SPDE from measurements. We begin by considering the case of internal measurements. The first question is: can the initial value be uniquely recovered from internal measurements?

Define a map $\mathcal{M}_1 : L^2_{\mathcal{F}_0}(\Omega; L^2(G)) \rightarrow L^2_{\mathbb{F}}(0, T; L^2(G_0))$ by

$$\mathcal{M}_1(y_0) = y|_{(0,T) \times G_0}$$

where y solves the equation (3.1.2). More precisely, we consider the following question:

Problem 3.1. *Can we determine y_0 from the measurement $\mathcal{M}_1(y_0)$?*

The problem described in Problem 3.1 involves determining the unknown initial value of the equation (3.1.2). As discussed in Chapter 1, in practical applications, our goal is not only to find y_0 from $\mathcal{M}_1(y_0)$, but also to ensure that y_0 varies continuously with $\mathcal{M}_1(y_0)$. Specifically, in the presence of measurement errors in $\mathcal{M}_1(y_0)$, we anticipate that the discrepancy between the function derived from this noisy measurement and the true initial value y_0 remains bounded by the measurement error in a suitable manner.

However, due to the regularization effect of equation (3.1.2), the inverse of the mapping $\mathcal{M}_1$ is not, in general, continuous. Therefore, it becomes necessary to either relax the constraints to achieve the desired error estimation. We will address this approach and study the following problem.

Problem 3.2 (Inverse state problem with an internal measurement. I). *For each $t \in (0, T]$, does there exist a constant $\mathcal{C}(t) > 0$ such that for any initial data $y_0, \hat{y}_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G))$,*

$$|y(t) - \hat{y}(t)|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))} \leq \mathcal{C}(t) |\mathcal{M}_1(y_0) - \mathcal{M}_1(\hat{y}_0)|_{L^2_{\mathbb{F}}(0, T; L^2(G_0))} \quad (3.1.3)$$

Here $y(\cdot)$ and $\hat{y}(\cdot)$ are solutions to (3.1.2) with the initial data y_0 and $\hat{y}_0$, respectively.

We will provide a positive answer to this problem in Section 3.3.

Remark 3.1.1. Problem 3.2 asks whether the solution $y(\cdot)$ of system (3.1.2) at $t \in (0, T]$ depends on the observation $y|_{(0, T] \times G_0}$ continuously within the space $L^2_{\mathcal{F}_t}(\Omega; L^2(G))$. As a consequence of the inequality (3.1.3), we can deduce that for any $t \in (0, T]$, $y(t; y_0)$ can be uniquely determined by the measurement $\mathcal{M}_1(y_0)$. This, combined with the continuity of $y(\cdot; y_0)$ with respect to the time variable, implies that $\mathcal{M}_1(y_0)$ can uniquely determine y_0 . Therefore, if the answer to Problem 3.2 is affirmative, we immediately obtain a positive result to Problem 3.1. On the other hand, in the inequality (3.1.3), the constant $\mathcal{C}(t)$ depends on time and in general becomes unbounded as $t \rightarrow 0^+$. Consistently with this, the continuous dependence of y_0 on $\mathcal{M}_1(y_0)$ in the space $L^2_{\mathcal{F}_0}(\Omega; L^2(G))$ fails.

However, even after relaxing the constraints, it may still be challenging to obtain stability results. To overcome this difficulty, one can incorporate additional a priori information about the initial value to achieve the desired stability estimate.

For example, consider the problem with terminal measurements, also known as the backward problem, which concerns determining the solution y to equation (3.1.2) given the data $y(T)$ at time $t = T$. Let us present it below.

Problem 3.3 (Inverse state problem with a terminal measurement). *Can we determine $y(t_0)$ in G , $\mathbb{P}$ -a.s. for $t_0 \in [0, T)$ from $y(T)$?*

Stochastic parabolic equations are ill-posed backward in time. Minor errors in measuring the terminal data may cause huge deviations in the final results, i.e., there is no stability for [Problem 3.3](#). Nevertheless, if we assume an *a priori* bound for $y(0)$ (an assumption aligned with practicality), then stability can be reestablished in some sense. Conditional stability is the term used to describe such a kind of stability.

In a general framework, the conditional stability problem (for two solutions y and $\hat{y}$ to the equation [\(3.1.2\)](#)) can be formulated as follows:

For given $t_0 \in (0, T]$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $M > 0$, find (if possible) a nonnegative function $\beta \in C[0, +\infty)$ such that $\lim_{\eta \rightarrow 0} \beta(\eta) = 0$ and that

$$|y(t_0) - \hat{y}(t_0)|_{L^2_{\mathcal{F}_{t_0}}(\Omega; H^{\alpha_1}(G))} \leq \beta(|y(T) - \hat{y}(T)|_{L^2_{\mathcal{F}_T}(\Omega; H^{\alpha_2}(G))}), \quad \forall y_0, \hat{y}_0 \in U_{M, \alpha_1}, \quad (3.1.4)$$

where

$$U_{M, \alpha_1} \triangleq \{f \in L^2_{\mathcal{F}_0}(\Omega; H^{\alpha_1}(G)) \mid |f|_{L^2_{\mathcal{F}_0}(\Omega; H^{\alpha_1}(G))} \leq M\}. \quad (3.1.5)$$

Remark 3.1.2. Here we expect the existence of β with the assumption that y_0 belongs to a special set U_{M, α_1} , which means that y_0 enjoys an *a priori* bound in some sense. In general, β depends on t_0 , M , α_1 , and α_2 . Hence, the stability result inferred from β depends on the choice of the initial data, leading to its designation as “conditional stability.”

Remark 3.1.3. The requirement that $\lim_{\eta \rightarrow 0} \beta(\eta) = 0$ is used to ensure the continuous dependence of $y(t_0)$ with respect to $y(T)$. Once β exists, it is not unique. To illustrate this, consider $\tilde{\beta}(x) = \beta(x) + x$, which is an alternative function that satisfies the desired properties.

After establishing uniqueness and (conditional) stability, a further important question is how to actually reconstruct the unknown component, i.e., the so-called reconstruction problem. Taking the backward problem as an example, if the answer to [Problem 3.3](#) is affirmative, i.e., the state y can be uniquely determined by the terminal measurement $y(T)$, how can one reconstruct the state y ?

Let us consider the following stochastic parabolic equation:

$$\begin{cases} dy - \sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt = (b_1 \cdot \nabla y + b_2 y + f) dt + (b_3 y + g) dW(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \quad (3.1.6)$$

where

$$\begin{cases} b_1 \in L^\infty_{\mathbb{F}}(0, T; W^{1, \infty}(G; \mathbb{R}^n)), & b_2 \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G)), \\ b_3 \in L^\infty_{\mathbb{F}}(0, T; W^{1, \infty}(G)), \\ f \in L^2_{\mathbb{F}}(0, T; L^2(G)), & g \in L^2_{\mathbb{F}}(0, T; H^1_0(G)). \end{cases}$$

In [Section 3.5](#), we answer the following problem.

Problem 3.4 (Reconstruct the state with a terminal measurement). *Find a function $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^1_0(G))$ that satisfies (3.1.6) with $y(T) = y_T$ which is previously known.*

Sometimes, even though we can observe part of the solution, the information about the system is incomplete. For example, if the boundary conditions satisfied by the stochastic parabolic equation are completely unknown, but internal measurements are still obtained, it is difficult to recover the initial value. Nevertheless, we can still obtain partial information about the solution in a larger domain.

Consider the following stochastic parabolic equation:

$$d\tilde{y} - \sum_{j,k=1}^n (b^{jk}\tilde{y}_{x_j})_{x_k} dt = \tilde{a}_1 \cdot \nabla \tilde{y} dt + \tilde{a}_2 \tilde{y} dt + \tilde{a}_3 \tilde{y} dW(t) \quad \text{in } Q. \quad (3.1.7)$$

Here, $\tilde{a}_1 \in L^\infty_{\mathbb{F}}(0, T; L^\infty_{\text{loc}}(G; \mathbb{R}^n))$, $\tilde{a}_2 \in L^\infty_{\mathbb{F}}(0, T; L^\infty_{\text{loc}}(G))$ and $\tilde{a}_3 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}_{\text{loc}}(G))$.

We first recall the definition of the solution to the equation (3.1.7).

Definition 3.1.1. We call a stochastic process $\tilde{y} \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_{\text{loc}}(G))) \cap L^2_{\mathbb{F}}(0, T; H^1_{\text{loc}}(G))$ a solution to the equation (3.1.7) if for any $t \in [0, T]$, $\hat{G} \subset \subset G$ and $\rho \in H^1_0(\hat{G})$, it holds that

$$\begin{aligned} & \int_{\hat{G}} \tilde{y}(t, x) \rho(x) dx - \int_{\hat{G}} \tilde{y}(0, x) \rho(x) dx \\ &= \int_0^t \int_{\hat{G}} \left[- \sum_{j,k=1}^n b^{jk} \tilde{y}_{x_j}(s, x) \rho_{x_k}(x) + \tilde{a}_1 \cdot \nabla \tilde{y}(s, x) \rho(x) + \tilde{a}_2 \tilde{y}(s, x) \rho(x) \right] dx ds \\ &+ \int_0^t \int_{\hat{G}} \tilde{a}_3 \tilde{y}(s, x) \rho(x) dx dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (3.1.8)$$

We consider the following problem

Problem 3.5 (Inverse state problem with an internal measurement. II). *Can we determine $\tilde{y}$ in $(0, T) \times G$ uniquely from $\tilde{y}|_{(0,T) \times G_0}$?*

Remark 3.1.4. Please note that the problem described in [Problem 3.5](#) remains an inverse state problem. It arises when the physical process is governed by a stochastic parabolic equation, but boundary value of the solution is unknown. We can determine $\tilde{y}(\cdot)$ in Q uniquely. However, unlike the inverse state problem for the equation (3.1.2), we will lose the continuous dependence of $\tilde{y}(t)$ ($t \in (0, T]$) with respect to $\tilde{y}|_{(0,T) \times G_0}$ in $L^2_{\mathcal{F}_t}(\Omega; L^2(G))$. In such case, instead of the whole domain G , we will determine solutions over a subset of whole domain G (see [Theorem 3.6.1](#)).

Finally, if it is difficult to access the interior of the system, we may consider inverse state problems with boundary measurements. For example, this situation arises when the state inside a nuclear reactor or a cell cannot be measured directly. In such cases, measurements can only be taken on part of the boundary of the domain.

Let Γ_0 be a given nonempty open subset of ∂G . Consider the following stochastic parabolic equation:

$$\begin{cases} dy - \sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y) dt + a_3 y dW(t) & \text{in } Q, \\ y = g_1 & \text{on } (0, T) \times \Gamma_0, \\ \frac{\partial y}{\partial \nu} = g_2 & \text{on } (0, T) \times \Gamma_0, \end{cases} \quad (3.1.9)$$

where

$$\begin{aligned} a_1 &\in L_{\mathbb{F}}^{\infty}(0, T; W^{1,\infty}(G; \mathbb{R}^n)), \\ a_2 &\in L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(G)), \\ a_3 &\in L_{\mathbb{F}}^{\infty}(0, T; W^{1,\infty}(G)). \end{aligned} \quad (3.1.10)$$

Let $\mathbf{G}_{\Gamma_0} \triangleq \{G' \subset G \mid \partial G' \cap \partial G \subset \Gamma_0\}$ and

$$H_{\Gamma_0}^2(G) \triangleq \left\{ \eta \in H_{loc}^2(G) \mid \eta|_{G'} \in H^2(G'), \forall G' \in \mathbf{G}_{\Gamma_0} \right\}.$$

We consider the following problem

Problem 3.6 (Inverse state problem with a boundary measurement). *Can we determine y in Q uniquely by the boundary measurements $y|_{(0,T) \times \Gamma_0}$ and $\frac{\partial y}{\partial \nu}|_{(0,T) \times \Gamma_0}$?*

The answer to this problem is provided in [Section 3.7](#).

Remark 3.1.5. In [Problem 3.6](#), we only do the measurement on $(0, T) \times \Gamma_0$. Hence, similar to [Problem 3.5](#), we can determine $y(\cdot)$ in $(0, T) \times G$ uniquely but we will lose the continuous dependence of $y(t)$ ($t \in (0, T]$) with respect to $y|_{(0,T) \times \Gamma_0}$ and $\frac{\partial y}{\partial \nu}|_{(0,T) \times \Gamma_0}$. In such case, instead of the whole domain Q , we will determine solutions over a subset of whole domain Q (see [Theorem 3.7.1](#)).

Remark 3.1.6. If we have the full information of y on $(0, T) \times \Gamma$, then we can get a better stability result for [Problem 3.6](#), which is similar to the one stated in [Problem 3.2](#). The proof is also very similar. We leave the proof of that result to interested readers.

Similarly, once conditional stability is established, one can also consider the reconstruction problem under boundary observations.

This lead to the following problem:

Problem 3.7 (Reconstruct the state with boundary measurements). *Find the unknown solution $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_{loc}(G))) \cap L^2_{\mathbb{F}}(0, T; H^2_{\Gamma_0}(G))$ to the equation (3.1.9) via $y|_{(0,T) \times \Gamma_0}$ and $\frac{\partial y}{\partial \nu}|_{(0,T) \times \Gamma_0}$.*

3.1.2 Inverse source problems

By measuring partial information of the solution, we can not only determine the state but also the source terms of the equation. However, compared to inverse state problems, research in this area is much less developed and more challenging.

First, for the linear case, we consider whether it is possible to determine the source term in the diffusion term.

Consider the following stochastic parabolic equation:

$$\begin{cases} dy - \sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y) dt + (a_3 y + g) dW(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \quad (3.1.11)$$

where a_1, a_2 and a_3 satisfy (3.1.10).

Problem 3.8 (Inverse source problem with a terminal measurement). *For any $t_0 \in (0, T)$, does $y(T)$ uniquely determine $(y(t_0), g)$?*

We will provide the answer to this problem in [Section 3.9](#).

To determine the source term in the drift term is even more challenging. Under the assumption that the source term has a special form, we may consider the following inverse source problem with boundary observation.

Denote $x = (x_1, x') \in \mathbb{R}^n$, where $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Let $G = (0, l) \times G'$, where $l > 0$ and $G' \subset \mathbb{R}^{n-1}$ be a bounded domain with a C^2 boundary. Consider the following stochastic parabolic equation:

$$\begin{cases} dy - \sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y + h(t, x') R(t, x)) dt \\ \quad + g(t, x) dW(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \end{cases} \quad (3.1.12)$$

where $a_1 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G; \mathbb{R}^n))$, $a_2 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G))$, $hR \in L^2_{\mathbb{F}}(0, T; L^2(G))$. Note that R is a deterministic function and h does not depend on x_1 .

Let us first assume that $g = a_3 y$, where $a_3 \in L^\infty_{\mathbb{F}}(0, T; W^{2,\infty}(G))$, and $y(0) = y_0$ in G for a prescribed initial datum y_0 . Consider the following problem, whose answer will be provided in [Section 3.10](#):

Problem 3.9 (Inverse source problem with a boundary measurement). *For a given R and initial datum y_0 , determine the source function $h(\cdot)$ by means of the measurement $\frac{\partial y}{\partial \nu} \Big|_{\Sigma}$.*

If more measurements are available, it is also possible to simultaneously determine the source terms in both the drift and diffusion terms.

Assume $g \in L^2_{\mathbb{F}}(0, T; H^1(G))$ and the value of $y(0)$ is initially unknown.

Consider the following problem:

Problem 3.10. *For a given R , determine the source function $h(\cdot)$ and $g(\cdot, \cdot)$ by means of the measurement $\frac{\partial y}{\partial \nu} \Big|_{\Sigma}$ and $y(T)$.*

We will provide the answer to this problem in [Section 3.11](#).

The rest of this chapter is organized as follows. In [Section 3.2](#), we present a fundamental pointwise weighted identity for second order stochastic parabolic-like operators, which is used to derive Carleman estimates for stochastic parabolic equations. Sections [3.3–3.11](#) are devoted to studying Problems [3.1–3.10](#). At last, in [Section 3.12](#), we give some comments for the content in this chapter and some related open problems.

3.2 A fundamental weighted identity

This section presents a fundamental pointwise weighted identity for second order stochastic parabolic-like operators, which is used to derive Carleman estimates for stochastic parabolic equations.

Before we state our weighted identity, let's start with an intuitive example to understand its key idea.

Proposition 3.2.1. *Let $\tau > 0$. It holds that*

$$2\tau \int_{-1}^1 |u|^2 e^{\tau t^2} dt \leq \int_{-1}^1 \left| \frac{du}{dt} \right|^2 e^{\tau t^2} dt, \quad \forall u \in C_0^\infty(-1, 1) \quad (3.2.1)$$

Proof. Set $v(t) = u(t)e^{\tau t^2/2}$. Then we have

$$\begin{aligned} |u'(t)|^2 e^{\tau t^2} &= |v'(t) - \tau t v(t)|^2 = v'(t)^2 - 2\tau t v(t) v'(t) + \tau^2 t^2 v(t)^2 \\ &= |v'(t) + \tau t v(t)|^2 - 4\tau t v(t) v'(t) \\ &= |v'(t) + \tau t v(t)|^2 + 2\tau |v(t)|^2 - 2\tau (t v(t))^2 \end{aligned} \quad (3.2.2)$$

Integrate this over $[-1, 1]$, noting that $v \in C_0^\infty(-1, 1)$, we get [\(3.2.1\)](#). $\square$

The proof of [\(3.2.1\)](#) is very simple, yet it reveals the two fundamental features in deriving Carleman estimates.

First, the introduction of the weight function $e^{\tau t^2/2}$. The effectiveness of Carleman estimates depends crucially on a clever choice of weight functions, which

differs for different types of equations. The weight function serves to amplify or suppress certain parts of the solution based on the geometry and the nature of the differential operator.

Second, the pointwise weighted identity (3.2.2). This identity demonstrates the core mechanism of Carleman estimates: by multiplying with the weighted multiplier $e^{\tau t^2}$, the principal operator $\frac{d}{dt}$ can be decomposed into:

- A “divergence” term: $-2\tau(tv(t)^2)'$, which typically contributes boundary terms after applying the divergence theorem and vanishes here due to the compact support of v in $(-1, 1)$.
- An “energy” term: $|v'(t) + \tau tv(t)|^2 + 2\tau|v(t)|^2$, which provides the left-hand side of (3.2.1).

This decomposition is the essence of weighted identities: they transform the original differential expression into a form where energy methods can extract quantitative information about the solution. The specific structure of the energy term allows us to control both the function v and its derivative simultaneously, which is crucial for applications to inverse problems.

With the example in place, we are now ready to present the general weighted identity for second order stochastic parabolic-like operators. This identity forms the basis for deriving Carleman estimates, which are cornerstones in the study of inverse problems for stochastic parabolic equations.

Note that the identity is stated in a general form. By choosing different functions ℓ and Ψ , it can be adapted to various cases, such as different observation regions or boundary conditions. Nevertheless, the essential idea remains analogous to (3.2.1).

Theorem 3.2.2. *Let*

$$b^{jk} = b^{kj} \in L^2_{\mathbb{F}}(\Omega; C^1([0, T]; W^{2,\infty}(G))), \quad j, k = 1, 2, \dots, n, \quad (3.2.3)$$

$\ell \in C^{2,3}((0, T) \times G)$ and $\Psi \in C^{1,2}((0, T) \times G)$. Let u be an $H^2(G)$ -valued continuous Itô process. Set $\theta = e^\ell$ and $w = \theta u$. Then, for any $t \in (0, T)$ and a.e. $(x, \omega) \in G \times \Omega$,

$$\begin{aligned} & 2\theta \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right] \left[du - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right] + 2 \sum_{j,k=1}^n (b^{jk} w_{x_j} dw)_{x_k} \\ & + 2 \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n (2b^{jk} b^{j'k'} \ell_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk} b^{j'k'} \ell_{x_j} w_{x_{j'}} w_{x_{k'}}) \right. \\ & \quad \left. + \Psi b^{jk} w_{x_j} w - b^{jk} \left(\mathcal{A} \ell_{x_j} + \frac{\Psi_{x_j}}{2} \right) w^2 \right]_{x_k} dt \\ & = 2 \sum_{j,k=1}^n \ell^{jk} w_{x_j} w_{x_k} dt + \mathcal{B} w^2 dt + d \left(\sum_{j,k=1}^n b^{jk} w_{x_j} w_{x_k} + \mathcal{A} w^2 \right) - \theta^2 \mathcal{A}(du)^2, \end{aligned} \quad (3.2.4)$$

$$+ 2 \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right]^2 dt - \theta^2 \sum_{j,k=1}^n b^{jk} (du_{x_j} + \ell_{x_j} du) (du_{x_k} + \ell_{x_k} du)$$

where

$$\begin{cases} \mathcal{A} \triangleq - \sum_{j,k=1}^n (b^{jk} \ell_{x_j} \ell_{x_k} - b_{x_k}^{jk} \ell_{x_j} - b^{jk} \ell_{x_j x_k}) - \Psi - \ell_t, \\ \mathcal{B} \triangleq 2 \left[\mathcal{A} \Psi - \sum_{j,k=1}^n (\mathcal{A} b^{jk} \ell_{x_j})_{x_k} \right] - \mathcal{A}_t - \sum_{j,k=1}^n (b^{jk} \Psi_{x_k})_{x_j}, \\ c^{jk} \triangleq \sum_{j',k'=1}^n \left[2b^{jk'} (b^{j'k} \ell_{x_{j'}})_{x_{k'}} - (b^{jk} b^{j'k'} \ell_{x_{j'}})_{x_{k'}} \right] - \frac{b_t^{jk}}{2} + \Psi b^{jk}. \end{cases} \quad (3.2.5)$$

Remark 3.2.1. Here, du is a formal notation for the stochastic differential, indicating that u is an $H^2(G)$ -valued continuous Itô process (see [Definition 2.4.1](#)). Suppose that u satisfies

$$u(t) = u(0) + \int_0^t \phi(s) ds + \int_0^t \Phi(s) dW(s), \quad \text{a.s.,} \quad \forall t \in [0, T],$$

where $\phi(\cdot) \in L^1_{\mathbb{F}}(0, T; H^2(G))$ and $\Phi(\cdot) \in L^2_{\mathbb{F}}(0, T; H^2(G))$. Then, for $j = 1, \dots, n$, the stochastic differential du_{x_j} is given by

$$u_{x_j}(t) = u_{x_j}(0) + \int_0^t \phi_{x_j}(s) ds + \int_0^t \Phi_{x_j}(s) dW(s), \quad \text{a.s.,} \quad \forall t \in [0, T].$$

Therefore, [\(3.2.4\)](#) should be interpreted in the sense of integral identities.

Also, $(du)^2$ is a formal notation. The term $\int_0^t (du)^2$ denotes the quadratic variation process of the Itô process u . Specifically, (see [\[2, Section 9.3\]](#)),

$$\int_0^t (du)^2 = \int_0^t (\Phi(s))^2 ds, \quad \text{a.s.,} \quad \forall t \in [0, T].$$

Remark 3.2.2. Compared with the deterministic case, the main difference in the stochastic setting is the appearance of the quadratic variation terms

$$-\theta^2 \mathcal{A}(du)^2 - \theta^2 \sum_{j,k=1}^n b^{jk} (du_{x_j} + \ell_{x_j} du) (du_{x_k} + \ell_{x_k} du)$$

which vanish in the deterministic case.

If the quadratic variation terms have a “good” sign, this can lead to stronger results in the stochastic setting. For example, one can obtain determination results for the source in diffusion term, see [Problem 3.10](#) and [Problem 4.4](#). On the other hand, if these terms have a “bad” sign, additional difficulties arise and extra effort is required to eliminate them. For instance, compared to the deterministic case, it is necessary to modify the choice of the function Ψ in [\(3.2.4\)](#) in order to obtain the corresponding Carleman estimates.

Proof. The proof is divided into four steps.

Step 1. Since $\theta = e^\ell$ and $w = \theta u$, we have that

$$\theta du = dw - \ell_t w dt,$$

and that

$$\begin{aligned} & \theta \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} \\ &= \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} - 2 \sum_{j,k=1}^n b^{jk} \ell_{x_k} w_{x_j} + \sum_{j,k=1}^n (b^{jk} \ell_{x_j} \ell_{x_k} - b_{x_k}^{jk} \ell_{x_j} - b^{jk} \ell_{x_j x_k}) w. \end{aligned} \quad (3.2.6)$$

Put

$$I = - \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w, \quad I_2 = dw + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_k} w_{x_j} dt, \quad I_3 = \Psi w dt. \quad (3.2.7)$$

Combining (3.2.6) and (3.2.7), we obtain

$$2\theta I \left(du - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right) = 2I^2 dt + 2II_2 + 2II_3. \quad (3.2.8)$$

Step 2. In this step, we handle the term II_2 . Applying Itô's formula (Remark 2.4.1), we obtain

$$\begin{aligned} & 2 \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right] dw \\ &= -2 \sum_{j,k=1}^n (b^{jk} w_{x_j} dw)_{x_k} + d \left(\sum_{j,k=1}^n b^{jk} w_{x_j} w_{x_k} + \mathcal{A}w^2 \right) - \sum_{j,k=1}^n b_t^{jk} w_{x_j} w_{x_k} dt \\ & \quad - \mathcal{A}_t w^2 dt - \sum_{j,k=1}^n b^{jk} dw_{x_j} dw_{x_k} - \mathcal{A}(dw)^2. \end{aligned} \quad (3.2.9)$$

Thanks to (3.2.3), we obtain

$$\begin{aligned} & 4 \sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_{x_{k'}} w_{x_j} w_{x_k x_{j'}} \\ &= 2 \sum_{j,k,j',k'=1}^n \left[(b^{jk} b^{j'k'} \ell_{x_k} w_{x_{j'}} w_{x_{k'}})_{x_j} - (b^{jk} b^{j'k'} \ell_{x_{k'}})_{x_{j'}} w_{x_j} w_{x_k} \right]. \end{aligned}$$

Hence, it holds that

$$4 \sum_{j',k'=1}^n \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right] b^{j'k'} \ell_{x_{k'}} w_{x_{j'}}$$

$$\begin{aligned}
&= -2 \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n (2b^{jk}b^{j'k'}\ell_{x_{j'}}w_{x_j}w_{x_{k'}} - b^{jk}b^{j'k'}\ell_{x_j}w_{x_{j'}}w_{x_{k'}}) - \mathcal{A}b^{jk}\ell_{x_j}w^2 \right]_{x_k} \\
&\quad + 2 \sum_{j,k,j',k'=1}^n [2b^{jk'}(b^{j'k}\ell_{x_{j'}})_{x_{k'}} - (b^{jk}b^{j'k'}\ell_{x_{k'}})_{x_{j'}}]w_{x_j}w_{x_k} \\
&\quad - 2 \sum_{j,k=1}^n (\mathcal{A}b^{jk}\ell_{x_j})_{x_k}w^2.
\end{aligned} \tag{3.2.10}$$

Step 3. In this step, we deal with II_3 . Thanks to (3.2.7), we obtain

$$\begin{aligned}
2II_3 &= 2 \left[- \sum_{j,k=1}^n (b^{jk}w_{x_j})_{x_k} + \mathcal{A}w \right] \Psi w dt \\
&= -2 \sum_{j,k=1}^n (b^{jk}\Psi w_{x_j}w - \frac{1}{2}b^{jk}\Psi_{x_j}w^2)_{x_k} dt + 2 \sum_{j,k=1}^n \Psi b^{jk}w_{x_j}w_{x_k} dt \\
&\quad + \left(2\mathcal{A}\Psi - \sum_{j,k=1}^n (b^{jk}\Psi_{x_j})_{x_k} \right) w^2 dt.
\end{aligned} \tag{3.2.11}$$

Step 4. Combining (3.2.8)–(3.2.11), and noting that

$$\begin{aligned}
&\sum_{j,k=1}^n b^{jk}dw_{x_j}dw_{x_k} + \mathcal{A}(dw)^2 \\
&= \sum_{j,k=1}^n \theta^2 b^{jk}(du_{x_j} + \ell_{x_j}du)(du_{x_k} + \ell_{x_k}du) + \theta^2 \mathcal{A}(du)^2,
\end{aligned}$$

we immediately have the desired equality (3.2.4). $\square$

3.3 Solution to inverse state problem with an internal measurement I

In this section, assume additionally that Γ is of class C^4 , as required by the weight construction and the Carleman estimate used below.

This section focuses on the following problems:

- **Problem 3.1:** Can the initial value y_0 be uniquely determined from the internal measurement $\mathcal{M}_1(y_0) = y|_{(0,T) \times G_0}$?
- **Problem 3.2:** Does the solution $y(t)$ depend continuously on the measurement, as expressed by the estimate (3.1.3)?

The following theorem, which is a kind of observability estimate, provides a straightforward extension of [68, Theorem 2.4], presenting a positive answer to these problems.

Theorem 3.3.1. *For each $t \in (0, T]$, there exists a constant $\mathcal{C}(t) > 0$ such that the inequality (3.1.3) holds for any $y_0, \hat{y}_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G))$.*

Remark 3.3.1. Theorem 3.3.1 asserts that the state $y(t)$ of system (3.1.2) (at $t \in (0, T]$) is uniquely determined from the observation $y|_{(0, T] \times G_0}$, $\mathbb{P}$ -a.s., and continuously depends on it. As a result of the arbitrariness of $t \in (0, T]$ in (3.1.3), we deduce that the answer to Problem 3.1 is certainly positive.

In order to prove Theorem 3.3.1, we present a global Carleman estimate for the following stochastic parabolic equation:

$$\begin{cases} dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt = f dt + g dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0 & \text{in } G, \end{cases} \quad (3.3.1)$$

where $z_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G))$, $f \in L^2_{\mathbb{F}}(0, T; L^2(G))$ and $g \in L^2_{\mathbb{F}}(0, T; H^1(G))$.

Let us first recall the following well-known result.

Lemma 3.3.2. [68, Lemma 5.1] *For any nonempty open subset G_1 of G , there is a $\psi \in C^4(\bar{G})$ such that $\psi > 0$ in G , $\psi = 0$ on Γ , and $|\nabla \psi(x)| > 0$ for all $x \in \bar{G} \setminus G_1$.*

Fix a nonempty open set $G_1 \subset\subset G_0$. With the function ψ given in Theorem 3.3.2, we choose the weighted functions in Theorem 3.2.2 as follows:

$$\theta = e^\ell, \quad \ell = \lambda\alpha, \quad \alpha(t, x) = \frac{e^{\mu\psi(x)} - e^{2\mu|\psi|_{C(\bar{G})}}}{t(T-t)}, \quad \varphi(t, x) = \frac{e^{\mu\psi(x)}}{t(T-t)}, \quad (3.3.2)$$

where $\lambda > 1$, $\mu > 1$. Further, let

$$\Psi = 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j x_k}. \quad (3.3.3)$$

For $j, k = 1, 2, \dots, n$, it is straightforward to verify that

$$\ell_t = \lambda\alpha_t, \quad \ell_{x_j} = \lambda\mu\varphi\psi_{x_j}, \quad \ell_{x_j x_k} = \lambda\mu^2\varphi\psi_{x_j}\psi_{x_k} + \lambda\mu\varphi\psi_{x_j x_k} \quad (3.3.4)$$

and that

$$\alpha_t = \varphi^2 O(e^{2\mu|\psi|_{C(\bar{G})}}), \quad \varphi_t = \varphi^2 O(e^{\mu|\psi|_{C(\bar{G})}}). \quad (3.3.5)$$

Here and in what follows, for a positive constant r , we denote $O(\mu^r)$ as a function of order μ^r for large μ . We use $O(e^{\mu|\psi|_{C(\bar{G})}})$ in a similar way.

We state the Carleman estimate for the equation (3.3.1) as follows:

Theorem 3.3.3. *There is a constant $\mu_0 = \mu_0(G, G_0, (b^{jk})_{n \times n}) > 0$ such that for all $\mu \geq \mu_0$, one can find two constants $\mathcal{C} = \mathcal{C}(\mu) > 0$ and $\lambda_0 = \lambda_0(\mu) > 0$ such that for any $\lambda \geq \lambda_0$, the solution $z \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^1_0(G))$, to (3.3.1) satisfies*

$$\begin{aligned} & \lambda^3 \mu^4 \mathbb{E} \int_Q \varphi^3 \theta^2 z^2 dx dt + \lambda \mu^2 \mathbb{E} \int_Q \varphi \theta^2 |\nabla z|^2 dx dt \\ & \leq \mathcal{C} \mathbb{E} \left[\int_Q \theta^2 (f^2 + |\nabla g|^2 + \lambda^2 \mu^2 \varphi^2 g^2) dx dt + \lambda^3 \mu^4 \int_{Q_0} \varphi^3 \theta^2 z^2 dx dt \right]. \end{aligned} \quad (3.3.6)$$

Proof. The proof is divided into four steps.

Step 1. In this step, we estimate the order of the terms in the equality (3.2.4) with respect to λ and μ .

Combining (3.0.1), (3.2.5), (3.3.3) and (3.3.4), we have

$$\ell_{x_j x_k} = \lambda \mu^2 \varphi \psi_{x_j} \psi_{x_k} + \lambda \varphi O(\mu)$$

and

$$\begin{aligned} & \sum_{j,k=1}^n c^{jk} w_{x_j} w_{x_k} \\ & = 2\lambda \mu^2 \varphi \left(\sum_{j,k=1}^n b^{jk} \psi_{x_j} w_{x_k} \right)^2 + \lambda \mu^2 \varphi \left(\sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k} \right) \left(\sum_{j,k=1}^n b^{jk} w_{x_j} w_{x_k} \right) \\ & \quad + \lambda \varphi O(\mu) |\nabla w|^2 \\ & \geq (s_0^2 \lambda \mu^2 \varphi |\nabla \psi|^2 + \lambda \varphi O(\mu)) |\nabla w|^2. \end{aligned} \quad (3.3.7)$$

Thanks to (3.2.5) and (3.3.3)–(3.3.5), we obtain that

$$\begin{aligned} \mathcal{A} & = - \sum_{j,k=1}^n (b^{jk} \ell_{x_j} \ell_{x_k} - b^{jk} \ell_{x_k} \ell_{x_j} + b^{jk} \ell_{x_j x_k}) - \ell_t \\ & = -\lambda^2 \mu^2 \varphi^2 \sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k} + \lambda \varphi^2 O(e^{2\mu|\psi|_{C(\overline{G})}}). \end{aligned} \quad (3.3.8)$$

Next, we estimate $\mathcal{B}$ term by term (referring to the definition of $\mathcal{B}$ in (3.2.5)). Thanks to (3.3.3) and (3.3.4), we see that

$$\Psi = 2\lambda \mu^2 \varphi \sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k} + \lambda \varphi O(\mu). \quad (3.3.9)$$

This, together with (3.3.8), implies that

$$\mathcal{A}\Psi = -2\lambda^3 \mu^4 \varphi^3 \left(\sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k} \right)^2 + \lambda^3 \varphi^3 O(\mu^3) + \lambda^2 \varphi^3 O(\mu^2 e^{2\mu|\psi|_{C(\overline{G})}}). \quad (3.3.10)$$

From (3.3.3), we obtain that

$$\sum_{j,k=1}^n (\mathcal{A}b^{jk}\ell_{x_j})_{x_k} = \sum_{j,k=1}^n b^{jk}\mathcal{A}_{x_k}\ell_{x_j} + \sum_{j,k=1}^n \mathcal{A}b_{x_k}^{jk}\ell_{x_j} + \frac{1}{2}\mathcal{A}\Psi. \quad (3.3.11)$$

Combining (3.2.5), (3.3.2) and (3.3.8), we get that

$$\mathcal{A}_{x_k} = -2\lambda^2\mu^3\varphi^2 \sum_{j',k'=1}^n b^{j'k'}\psi_{x_{j'}}\psi_{x_{k'}}\psi_{x_k} + \lambda^2\varphi^2O(\mu^2) + \lambda\varphi^2O(\mu e^{2\mu|\psi|_{C(\overline{G})}}).$$

This, together with (3.3.4), yields that

$$\begin{aligned} \sum_{j,k=1}^n b^{jk}\mathcal{A}_{x_k}\ell_{x_j} &= -2\lambda^3\mu^4\varphi^3 \left(\sum_{j,k=1}^n b^{jk}\psi_{x_j}\psi_{x_k} \right)^2 + \lambda^3\varphi^3O(\mu^3) \\ &\quad + \lambda^2\varphi^3O(\mu^2e^{2\mu|\psi|_{C(\overline{G})}}). \end{aligned} \quad (3.3.12)$$

From (3.3.4) and (3.3.8), we get that

$$\sum_{j,k=1}^n \mathcal{A}b_{x_k}^{jk}\ell_{x_j} = \lambda^3\varphi^3O(\mu^3) + \lambda^2\varphi^3O(\mu e^{2\mu|\psi|_{C(\overline{G})}}). \quad (3.3.13)$$

By (3.3.5) and (3.3.8), we find that

$$\mathcal{A}_t = \lambda^2\varphi^3O(\mu^2e^{2\mu|\psi|_{C(\overline{G})}}) + \lambda\varphi^3O(e^{2\mu|\psi|_{C(\overline{G})}}). \quad (3.3.14)$$

From (3.3.2) and (3.3.9), we obtain

$$\sum_{j,k=1}^n (b^{jk}\Psi_{x_k})_{x_j} = 2\lambda\mu^4\varphi \left(\sum_{j,k=1}^n b^{jk}\psi_{x_j}\psi_{x_k} \right)^2 + \lambda\varphi O(\mu^3). \quad (3.3.15)$$

Combining (3.3.10)–(3.3.15), we have that

$$\mathcal{B} \geq 2s_0^2\lambda^3\mu^4\varphi^3|\nabla\psi|^4 + \lambda^3\varphi^3O(\mu^3) + \lambda^2\varphi^3O(\mu^2e^{2\mu|\psi|_{C(\overline{G})}}) + \lambda\varphi^3O(e^{2\mu|\psi|_{C(\overline{G})}}). \quad (3.3.16)$$

Step 2. In this step, we apply Theorem 3.2.2 to the equation (3.3.1) and handle the boundary terms.

For the direct proof below, first take $b^{jk} \in C^2(\overline{G})$ and smooth data compatible with the homogeneous boundary condition, so that the weighted identity applies. Approximation gives the estimate for $g \in L_{\mathbb{F}}^2(0, T; H_0^1(G))$. For the full stated class $b^{jk} \in C^1(\overline{G})$ and $g \in L_{\mathbb{F}}^2(0, T; H^1(G))$, (3.3.6) also follows from the stronger estimate in [51, Theorem 1.1], recalled after this proof.

Let us choose $u = z$ in [Theorem 3.2.2](#). Integrating the equality (3.2.4) over Q , taking mathematical expectation in both sides, and noting (3.3.7), (3.3.8) and (3.3.16), we conclude that there is a constant $c_0 > 0$ such that

$$\begin{aligned}
& 2\mathbb{E} \int_Q \theta \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right] \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\
& + 2\mathbb{E} \int_Q \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n \left(2b^{jk} b^{j'k'} \ell_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk} b^{j'k'} \ell_{x_j} w_{x_{j'}} w_{x_{k'}} \right) + \Psi b^{jk} w_{x_j} w \right. \\
& \quad \left. - b^{jk} \left(\mathcal{A} \ell_{x_j} + \frac{\Psi_{x_j}}{2} \right) w^2 \right]_{x_k} dx dt + 2\mathbb{E} \int_Q \sum_{j,k=1}^n (b^{jk} w_{x_j} dw)_{x_k} dx \\
& \geq 2c_0 \mathbb{E} \int_Q \left[\varphi(\lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu)) |\nabla w|^2 + \varphi^3 \left(\lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) \right) \right. \\
& \quad \left. + \lambda^2 O(\mu^2 e^{2\mu|\psi|_{C(\overline{G})}}) + \lambda O(e^{2\mu|\psi|_{C(\overline{G})}}) w^2 \right] dx dt \\
& + 2\mathbb{E} \int_Q \left| - \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right|^2 dx dt \\
& - \mathbb{E} \int_Q \theta^2 \sum_{j,k=1}^n b^{jk} (dz_{x_j} + \ell_{x_j} dz) (dz_{x_k} + \ell_{x_k} dz) dx - \mathbb{E} \int_Q \theta^2 \mathcal{A}(dz)^2 dx,
\end{aligned} \tag{3.3.17}$$

where we use the fact that, for all $x \in G$, $\lim_{t \rightarrow 0+} \alpha = \lim_{t \rightarrow T-} \alpha = -\infty$.

By (3.3.1), we obtain

$$\begin{aligned}
& 2\mathbb{E} \int_Q \theta \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right] \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\
& \leq \mathbb{E} \int_Q \left| - \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right|^2 dt dx + \mathbb{E} \int_Q \theta^2 f^2 dt dx.
\end{aligned} \tag{3.3.18}$$

Next, we deal with divergence terms. Since $z = 0$ on Σ , we get

$$w|_{\Sigma} = 0 \text{ and } w_{x_j}|_{\Sigma} = \theta z_{x_j}|_{\Sigma} = \theta \frac{\partial z}{\partial \nu} \nu_j \Big|_{\Sigma}, \quad \text{for } j = 1, \dots, n.$$

Similarly, by [Theorem 3.3.2](#), we get

$$\ell_{x_j} = \lambda \mu \varphi \psi_{x_j} = \lambda \mu \varphi \frac{\partial \psi}{\partial \nu} \nu_j \quad \text{for } j = 1, \dots, n, \text{ and } \quad \frac{\partial \psi}{\partial \nu} < 0 \quad \text{on } \Sigma.$$

Therefore, via integration by parts and noting $w = 0$ on Σ , we obtain that

$$2\mathbb{E} \int_Q \sum_{j,k=1}^n (b^{jk} w_{x_j} dw)_{x_k} dx = 2\mathbb{E} \int_{\Sigma} \sum_{j,k=1}^n b^{jk} w_{x_j} \nu_k dw dx = 0, \tag{3.3.19}$$

and that

$$\begin{aligned}
& 2\mathbb{E} \int_Q \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n \left(2b^{jk}b^{j'k'} \ell_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk}b^{j'k'} \ell_{x_j} w_{x_{j'}} w_{x_{k'}} \right) \right. \\
& \quad \left. + \Psi b^{jk} w_{x_j} w - b^{jk} \left(\mathcal{A} \ell_{x_j} + \frac{\Psi_{x_j}}{2} \right) w^2 \right]_{x_k} dt dx \quad (3.3.20) \\
& = 2\lambda\mu \mathbb{E} \int_{\Sigma} \varphi \theta^2 \frac{\partial \psi}{\partial \nu} \left(\frac{\partial z}{\partial \nu} \right)^2 \left(\sum_{j,k=1}^n b^{jk} \nu_j \nu_k \right)^2 d\Gamma dt \leq 0.
\end{aligned}$$

Since z is a $H^2(G)$ -valued Itô process, for $j, k = 1, \dots, n$, we obtain that z_{x_j} is a H^1 -valued Itô process. Hence, for $j, k = 1, \dots, n$, we have

$$\int_0^t dz_{x_j} dz_{x_k} = \int_0^t g_{x_j} g_{x_k} ds, \text{ for } (t, x) \in [0, T] \times G, \text{ } \mathbb{P}\text{-a.s.}$$

Thanks to it and Remark 2.4.1, we have

$$\begin{aligned}
& \mathbb{E} \int_Q \theta^2 \sum_{j,k=1}^n b^{jk} (dz_{x_j} + \ell_{x_j} dz) (dz_{x_k} + \ell_{x_k} dz) dx + \mathbb{E} \int_Q \theta^2 \mathcal{A} (dz)^2 dx \\
& = \mathbb{E} \int_Q \theta^2 \sum_{j,k=1}^n b^{jk} (g_{x_j} + \ell_{x_j} g) (g_{x_k} + \ell_{x_k} g) dx dt + \mathbb{E} \int_Q \theta^2 \mathcal{A} g^2 dx dt. \quad (3.3.21)
\end{aligned}$$

Combining (3.3.17)–(3.3.21), we arrive at

$$\begin{aligned}
& 2c_0 \mathbb{E} \int_Q \left[\varphi (\lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu)) |\nabla w|^2 + \varphi^3 \left(\lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) \right) \right. \\
& \quad \left. + \lambda^2 O(\mu^2 e^{2\mu|\psi|_{C(\overline{G})}}) + \lambda O(e^{2\mu|\psi|_{C(\overline{G})}}) w^2 \right] dx dt \quad (3.3.22) \\
& \leq \mathbb{E} \int_Q \theta^2 \left[f^2 + \sum_{j,k=1}^n b^{jk} (g_{x_j} + \ell_{x_j} g) (g_{x_k} + \ell_{x_k} g) + \mathcal{A} g^2 \right] dx dt.
\end{aligned}$$

Step 3. By Theorem 3.3.2, we have $\min_{x \in G \setminus G_1} |\nabla \psi| > 0$. Hence, there exists a $\mu_1 > 0$ such that for all $\mu \geq \mu_1$, one can find a constant $\lambda_1 = \lambda_1(\mu)$ such that for any $\lambda \geq \lambda_1$, it holds that

$$\begin{aligned}
& 2\mathbb{E} \int_Q \left[\varphi (\lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu)) |\nabla w|^2 + \varphi^3 \left(\lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) \right) \right. \\
& \quad \left. + \lambda^2 O(\mu^2 e^{2\mu|\psi|_{C(\overline{G})}}) + \lambda O(e^{2\mu|\psi|_{C(\overline{G})}}) w^2 \right] dx dt \\
& \geq c_1 \lambda \mu^2 \mathbb{E} \int_0^T \int_{G \setminus G_1} \varphi \left(|\nabla w|^2 + \lambda^2 \mu^2 \varphi^2 w^2 \right) dx dt \quad (3.3.23) \\
& \quad - C \lambda \mu^2 \mathbb{E} \int_0^T \int_{G_1} \varphi \left(|\nabla w|^2 + \lambda^2 \mu^2 \varphi^2 w^2 \right) dx dt,
\end{aligned}$$

where $c_1 \triangleq \min \left(\min_{x \in G \setminus G_1} |\nabla \psi|^2, \min_{x \in G \setminus G_1} |\nabla \psi|^4 \right)$.

Note that

$$\frac{1}{C} \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 z^2) \leq |\nabla w|^2 + \lambda^2 \mu^2 \varphi^2 w^2 \leq C \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 z^2).$$

Combining this with (3.3.22) and (3.3.23), implies that

$$\begin{aligned} & \lambda \mu^2 \mathbb{E} \int_Q \varphi \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 z^2) dx dt \\ & \leq C \left[\mathbb{E} \int_Q \theta^2 (f^2 + |\nabla g|^2 + \lambda^2 \mu^2 \varphi^2 g^2) dx dt \right. \\ & \quad \left. + \lambda \mu^2 \mathbb{E} \int_0^T \int_{G_1} \varphi \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 z^2) dx dt \right]. \end{aligned} \quad (3.3.24)$$

Step 4. In this step, we get rid of the term containing ∇z in the right hand side of (3.3.24) and complete the proof of (3.3.6).

Choosing a cut-off function $\zeta \in C_0^\infty(G_0)$ satisfied $0 \leq \zeta \leq 1$ and $\zeta = 1$ in G_1 . Noting that

$$d(\theta^2 \varphi z^2) = z^2 (\theta^2 \varphi)_t dt + 2\theta^2 \varphi z dz + \theta^2 \varphi (dz)^2,$$

and $\lim_{t \rightarrow 0^+} (\theta^2 \varphi)(t, \cdot) = \lim_{t \rightarrow T^-} (\theta^2 \varphi)(t, \cdot) = 0$, utilizing (3.3.1) and Divergence Theorem, we obtain that

$$\begin{aligned} 0 &= \mathbb{E} \int_{Q_0} \zeta^2 \left[z^2 (\theta^2 \varphi)_t dt + 2\theta^2 \varphi z dz + \theta^2 \varphi (dz)^2 \right] dx \\ &= \mathbb{E} \int_{Q_0} \theta^2 \left[\zeta^2 z^2 (\varphi_t + 2\lambda \varphi \alpha_t) - 2\zeta^2 \varphi \sum_{j,k=1}^n b^{jk} z_{x_j} z_{x_k} \right. \\ & \quad \left. - 2\mu \zeta^2 \varphi (1 + 2\lambda \varphi) z \sum_{j,k=1}^n b^{jk} z_{x_j} \psi_{x_k} \right. \\ & \quad \left. - 4\zeta \varphi z \sum_{j,k=1}^n b^{jk} z_{x_j} \zeta_{x_k} + 2\zeta^2 \varphi f z + \zeta^2 \varphi g^2 \right] dx dt. \end{aligned}$$

This, along with (3.3.5), yields that there exists a $\mu_2 > 0$ such that for all $\mu \geq \mu_2$, one can find a constant $\lambda_2 = \lambda_2(\mu)$ so that for any $\lambda \geq \lambda_2$ and $\varepsilon > 0$, it holds

$$\begin{aligned} & 2\mathbb{E} \int_{Q_0} \theta^2 \zeta^2 \varphi \sum_{j,k=1}^n b^{jk} z_{x_j} z_{x_k} dx dt \\ & \leq \varepsilon \mathbb{E} \int_{Q_0} \theta^2 \zeta^2 \varphi |\nabla z|^2 dx dt + \frac{C}{\varepsilon} \mathbb{E} \int_{Q_0} \theta^2 \left(\frac{1}{\lambda^2 \mu^2} f^2 + \lambda^2 \mu^2 \varphi^3 z^2 \right) dx dt \\ & \quad + \mathbb{E} \int_{Q_0} \theta^2 \varphi g^2 dx dt. \end{aligned}$$

By this and (3.0.1), choosing ε sufficiently small, we conclude that

$$\begin{aligned} & \mathbb{E} \int_0^T \int_{G_1} \theta^2 \varphi |\nabla z|^2 dx dt \\ & \leq \mathcal{C} \left[\mathbb{E} \int_Q \theta^2 \left(\frac{1}{\lambda^2 \mu^2} f^2 + \varphi g^2 \right) dx dt + \lambda^2 \mu^2 \mathbb{E} \int_{Q_0} \theta^2 \varphi^3 z^2 dx dt \right]. \end{aligned} \quad (3.3.25)$$

Finally, combining (3.3.24) and (3.3.25), choosing $\mu_0 = \max\{\mu_1, \mu_2\}$ and $\lambda_0 = \max\{\lambda_1, \lambda_2\}$, we obtain (3.3.6). $\square$

Remark 3.3.2. By [51, Theorem 1.1], for $g \in L^2_{\mathbb{F}}(0, T; L^2(G))$, the term $|\nabla g|^2$ can be omitted from the right hand side of (3.3.6). Here μ is fixed sufficiently large; the weights φ and $[t(T-t)]^{-1}$ in that theorem are comparable, with constants depending on μ . The method used by the author is different from the pointwise weighted identity approach adopted in this book. The author first transforms the proof of the Carleman estimate for forward stochastic parabolic equations into a controllability problem for a backward stochastic parabolic equation. Subsequently, this controllability problem is reduced to the Carleman estimate for a random parabolic equation. Through this double duality argument, the Carleman estimate for the forward stochastic parabolic equation is reduced to the known Carleman estimate for deterministic parabolic equations. This result allows for a relaxation of the regularity assumptions on the coefficients in certain inverse problems.

Proof of Theorem 3.3.1. Let $z = y - \hat{y}$. From (3.1.2), we obtain that z satisfies

$$\begin{cases} dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt = [F(\nabla y, y) - F(\nabla \hat{y}, \hat{y})] dt \\ \quad + [K(y) - K(\hat{y})] dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = y_0 - \hat{y}_0 & \text{in } G. \end{cases}$$

By the Lipschitz continuity of F and K (see (3.1.1)), we have $F(\nabla y, y) - F(\nabla \hat{y}, \hat{y}) \in L^2_{\mathbb{F}}(0, T; L^2(G))$ and $K(y) - K(\hat{y}) \in L^2_{\mathbb{F}}(0, T; L^2(G))$. Hence, z fulfills the equation (3.3.1) with

$$f = F(\nabla y, y) - F(\nabla \hat{y}, \hat{y}), \quad g = K(y) - K(\hat{y}). \quad (3.3.26)$$

Fix μ sufficiently large. By the preceding remark, there exists $\lambda_0 = \lambda_0(\mu) > 0$ such that for all $\lambda \geq \lambda_0$, it holds that

$$\begin{aligned} & \lambda^3 \mu^4 \mathbb{E} \int_Q \varphi^3 \theta^2 z^2 dx dt + \lambda \mu^2 \mathbb{E} \int_Q \varphi \theta^2 |\nabla z|^2 dx dt \\ & \leq \mathcal{C} \mathbb{E} \left[\int_Q \theta^2 (f^2 + \lambda^2 \mu^2 \varphi^2 g^2) dx dt + \lambda^3 \mu^4 \int_{Q_0} \varphi^3 \theta^2 z^2 dx dt \right]. \end{aligned}$$

From (3.1.1) and (3.3.26), we have

$$\mathbb{E} \int_Q \theta^2 |f|^2 dx dt \leq 2L^2 \mathbb{E} \int_Q \theta^2 (|\nabla z|^2 + |z|^2) dx dt,$$

and

$$\mathbb{E} \int_Q \theta^2 \lambda^2 \mu^2 \varphi^2 |g|^2 dx dt \leq L^2 \mathbb{E} \int_Q \theta^2 \lambda^2 \mu^2 \varphi^2 |z|^2 dx dt.$$

Hence,

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 (\lambda^3 \mu^4 \varphi^3 |z|^2 + \lambda \mu^2 \varphi |\nabla z|^2) dx dt \\ & \leq C \left\{ L^2 \mathbb{E} \int_Q \theta^2 [|\nabla z|^2 + (1 + \lambda^2 \mu^2 \varphi^2) |z|^2] dx dt + \lambda^3 \mu^4 \mathbb{E} \int_{Q_0} \varphi^3 \theta^2 z^2 dx dt \right\}. \end{aligned}$$

Keeping μ fixed and choosing $\lambda \geq \lambda_0(\mu)$ sufficiently large, depending also on L , we absorb the lower-order terms and obtain

$$\mathbb{E} \int_Q \theta^2 (\lambda^3 \mu^4 \varphi^3 |z|^2 + \lambda \mu^2 \varphi |\nabla z|^2) dx dt \leq C \lambda^3 \mu^4 \mathbb{E} \int_{Q_0} \varphi^3 \theta^2 z^2 dx dt. \quad (3.3.27)$$

From (3.3.2), we find that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 (\varphi^3 |z|^2 + \varphi |\nabla z|^2) dx dt \\ & \geq \min_{x \in \overline{G}} \left(\min\{\varphi^3(T/2, x), \varphi(T/2, x)\} \theta^2(T/4, x) \right) \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_G (|z|^2 + |\nabla z|^2) dx dt \end{aligned} \quad (3.3.28)$$

and that

$$\mathbb{E} \int_{Q_0} \varphi^3 \theta^2 z^2 dx dt \leq \max_{(x,t) \in \overline{Q}} (\varphi^3(t, x) \theta^2(t, x)) \mathbb{E} \int_{Q_0} z^2 dx dt. \quad (3.3.29)$$

Using a standard energy estimate, from (3.1.1), (3.3.1) and (3.3.26), we can obtain that for any $0 \leq s \leq t \leq T$,

$$\mathbb{E} |z(t)|_{L^2(G)}^2 \leq e^{CL^2} \mathbb{E} |z(s)|_{L^2(G)}^2. \quad (3.3.30)$$

For any fixed $t \in (0, T]$, apply the preceding estimates with T replaced by t , and integrate (3.3.30) over $s \in [t/4, 3t/4]$. Combining this with (3.3.27)–(3.3.29) and $(0, t) \times G_0 \subset Q_0$, we obtain (3.1.3). $\square$

3.4 Solution to inverse state problem with a terminal measurement

This section focuses on the following problem:

- **Problem 3.3:** Can we determine $y(t_0)$ in G , $\mathbb{P}$ -a.s. for $t_0 \in [0, T]$ from $y(T)$, where y solves (3.1.2)?

In this section, we consider solutions to (3.1.2) satisfying

$$y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G))) \cap L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G)).$$

For the linear equation (3.1.6), its stated coefficient and forcing assumptions together with $y_0 \in L^2_{\mathcal{F}_0}(\Omega; H^1_0(G))$ ensure this regularity. Indeed, applying Itô's formula to the Dirichlet energy $\int_G \sum_{j,k} b^{jk} y_{x_j} y_{x_k} dx$ and using elliptic regularity gives

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} |y(t)|^2_{H^1(G)} + \mathbb{E} \int_0^T |y(t)|^2_{H^2(G)} dt \\ & \leq \mathcal{C}(|y_0|^2_{L^2_{\mathcal{F}_0}(\Omega; H^1(G))} + |f|^2_{L^2_{\mathbb{F}}(0, T; L^2(G))} + |g|^2_{L^2_{\mathbb{F}}(0, T; H^1(G))}). \end{aligned}$$

A Galerkin approximation justifies this calculation for the stated data.

For the results in this section, we additionally assume that $K(t, \cdot, \cdot)$ maps $H^1_0(G)$ into itself and that, for some $L_K > 0$ and almost every (t, ω) ,

$$|K(t, \cdot, u) - K(t, \cdot, v)|_{H^1(G)} \leq L_K |u - v|_{H^1(G)}, \quad u, v \in H^1_0(G). \quad (3.4.1)$$

This condition holds, for example, for $K(t, x, \zeta) = a_3(t, x)\zeta + K(t, x, 0)$ with $a_3 \in L^\infty_{\mathbb{F}}(0, T; W^{1, \infty}(G))$ and the standing assumption $K(\cdot, \cdot, 0) \in L^2_{\mathbb{F}}(0, T; H^1_0(G))$. In this case, $\nabla(K(y) - K(\hat{y})) = a_3 \nabla(y - \hat{y}) + (y - \hat{y}) \nabla a_3$. The constants below may also depend on L_K . We prove two results for the conditional stability of Problem 3.3. Recall that for any $M > 0$ and $\alpha_1 \geq 0$, we define

$$U_{M, \alpha_1} \triangleq \{f \in L^2_{\mathcal{F}_0}(\Omega; H^{\alpha_1}(G)) \mid |f|_{L^2_{\mathcal{F}_0}(\Omega; H^{\alpha_1}(G))} \leq M\}.$$

Then, the first one is

Theorem 3.4.1. *For any $t_0 \in (0, T]$, there exists a positive constant $\mathcal{C} = \mathcal{C}(t_0)$ such that for some constant $\gamma = \gamma(t_0) \in (0, 1)$ and for any $M > 0$, for all $y_0, \hat{y}_0 \in U_{M, 0}$ satisfying the standing regularity assumption, it holds that*

$$|y(t_0) - \hat{y}(t_0)|_{L^2_{\mathcal{F}_{t_0}}(\Omega; L^2(G))} \leq \mathcal{C} M^{1-\gamma} |y(T) - \hat{y}(T)|^\gamma_{L^2_{\mathcal{F}_T}(\Omega; H^1(G))}, \quad (3.4.2)$$

where $y(\cdot)$ (resp. $\hat{y}(\cdot)$) is the solution to (3.1.2) with respect to the initial datum y_0 (resp. $\hat{y}_0$).

Remark 3.4.1. For the stochastic heat equation $dv - \Delta v dt = av dt + bv dW(t)$ with zero Dirichlet boundary data, a related estimate with a local terminal observation is available when G is bounded and convex, $a \in L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(G))$ and $b \in L_{\mathbb{F}}^{\infty}(0, T; W^{1,\infty}(G))$. For any nonempty open $G_0 \Subset G$, [57, Theorem 1.6] gives constants $\mathcal{C} > 0$ and $\delta \in (0, 1)$ such that

$$|v(T)|_{L_{\mathcal{F}_T}^2(\Omega; L^2(G))} \leq \mathcal{C} |v(0)|_{L_{\mathcal{F}_0}^2(\Omega; L^2(G))}^{1-\delta} |v(T)|_{L_{\mathcal{F}_T}^2(\Omega; L^2(G_0))}^{\delta}.$$

The convexity ensures $(x - x_0) \cdot \nu(x) \geq 0$ for $x_0 \in G_0$ and $x \in \partial G$, which is used to control the boundary terms in that proof.

Remark 3.4.2. From Theorem 3.4.1, we know the answer to Problem 3.3 is positive. Moreover, the inequality (3.4.2) shows that the function $\beta(\cdot)$ in (3.1.4) can be chosen as $\beta(\eta) = \mathcal{C} M^{1-\gamma} \eta^{\gamma}$ for $\eta \in [0, \infty)$.

By a standard energy estimate, Theorem 3.4.1 follows from the following result, which is a generalization of [53, Theorem 1.2].

Theorem 3.4.2. *For each $t_0 \in (0, T]$, there exist constants $\gamma = \gamma(t_0) \in (0, 1)$ and $\mathcal{C} = \mathcal{C}(t_0) > 0$ such that*

$$|y(t_0) - \hat{y}(t_0)|_{L_{\mathcal{F}_{t_0}}^2(\Omega; L^2(G))} \leq \mathcal{C} |y(\cdot) - \hat{y}(\cdot)|_{L_{\mathbb{F}}^2(0, T; L^2(G))}^{1-\gamma} |y(T) - \hat{y}(T)|_{L_{\mathcal{F}_T}^2(\Omega; H^1(G))}^{\gamma}, \quad (3.4.3)$$

where $y(\cdot)$ (resp. $\hat{y}(\cdot)$) solves the equation (3.1.2) with the initial datum y_0 (resp. $\hat{y}_0$).

In Theorem 3.4.1, the left hand side of the condition stability estimate (3.4.2) is $|y(t_0) - \hat{y}(t_0)|_{L_{\mathcal{F}_{t_0}}^2(\Omega; L^2(G))}$ and the right hand side is $|y(T) - \hat{y}(T)|_{L_{\mathcal{F}_T}^2(\Omega; H^1(G))}^{\gamma}$. This means that the unknown state is continuously dependent on the measurement in $L_{\mathcal{F}_{t_0}}^2(\Omega; L^2(G))$ norm when the measurement is in $L_{\mathcal{F}_T}^2(\Omega; H^1(G))$. In other words, the norm for the measured data is stronger than the unknown state. Generally speaking, the stronger of the norm for the measured data, the more accurate measurement should we do. Hence, it is natural to expect to relax the norm in the right hand side of (3.4.2) while enforcing a stronger norm in the left hand side of (3.4.2). This is done by the following result.

Theorem 3.4.3. *For any $t_0 \in (0, T)$, there exists a positive constant $\mathcal{C} = \mathcal{C}(t_0)$ such that for some generic constant $\gamma = \gamma(t_0, T) \in (0, 1)$ and for any $M > 0$, for all $y_0, \hat{y}_0 \in U_{M,1}$ satisfying the standing regularity assumption, letting*

$$\beta(\eta) = \mathcal{C}\eta + \mathcal{C}M^2 \exp[-3^{-\gamma} \ln^{\gamma}(\eta^{-1})], \quad 0 < \eta < 1, \quad (3.4.4)$$

with $\beta(0) = 0$ and $\beta(\eta) = \mathcal{C}(\eta + M^2)$ for $\eta \geq 1$, it holds that

$$|y(t_0) - \hat{y}(t_0)|_{L_{\mathcal{F}_{t_0}}^2(\Omega; H^1(G))}^2 \leq \beta\left(|y(T) - \hat{y}(T)|_{L_{\mathcal{F}_T}^2(\Omega; L^2(G))}\right), \quad (3.4.5)$$

where $y(\cdot)$ (resp. $\hat{y}(\cdot)$) is the solution to (3.1.2) with respect to the initial datum y_0 (resp. $\hat{y}_0$).

3.4.1 Proof of the conditional stability I

In this subsection, we will prove the first result of conditional stability, i.e., [Theorem 3.4.2](#).

Let $s, \lambda \in (0, +\infty)$, $t \in (0, +\infty)$, and $\psi \in C^\infty(\mathbb{R})$ with $\psi_t \geq 1$. Put

$$\varphi = e^{\lambda\psi}, \quad \theta = e^{s\varphi}. \quad (3.4.6)$$

As a consequence of [Theorem 3.2.2](#), we have the following result.

Proposition 3.4.4. *Assume that u is an $H^2(\mathbb{R}^n)$ -valued continuous Itô process. Put $v = \theta u$ (with θ given by (3.4.6)). Then,*

$$\begin{aligned} & -\theta \left[\sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + s\lambda\varphi\psi_t v \right] \left[du - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right] \\ & + \frac{1}{4}\lambda\theta v \left[du - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right] \\ & = - \sum_{j,k=1}^n \left(b^{jk} v_{x_j} dv + \frac{1}{4}\lambda b^{jk} v_{x_j} v dt \right)_{x_k} + \frac{1}{2}d \left(\sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} - s\lambda\varphi\psi_t v^2 + \frac{1}{4}\lambda v^2 \right) \\ & + \left(\frac{1}{4}\lambda \sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} dt - \frac{1}{2} \sum_{j,k=1}^n b^{jk} dv_{x_j} dv_{x_k} \right) + \frac{1}{2}s\lambda^2\varphi\psi_t^2 v^2 dt \\ & + \frac{1}{2}s\lambda\varphi\psi_{tt} v^2 dt - \frac{1}{4}s\lambda^2\psi_t\varphi v^2 dt + \frac{1}{2}s\lambda\varphi\psi_t (dv)^2 - \frac{1}{8}\lambda(dv)^2 \\ & + \left[\sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + s\lambda\varphi\psi_t v \right]^2 dt. \end{aligned} \quad (3.4.7)$$

Proof. Noting that ψ is independent of the x -variable, choosing $\Psi = -\frac{1}{4}\lambda$ in [Theorem 3.2.2](#), the proof is straightforward. $\square$

We have the following Carleman estimate for [\(3.3.1\)](#), which is a generalization of Theorem 2.1 in [\[78\]](#):

Theorem 3.4.5. *Let $\delta \in [0, T)$, and let φ and θ be given in (3.4.6). There exist constants $\mathcal{C} > 0$ and $\lambda_1 > 0$ such that for all $\lambda \geq \lambda_1$, there exists an $s_0(\lambda) > 0$ so that for all $s \geq s_0(\lambda)$, for any solution $z \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G)))$ to the equation [\(3.3.1\)](#) with $g \in L^2_{\mathbb{F}}(0, T; H^1_0(G))$, it holds that*

$$\begin{aligned} & \lambda \mathbb{E} \int_{\delta}^T \int_G \theta^2 |\nabla z|^2 dx dt + s\lambda^2 \mathbb{E} \int_{\delta}^T \int_G \varphi \theta^2 z^2 dx dt + s\lambda \mathbb{E} \int_{\delta}^T \int_G \varphi \theta^2 g^2 dx dt \\ & \leq \mathcal{C} \mathbb{E} \left[\theta^2(T) |\nabla z(T)|^2_{L^2(G)} + \theta^2(\delta) |\nabla z(\delta)|^2_{L^2(G)} + s\lambda\varphi(T) \theta^2(T) |z(T)|^2_{L^2(G)} \right. \\ & \quad \left. + s\lambda\varphi(\delta) \theta^2(\delta) |z(\delta)|^2_{L^2(G)} + \int_{\delta}^T \int_G \theta^2 (f^2 + |\nabla g|^2) dx dt \right]. \end{aligned} \quad (3.4.8)$$

Remark 3.4.3. If we replace the assumption of ψ_t with $|\psi_t| \geq 1$, then one can follow the proof of (3.4.8) step by step to get

$$\begin{aligned} & \lambda \mathbb{E} \int_{\delta}^T \int_G \theta^2 |\nabla z|^2 dx dt + s \lambda^2 \mathbb{E} \int_{\delta}^T \int_G \varphi \theta^2 z^2 dx dt \\ & \leq \mathcal{C} \mathbb{E} \left[\theta^2(T) |\nabla z(T)|_{L^2(G)}^2 + \theta^2(\delta) |\nabla z(\delta)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |z(T)|_{L^2(G)}^2 \right. \\ & \quad \left. + s \lambda \varphi(\delta) \theta^2(\delta) |z(\delta)|_{L^2(G)}^2 + \int_{\delta}^T \int_G \theta^2 (f^2 + |\nabla g|^2 + s \lambda \varphi g^2) dx dt \right]. \end{aligned} \quad (3.4.9)$$

Proof of Theorem 3.4.5. The calculation below is first made for smooth solutions and then extended by Dirichlet spectral approximation, using $g \in L_{\mathbb{F}}^2(0, T; H_0^1(G))$. Applying Theorem 3.4.4 to the equation (3.3.1) with $u = z$, integrating the equality (3.4.7) on $[\delta, T] \times G$, and taking mathematical expectation on both sides, we obtain that

$$\begin{aligned} & - \mathbb{E} \int_{\delta}^T \int_G \theta \left[\sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + s \lambda \varphi \psi_t v \right] \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\ & + \frac{1}{4} \lambda \mathbb{E} \int_{\delta}^T \int_G \theta v \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\ & = - \mathbb{E} \int_{\delta}^T \int_G \sum_{j,k=1}^n \left(b^{jk} v_{x_j} dv + \frac{1}{4} \lambda b^{jk} v_{x_j} v dt \right)_{x_k} dx \\ & + \frac{1}{2} \mathbb{E} \int_{\delta}^T \int_G d \left(\sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} - s \lambda \varphi \psi_t v^2 + \frac{1}{4} \lambda v^2 \right) dx \\ & + \mathbb{E} \int_{\delta}^T \int_G \left(\frac{1}{4} \lambda \sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} dt - \frac{1}{2} \sum_{j,k=1}^n b^{jk} dv_{x_j} dv_{x_k} \right) dx \\ & + \mathbb{E} \int_{\delta}^T \int_G \left[\frac{1}{2} s \lambda^2 \varphi \psi_t^2 v^2 dt + \frac{1}{2} s \lambda \varphi \psi_{tt} v^2 dt - \frac{1}{4} s \lambda^2 \varphi \psi_t v^2 dt \right. \\ & \quad \left. + \frac{1}{2} s \lambda \varphi \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \right] dx \\ & + \mathbb{E} \int_{\delta}^T \int_G \left[\sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + s \lambda \varphi \psi_t v \right]^2 dx dt. \end{aligned} \quad (3.4.10)$$

Now we estimate the terms in the right hand side of (3.4.10) one by one. Noting that $v|_{\Sigma} = z|_{\Sigma} = 0$, thanks to Divergence Theorem, we have

$$\begin{aligned} & - \mathbb{E} \int_{\delta}^T \int_G \sum_{j,k=1}^n \left(b^{jk} v_{x_j} dv + \frac{1}{4} \lambda b^{jk} v_{x_j} v dt \right)_{x_k} dx \\ & = - \mathbb{E} \int_{\delta}^T \int_{\Gamma} \sum_{j,k=1}^n b^{jk} \left(v_{x_j} dv + \frac{1}{4} \lambda v_{x_j} v dt \right) \nu^k d\Gamma = 0. \end{aligned} \quad (3.4.11)$$

The second one reads as follows:

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \int_{\delta}^T \int_G d \left(\sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} - s \lambda \varphi \psi_t v^2 + \frac{1}{4} \lambda v^2 \right) dx \\ & \geq -\mathcal{C} \mathbb{E} \left(|\nabla v(T)|_{L^2(G)}^2 + |\nabla v(\delta)|_{L^2(G)}^2 + s \lambda \varphi(T) |v(T)|_{L^2(G)}^2 + s \lambda \varphi(\delta) |v(\delta)|_{L^2(G)}^2 \right). \end{aligned} \quad (3.4.12)$$

For the third one, thanks to (3.0.1) and (3.3.1), it holds that

$$\begin{aligned} & \mathbb{E} \int_{\delta}^T \int_G \left(\frac{1}{4} \lambda \sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} dt - \frac{1}{2} \sum_{j,k=1}^n b^{jk} dv_{x_j} dv_{x_k} \right) dx \\ & \geq \frac{1}{4} \lambda \mathbb{E} \int_{\delta}^T \int_G s_0 |\nabla v|^2 dx dt - \mathcal{C} \mathbb{E} \int_{\delta}^T \int_G \theta^2 |\nabla g|^2 dx dt. \end{aligned} \quad (3.4.13)$$

Noting that $\psi_t \geq 1$, there exists a $\lambda_0 > 0$ such that, for $\lambda \geq \lambda_0$ and $s \min_{[0,T]} \varphi \geq 1$, it holds that

$$\begin{aligned} & \mathbb{E} \int_{\delta}^T \int_G \left[\frac{1}{2} s \lambda^2 \varphi \psi_t^2 v^2 dt + \frac{1}{2} s \lambda \varphi \psi_{tt} v^2 dt - \frac{1}{4} s \lambda^2 \varphi \psi_t v^2 dt \right. \\ & \quad \left. + \frac{1}{2} s \lambda \varphi \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \right] dx \\ & \geq \frac{1}{8} s \lambda^2 \mathbb{E} \int_{\delta}^T \int_G \varphi v^2 dx dt + \frac{1}{4} s \lambda \mathbb{E} \int_{\delta}^T \int_G \varphi \theta^2 g^2 dx dt. \end{aligned} \quad (3.4.14)$$

Now, we estimate the terms in the left hand side of (3.4.10). Thanks to (3.3.1), we obtain

$$\begin{aligned} & -\mathbb{E} \int_{\delta}^T \int_G \theta \left[\sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + s \lambda \varphi \psi_t v \right] \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\ & \leq \mathbb{E} \int_{\delta}^T \int_G \left[\sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + s \lambda \varphi \psi_t v \right]^2 dx dt + \mathbb{E} \int_{\delta}^T \int_G \theta^2 f^2 dx dt, \end{aligned} \quad (3.4.15)$$

and

$$\begin{aligned} & \frac{1}{4} \lambda \mathbb{E} \int_{\delta}^T \int_G \theta v \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\ & \leq \frac{1}{64} \lambda^2 \mathbb{E} \int_{\delta}^T \int_G v^2 dx dt + \mathbb{E} \int_{\delta}^T \int_G \theta^2 f^2 dx dt. \end{aligned} \quad (3.4.16)$$

From (3.4.10)–(3.4.16), we find

$$\frac{s_0}{4} \lambda \mathbb{E} \int_{\delta}^T \int_G |\nabla v|^2 dx dt + \frac{1}{4} s \lambda \mathbb{E} \int_{\delta}^T \int_G \varphi \theta^2 g^2 dx dt$$

$$\begin{aligned}
& -\mathcal{C}\mathbb{E} \int_{\delta}^T \int_G \theta^2 (|\nabla g|^2 + |f|^2) dxdt \\
& + \mathbb{E} \int_{\delta}^T \int_G \left(\frac{1}{8} s \lambda^2 \varphi - \frac{1}{64} \lambda^2 \right) v^2 dxdt \\
& \leq \mathcal{C}\mathbb{E} \left(|\nabla v(T)|_{L^2(G)}^2 + |\nabla v(\delta)|_{L^2(G)}^2 + s \lambda \varphi(T) |v(T)|_{L^2(G)}^2 + s \lambda \varphi(\delta) |v(\delta)|_{L^2(G)}^2 \right).
\end{aligned} \tag{3.4.17}$$

Hence, there exists a $\lambda_1 \geq \lambda_0$ such that for all $\lambda \geq \lambda_1$, one can find an $s_0(\lambda) > 0$ so that for all $s \geq s_0(\lambda)$, it holds

$$\begin{aligned}
& \lambda \mathbb{E} \int_{\delta}^T \int_G |\nabla v|^2 dxdt + s \lambda \mathbb{E} \int_{\delta}^T \int_G \varphi \theta^2 g^2 dxdt + s \lambda^2 \mathbb{E} \int_{\delta}^T \int_G \varphi v^2 dxdt \\
& \leq \mathcal{C}\mathbb{E} \left[|\nabla v(T)|_{L^2(G)}^2 + |\nabla v(\delta)|_{L^2(G)}^2 + s \lambda \varphi(T) |v(T)|_{L^2(G)}^2 \right. \\
& \quad \left. + s \lambda \varphi(\delta) |v(\delta)|_{L^2(G)}^2 \right] + \mathcal{C}\mathbb{E} \int_{\delta}^T \int_G \theta^2 (|\nabla g|^2 + |f|^2) dxdt,
\end{aligned}$$

which implies inequality (3.4.8) immediately. $\square$

Now, we are in a position to prove [Theorem 3.4.2](#).

Proof of Theorem 3.4.2. Choose t_1 and t_2 such that $0 < t_1 < t_2 < t_0$. Set $\rho \in C^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$ and

$$\rho = \begin{cases} 1, & t \geq t_2, \\ 0, & t \leq t_1. \end{cases} \tag{3.4.18}$$

Let $z = \rho(y - \hat{y})$. Then, z solves

$$\begin{cases} dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt = [\rho(F(\nabla y, y) - F(\nabla \hat{y}, \hat{y})) + \rho_t(y - \hat{y})] dt \\ \quad + \rho(K(y) - K(\hat{y})) dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{in } G. \end{cases} \tag{3.4.19}$$

Applying [Theorem 3.4.5](#) with $\psi = t$ and $\delta = 0$ to the equation (3.4.19), for $\lambda \geq \lambda_1$ and $s \geq s_0(\lambda)$, we have

$$\begin{aligned}
& \lambda \mathbb{E} \int_Q \theta^2 |\nabla z|^2 dxdt + s \lambda^2 \mathbb{E} \int_Q \theta^2 \varphi |z|^2 dxdt \\
& \leq \mathcal{C}\mathbb{E} \left[\theta^2(T) |\nabla z(T)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |z(T)|_{L^2(G)}^2 \right. \\
& \quad + \int_Q \theta^2 |\rho_t(y - \hat{y})|^2 dxdt + \int_Q \theta^2 |\rho(F(\nabla y, y) - F(\nabla \hat{y}, \hat{y}))|^2 dxdt \\
& \quad \left. + \int_Q \theta^2 |\rho \nabla(K(y) - K(\hat{y}))|^2 dxdt \right].
\end{aligned} \tag{3.4.20}$$

From the choice of ρ and the fact that $\theta(t) \leq \theta(s)$ for $t \leq s$, we get that

$$\mathbb{E} \int_Q \theta^2 |\rho_t(y - \hat{y})|^2 dx dt \leq \mathcal{C} \theta(t_2)^2 |y - \hat{y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2. \quad (3.4.21)$$

From (3.1.1) and (3.4.1), since ρ and θ depend only on time, we have

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 |\rho(F(\nabla y, y) - F(\nabla \hat{y}, \hat{y}))|^2 dx dt + \mathbb{E} \int_Q \theta^2 |\rho \nabla (K(y) - K(\hat{y}))|^2 dx dt \\ & \leq \mathcal{C} \mathbb{E} \int_Q \theta^2 (z^2 + |\nabla z|^2) dx dt. \end{aligned} \quad (3.4.22)$$

This, together with the inequality (3.4.20) and the fact that $\theta(t) \leq \theta(s)$ for $t \leq s$, implies that there exists a $\lambda_2 \geq \lambda_1$ such that for all $\lambda \geq \lambda_2$, we have

$$\begin{aligned} & \lambda \theta^2(t_0) \mathbb{E} \int_{t_0}^T \int_G |\nabla y - \nabla \hat{y}|^2 dx dt + s \lambda^2 \theta^2(t_0) \mathbb{E} \int_{t_0}^T \int_G \varphi |y - \hat{y}|^2 dx dt \\ & \leq \lambda \mathbb{E} \int_Q \theta^2 |\nabla z|^2 dx dt + s \lambda^2 \mathbb{E} \int_Q \theta^2 \varphi |z|^2 dx dt \\ & \leq \mathcal{C} \theta^2(t_2) |y - \hat{y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2 \\ & \quad + \mathcal{C} \mathbb{E} \left(\theta^2(T) |\nabla y(T) - \nabla \hat{y}(T)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |y(T) - \hat{y}(T)|_{L^2(G)}^2 \right). \end{aligned} \quad (3.4.23)$$

Thanks to Itô's formula, (3.1.1) and (3.1.2), we obtain that

$$\begin{aligned} & \mathbb{E} \int_G |y(t_0) - \hat{y}(t_0)|^2 dx \\ & = \mathbb{E} \int_G |y(T) - \hat{y}(T)|^2 dx - \mathbb{E} \int_{t_0}^T \int_G \{2(y - \hat{y})d(y - \hat{y}) + [d(y - \hat{y})]^2\} dx \\ & \leq \mathbb{E} \int_G |y(T) - \hat{y}(T)|^2 dx + \mathcal{C} \mathbb{E} \int_{t_0}^T \int_G |\nabla y - \nabla \hat{y}|^2 dx dt + \mathcal{C} \mathbb{E} \int_{t_0}^T \int_G |y - \hat{y}|^2 dx dt. \end{aligned} \quad (3.4.24)$$

Recalling that $\varphi \geq 1$, combining the inequalities (3.4.23) and (3.4.24), for any $\lambda \geq \lambda_2$ and $s \geq s_0(\lambda)$, we have

$$\begin{aligned} & \mathbb{E} \int_G |y(t_0) - \hat{y}(t_0)|^2 dx \\ & \leq \mathcal{C} \theta^2(t_2) \theta^{-2}(t_0) |y - \hat{y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2 \\ & \quad + \mathcal{C} \mathbb{E} \left(\theta^2(T) |\nabla y(T) - \nabla \hat{y}(T)|_{L^2(G)}^2 + s \lambda \varphi(T) \theta^2(T) |y(T) - \hat{y}(T)|_{L^2(G)}^2 \right). \end{aligned} \quad (3.4.25)$$

Now we fix $\lambda = \lambda_2$. It follows from the inequality (3.4.25) that

$$\mathbb{E} \int_G |y(t_0) - \hat{y}(t_0)|^2 dx$$

$$\leq \mathcal{C}(\theta^2(t_2)\theta^{-2}(t_0)|y - \hat{y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2 + (1+s)\theta^2(T)\mathbb{E}|y(T) - \hat{y}(T)|_{H^1(G)}^2). \quad (3.4.26)$$

Using $1 + s \leq 2e^s$ in (3.4.26), for all $s \geq s_0(\lambda_2)$, we obtain

$$\begin{aligned} \mathbb{E} \int_G |y(t_0) - \hat{y}(t_0)|^2 dx &\leq \mathcal{C}e^{-2s(e^{\lambda_2 t_0} - e^{\lambda_2 t_2})} |y - \hat{y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2 \\ &\quad + \mathcal{C}e^{\mathcal{C}s} \mathbb{E}|y(T) - \hat{y}(T)|_{H^1(G)}^2. \end{aligned} \quad (3.4.27)$$

Optimizing (3.4.27) over $s \geq s_0(\lambda_2)$, and using the forward energy estimate $|y(t_0) - \hat{y}(t_0)|_{L^2_{\mathcal{F}_{t_0}}(\Omega;L^2(G))} \leq \mathcal{C}(t_0)|y - \hat{y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}$ when the unconstrained minimizer is smaller than $s_0(\lambda_2)$, we obtain

$$|y(t_0) - \hat{y}(t_0)|_{L^2_{\mathcal{F}_{t_0}}(\Omega;L^2(G))} \leq \mathcal{C}|y - \hat{y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^{1-\gamma} |y(T) - \hat{y}(T)|_{L^2_{\mathcal{F}_T}(\Omega;H^1(G))}^{\gamma}, \quad (3.4.28)$$

with $\gamma = \frac{2(e^{\lambda_2 t_0} - e^{\lambda_2 t_2})}{\mathcal{C} + 2(e^{\lambda_2 t_0} - e^{\lambda_2 t_2})}$. This completes the proof of Theorem 3.4.2. $\square$

3.4.2 Proof of the conditional stability II

In this subsection, we will prove the second conditional stability result, i.e., Theorem 3.4.3. To this end, we need to establish the following Carleman weight function, which is different from (3.4.6).

Let $\lambda \in (0, +\infty)$ and $t \in [0, T]$. Put

$$\psi = (t+1)^\lambda, \quad \theta = e^\psi. \quad (3.4.29)$$

We have the following Carleman estimate for (3.3.1):

Theorem 3.4.6. *Let $\delta \in [0, T)$, and ψ and θ be given in (3.4.29). There exist constants $\mathcal{C} > 0$ and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ and for any solution z to (3.3.1) with $z(\delta) \in L^2_{\mathcal{F}_\delta}(\Omega; H_0^1(G))$ and $g \in L^2_{\mathbb{F}}(\delta, T; H_0^1(G))$, it holds that*

$$\begin{aligned} &\mathbb{E} \int_\delta^T \int_G \lambda^2(t+1)^{(\lambda-2)} \theta^2 z^2 dx dt + \mathbb{E} \int_\delta^T \int_G \lambda(t+1)^{-1} \theta^2 |\nabla z|^2 dx dt \\ &\leq \mathcal{C} \mathbb{E} \left[\lambda(T+1)^{\lambda-1} \theta^2(T) |z(T)|_{L^2(G)}^2 + \theta^2(\delta) |\nabla z(\delta)|_{L^2(G)}^2 \right. \\ &\quad \left. + \int_\delta^T \int_G \theta^2 (f^2 + g^2 + |\nabla g|^2) dx dt \right]. \end{aligned} \quad (3.4.30)$$

Proof. Let $\ell = \psi$, $w = \theta z$ and $\Psi = \frac{1}{4}\lambda(t+1)^{-1}$. As above, Dirichlet spectral approximation justifies the following calculation. Apply Theorem 3.2.2 to the equation (3.3.1) with $u = z$.

For $j = 1, \dots, n$, it is easy to verify that $\ell_{x_j} = 0$ and $\ell_t = \lambda(t+1)^{\lambda-1}$. Integrating the equality (3.2.4) on $[\delta, T] \times G$ and taking mathematical expectation in

both sides, we obtain that

$$\begin{aligned}
& -2\mathbb{E} \int_{\delta}^T \int_G \theta \left\{ \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] w \right\} \\
& \quad \times \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\
& = -2\mathbb{E} \int_{\delta}^T \int_G \sum_{j,k=1}^n \left[b^{jk} w_{x_j} dw + \frac{1}{4} \lambda(t+1)^{-1} b^{jk} w_{x_j} w dt \right]_{x_k} dx \\
& \quad + 2\mathbb{E} \int_{\delta}^T \int_G \sum_{j,k=1}^n \left[\frac{1}{4} \lambda(t+1)^{-1} b^{jk} - \frac{b_t^{jk}}{2} \right] w_{x_j} w_{x_k} dx dt \\
& \quad + \mathbb{E} \int_{\delta}^T \int_G \left[-\frac{1}{8} \lambda^2(t+1)^{-2} - \frac{1}{2} \lambda^2(t+1)^{\lambda-2} - \frac{1}{4} \lambda(t+1)^{-2} \right. \\
& \quad \quad \left. + \lambda(\lambda-1)(t+1)^{\lambda-2} \right] w^2 dx dt \\
& \quad + \mathbb{E} \int_{\delta}^T \int_G d \left\{ \sum_{j,k=1}^n b^{jk} w_{x_j} w_{x_k} - \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] w^2 \right\} dx \\
& \quad + 2\mathbb{E} \int_{\delta}^T \int_G \left\{ \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] w \right\}^2 dx dt \\
& \quad + \mathbb{E} \int_{\delta}^T \int_G \theta^2 \left[-\sum_{j,k=1}^n b^{jk} dz_{x_j} dz_{x_k} + \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] (dz)^2 \right] dx.
\end{aligned} \tag{3.4.31}$$

Now we handle the terms in (3.4.31). Since $z = w = 0$ on Σ , by Divergence Theorem, we have

$$-2\mathbb{E} \int_{\delta}^T \int_G \sum_{j,k=1}^n \left[b^{jk} w_{x_j} dw + \frac{1}{4} \lambda(t+1)^{-1} b^{jk} w_{x_j} w dt \right]_{x_k} dx = 0. \tag{3.4.32}$$

Noting (3.0.1), we get that

$$\begin{aligned}
& 2\mathbb{E} \int_{\delta}^T \int_G \sum_{j,k=1}^n \left[\frac{1}{4} \lambda(t+1)^{-1} b^{jk} - \frac{b_t^{jk}}{2} \right] w_{x_j} w_{x_k} dx dt \\
& \geq \frac{s_0}{2} \mathbb{E} \int_{\delta}^T \int_G \lambda(t+1)^{-1} |\nabla w|^2 dx dt - C \mathbb{E} \int_{\delta}^T \int_G |\nabla w|^2 dx dt.
\end{aligned} \tag{3.4.33}$$

Since $(t+1)^{-2} \leq (t+1)^{\lambda-2}$ on $[0, T]$, for $\lambda \geq 10$ we have

$$\begin{aligned}
& \mathbb{E} \int_{\delta}^T \int_G \left[-\frac{1}{8} \lambda^2(t+1)^{-2} - \frac{1}{2} \lambda^2(t+1)^{\lambda-2} - \frac{1}{4} \lambda(t+1)^{-2} \right. \\
& \quad \left. + \lambda(\lambda-1)(t+1)^{\lambda-2} \right] w^2 dx dt
\end{aligned}$$

$$\geq \mathbb{E} \int_{\delta}^T \int_G \frac{1}{4} \lambda^2 (t+1)^{\lambda-2} w^2 dx dt. \quad (3.4.34)$$

Thanks to (3.0.1) and $\lambda > 0$, we obtain that

$$\begin{aligned} & \mathbb{E} \int_{\delta}^T \int_G d \left\{ \sum_{j,k=1}^n b^{jk} w_{x_j} w_{x_k} - \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] w^2 \right\} dx \\ & \geq s_0 \mathbb{E} \int_G |\nabla w(T)|^2 dx - \frac{5}{4} \mathbb{E} \int_G \lambda(T+1)^{\lambda-1} |w(T)|^2 dx - \mathcal{C} \mathbb{E} \int_G |\nabla w(\delta)|^2 dx. \end{aligned} \quad (3.4.35)$$

Noting (3.3.1), we have that

$$\mathbb{E} \int_{\delta}^T \int_G \theta^2 \left(- \sum_{j,k=1}^n b^{jk} dz_{x_j} dz_{x_k} \right) dx \geq -\mathcal{C} \mathbb{E} \int_{\delta}^T \int_G \theta^2 |\nabla g|^2 dx dt. \quad (3.4.36)$$

The remaining quadratic variation term is nonnegative:

$$\mathbb{E} \int_{\delta}^T \int_G \theta^2 \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] (dz)^2 dx \geq 0. \quad (3.4.37)$$

As for the left hand side of (3.4.31), from (3.3.1), we have

$$\begin{aligned} & -2 \mathbb{E} \int_{\delta}^T \int_G \theta \left\{ \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] w \right\} \\ & \quad \times \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\ & \leq \mathbb{E} \int_{\delta}^T \int_G \left\{ \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \lambda(t+1)^{-1} \left[\frac{1}{4} + (t+1)^{\lambda} \right] w \right\}^2 dx dt \\ & \quad + \mathcal{C} \mathbb{E} \int_{\delta}^T \int_G \theta^2 f^2 dx dt. \end{aligned} \quad (3.4.38)$$

Combining (3.4.31)–(3.4.38), and noting that $w = \theta z$ and $\nabla w = \theta \nabla z$, we arrive that

$$\begin{aligned} & \mathcal{C} \mathbb{E} \int_{\delta}^T \int_G \theta^2 (f^2 + |\nabla g|^2) dx dt \\ & \geq \mathbb{E} \int_{\delta}^T \int_G \theta^2 [\lambda(t+1)^{-1} |\nabla z|^2 + \lambda^2 (t+1)^{\lambda-2} z^2] dx dt \\ & \quad - \mathcal{C} \mathbb{E} [\lambda(T+1)^{\lambda-1} \theta^2(T) |z(T)|_{L^2(G)}^2 + \theta^2(\delta) |\nabla z(\delta)|_{L^2(G)}^2] \\ & \quad - \mathcal{C} \mathbb{E} \int_{\delta}^T \int_G \theta^2 (|\nabla z|^2 + \lambda |z|^2) dx dt. \end{aligned} \quad (3.4.39)$$

From (3.4.39), we know there exists $\lambda_2 \geq \max\{10, \mathcal{C}(T+1)\}$, such that for all $\lambda \geq \lambda_2$, it holds that

$$\begin{aligned} & \mathcal{C}\mathbb{E} \int_{\delta}^T \int_G \theta^2 [f^2 + |\nabla g|^2] dx dt \\ & \geq \mathbb{E} \int_{\delta}^T \int_G \theta^2 \left[\lambda(t+1)^{-1} |\nabla z|^2 + \lambda^2(t+1)^{\lambda-2} z^2 \right] dx dt \\ & \quad - \mathcal{C}\mathbb{E} \left[\lambda(T+1)^{\lambda-1} \theta^2(T) |z(T)|_{L^2(G)}^2 + \theta^2(\delta) |\nabla z(\delta)|_{L^2(G)}^2 \right], \end{aligned}$$

which completes the proof. $\square$

Now, we are in a position to prove [Theorem 3.4.3](#).

Proof of Theorem 3.4.3. Let $v(x, t) = y(x, t) - \hat{y}(x, t)$. From (3.1.2), we have

$$\begin{cases} dv - \sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} dt = (F(\nabla y, y) - F(\nabla \hat{y}, \hat{y})) dt \\ \quad + (K(y) - K(\hat{y})) dW(t) & \text{in } Q, \\ v = 0 & \text{on } \Sigma. \end{cases}$$

From [Theorem 3.4.6](#) and (3.1.1), (3.4.1) and (3.4.29), for $\lambda \geq \lambda_0$, we obtain that

$$\begin{aligned} & \mathcal{C}\mathbb{E} \left[\lambda(T+1)^{\lambda-1} \theta^2(T) |v(T)|_{L^2(G)}^2 + \theta^2(0) |\nabla v(0)|_{L^2(G)}^2 \right] \\ & \geq e^{2(t_0+1)\lambda} \mathbb{E} \int_{t_0}^T \int_G (v^2 + |\nabla v|^2) dx dt, \end{aligned} \quad (3.4.40)$$

where (3.4.1) and the fact that θ depends only on time give

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 |F(\nabla y, y) - F(\nabla \hat{y}, \hat{y})|^2 dx dt + \mathbb{E} \int_Q \theta^2 |\nabla (K(y) - K(\hat{y}))|^2 dx dt \\ & \quad + \mathbb{E} \int_Q \theta^2 |K(y) - K(\hat{y})|^2 dx dt \\ & \leq \mathcal{C}\mathbb{E} \int_Q \theta^2 [v^2 + |\nabla v|^2] dx dt. \end{aligned}$$

It is clear that there exists $\lambda_1 \geq \lambda_0$ such that for all $\lambda \geq \lambda_1$, it holds that

$$\lambda(T+1)^{\lambda-1} e^{2(T+1)\lambda-2(t_0+1)\lambda} \leq e^{3(T+1)\lambda}. \quad (3.4.41)$$

Combining (3.4.40) and (3.4.41), for all $\lambda \geq \lambda_1$, we get that

$$|v|_{L^2_{\mathbb{F}}(t_0, T; H^1(G))}^2 \leq C_1 e^{3(T+1)\lambda} \mathbb{E} |v(T)|_{L^2(G)}^2 + C_2 e^{-2(t_0+1)\lambda} \mathbb{E} |\nabla v(0)|_{L^2(G)}^2. \quad (3.4.42)$$

Letting $s = (T + 1)^\lambda$, we have $(t_0 + 1)^\lambda = s^\gamma$ for $\gamma = \frac{\ln(t_0+1)}{\ln(T+1)}$. It follows from (3.4.42) that there exists $s_1 > 0$ such that for all $s \geq s_1$,

$$|v|_{L^2_{\mathbb{F}}(t_0, T; H^1(G))}^2 \leq C_1 e^{3s} \mathbb{E}|v(T)|_{L^2(G)}^2 + C_2 e^{-2s^\gamma} \mathbb{E}|\nabla v(0)|_{L^2(G)}^2. \quad (3.4.43)$$

For $0 < \eta = |v(T)|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))} \leq e^{-3s_1}$, choose $s = s_2 = -\frac{1}{6} \ln(\mathbb{E}|v(T)|_{L^2(G)}^2)$. Then

$$e^{3s} \mathbb{E}|v(T)|_{L^2(G)}^2 \leq |v(T)|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))}^2.$$

Together with (3.4.4) and (3.4.43), this proves the following estimate for small positive η . The case $\eta = 0$ follows by letting $s \rightarrow \infty$; for larger η , the forward energy estimate gives the same conclusion after increasing $\mathcal{C}$:

$$|v|_{L^2_{\mathbb{F}}(t_0, T; H^1(G))}^2 \leq \beta(|v(T)|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))}). \quad (3.4.44)$$

For each $t_1 \in (0, T]$ such that $t_0 < t_1$, since

$$\mathbb{E}|v(t_1)|_{H^1(G)}^2 \leq \mathcal{C} \mathbb{E}|v(t)|_{H^1(G)}^2, \quad \forall t_0 \leq t \leq t_1, \quad (3.4.45)$$

integrating (3.4.45) on $[t_0, t_1]$, we have

$$(t_1 - t_0) \mathbb{E}|v(t_1)|_{H^1(G)}^2 \leq \mathcal{C} \mathbb{E} \int_{t_0}^{t_1} |v(t)|_{H^1(G)}^2 dt.$$

This, together with (3.4.44), yields

$$|v(t_1)|_{L^2_{\mathcal{F}_{t_1}}(\Omega; H^1(G))}^2 \leq \mathcal{C} (t_1 - t_0)^{-1} \beta(|v(T)|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))}),$$

Taking the lower time in this argument to be $t_0/2$ and $t_1 = t_0$ proves (3.4.5), with $\gamma = \ln(1 + t_0/2)/\ln(1 + T)$. $\square$

3.5 Reconstruct the unknown state with a terminal measurement

This section focuses on the following problem:

- **Problem 3.4:** Find a function $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^1_0(G))$ that satisfies (3.1.6) with $y(T) = y_T$ which is previously known.

In the previous sections, we study whether the unknown state/source could be uniquely determined through suitable measurements. In such a context, we do not take into account the errors in the measurements. When addressing the problem of reconstructing the unknown state/source from the measurements, the measurement errors must be considered. Various methods of “regularization” have

been developed for inverse problems of deterministic PDEs to tackle this issue, all of which, in essence, utilize additional information about the unknown function. For instance, the method introduced by Tikhonov aims to minimize a quadratic functional that contains higher derivatives to reproduce the measured data. In this section, we employ this method to investigate the reconstruction problem for the unknown state with a terminal measurement.

Denote $\mathcal{H}_0 \triangleq L^2_{\mathcal{F}_0}(\Omega; H^2(G) \cap H^1_0(G))$. Suppose $y_T^* \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$ is the exact terminal value of the system (3.1.6) with the initial datum $y_0^* \in \mathcal{H}_0$. Let $y_T^\delta \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$ be the measured data satisfying

$$|y_T^* - y_T^\delta|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))} \leq \delta, \quad (3.5.1)$$

where $\delta > 0$ is the noise level of data.

Consider the following Tikhonov type functional:

$$\begin{aligned} \mathcal{J}_\tau(y_0) = & \mathbb{E} \int_G \left\{ y_0 + \int_0^T \left[\sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} + b_1 \cdot \nabla y + b_2 y + f \right] ds \right. \\ & \left. + \int_0^T (b_3 y + g) dW(s) - y_T^\delta \right\}^2 dx + \tau \mathbb{E} |y_0|_{H^2(G)}^2, \end{aligned} \quad (3.5.2)$$

where $y_0 \in \mathcal{H}_0$ and y solves (3.1.6) with the initial datum y_0 .

Since $\mathcal{J}_\tau(\cdot)$ is coercive, strictly convex and lower semi-continuous, we have the following result.

Proposition 3.5.1. *For every $\tau \in (0, 1)$, there exists a unique minimizer $\bar{y}_0 \in \mathcal{H}_0$ of the functional (3.5.2).*

We have the following error estimate.

Theorem 3.5.2. *For any $t_0 \in (0, T)$, there exists a positive constant $\mathcal{C} = \mathcal{C}(t_0)$ such that for $\gamma = \gamma(t_0, T) \in (0, 1)$ as stated in Theorem 3.4.3, for all $\delta \in (0, 1)$ and $\tau = \delta^2$, putting $\bar{y}_0^\delta$ as the minimizer of functional $\mathcal{J}_\tau(\cdot)$, then we have*

$$|y(t_0; \bar{y}_0^\delta) - y(t_0; y_0^*)|_{L^2_{\mathcal{F}_{t_0}}(\Omega; H^1(G))} \leq \mathcal{C} M \exp[-3^{-\gamma} [\ln^+(M^{-1/2} \delta^{-1})]^\gamma],$$

where $\ln^+ r = \max\{\ln r, 0\}$, $M = 1 + |y_0^*|_{\mathcal{H}_0}^2$, and $y(\cdot; \bar{y}_0^\delta)$ (resp. $y(\cdot; y_0^*)$) is the solution to (3.1.6) with respect to the initial datum $\bar{y}_0^\delta$ (resp. y_0^*).

Proof. For simplicity of notations, we denote

$$\begin{aligned} \mathcal{A}y_0 \triangleq & y_0 + \int_0^T \left[\sum_{j,k=1}^n (b^{jk} y_{x_j}(s; y_0))_{x_k} + b_1 \cdot \nabla y(s; y_0) + b_2 y(s; y_0) + f \right] ds \\ & + \int_0^T (b_3 y(s; y_0) + g) dW(s), \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

and define the bounded linear operator

$$\mathcal{B}y_0 \triangleq \mathcal{A}y_0 - \mathcal{A}0.$$

Thus $\mathcal{B}y_0$ is the terminal value of the homogeneous equation with initial datum y_0 , and $\mathcal{A}u - \mathcal{A}v = \mathcal{B}(u - v)$.

The Gâteaux derivation of $\mathcal{J}_\tau(\cdot)$ at y_0 reads

$$\mathcal{J}'_\tau(y_0)\phi = 2\mathbb{E} \int_G \mathcal{A}y_0 \mathcal{B}\phi dx - 2\mathbb{E} \int_G y_T^\delta \mathcal{B}\phi dx + 2\tau \mathbb{E} \langle y_0, \phi \rangle_{H^2(G)}, \quad \forall \phi \in \mathcal{H}_0.$$

Particularly, for the minimizer $\bar{y}_0^\delta$, we obtain

$$\mathbb{E} \int_G \mathcal{A}\bar{y}_0^\delta \mathcal{B}\phi dx + \tau \mathbb{E} \langle \bar{y}_0^\delta, \phi \rangle_{H^2(G)} = \mathbb{E} \int_G y_T^\delta \mathcal{B}\phi dx, \quad \forall \phi \in \mathcal{H}_0. \quad (3.5.3)$$

Recalling that y_0^* is the exact initial value, thanks to (3.1.6), we obtain

$$\mathbb{E} \int_G \mathcal{A}y_0^* \mathcal{B}\phi dx + \tau \mathbb{E} \langle y_0^*, \phi \rangle_{H^2(G)} = \mathbb{E} \int_G y_T^* \mathcal{B}\phi dx + \tau \mathbb{E} \langle y_0^*, \phi \rangle_{H^2(G)}, \quad \forall \phi \in \mathcal{H}_0. \quad (3.5.4)$$

From (3.5.3) and (3.5.4), we have

$$\begin{aligned} & \mathbb{E} \int_G \mathcal{B}(\bar{y}_0^\delta - y_0^*) \mathcal{B}\phi dx + \tau \mathbb{E} \langle (\bar{y}_0^\delta - y_0^*), \phi \rangle_{H^2(G)} \\ &= \mathbb{E} \int_G (y_T^\delta - y_T^*) \mathcal{B}\phi dx - \tau \mathbb{E} \langle y_0^*, \phi \rangle_{H^2(G)}, \quad \forall \phi \in \mathcal{H}_0. \end{aligned} \quad (3.5.5)$$

Choose $\phi = \bar{y}_0^\delta - y_0^*$ in (3.5.5). Noting that $\mathcal{B}\phi = \mathcal{A}\bar{y}_0^\delta - \mathcal{A}y_0^*$, using Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} & \mathbb{E} \int_G (\mathcal{A}\bar{y}_0^\delta - \mathcal{A}y_0^*)^2 dx + \tau \mathbb{E} |\bar{y}_0^\delta - y_0^*|_{H^2(G)}^2 \\ & \leq \frac{1}{2} \mathbb{E} |y_T^\delta - y_T^*|_{L^2(G)}^2 + \frac{1}{2} \mathbb{E} \int_G (\mathcal{A}\bar{y}_0^\delta - \mathcal{A}y_0^*)^2 dx + \frac{1}{2} \tau \mathbb{E} |y_0^*|_{H^2(G)}^2 \\ & \quad + \frac{1}{2} \tau \mathbb{E} |\bar{y}_0^\delta - y_0^*|_{H^2(G)}^2. \end{aligned} \quad (3.5.6)$$

Combining (3.5.1) and (3.5.6), recalling that $\tau = \delta^2$, we have

$$\mathbb{E} \int_G (\mathcal{A}\bar{y}_0^\delta - \mathcal{A}y_0^*)^2 dx + \delta^2 \mathbb{E} |\bar{y}_0^\delta - y_0^*|_{H^2(G)}^2 \leq \delta^2 (1 + \mathbb{E} |y_0^*|_{H^2(G)}^2).$$

Consequently,

$$\mathbb{E} |\bar{y}_0^\delta - y_0^*|_{H^2(G)}^2 \leq 1 + \mathbb{E} |y_0^*|_{H^2(G)}^2, \quad (3.5.7)$$

and

$$\mathbb{E}|y(T; \bar{y}_0^\delta) - y_T^*|_{L^2(G)}^2 \leq \mathbb{E} \int_G (\mathcal{A}\bar{y}_0^\delta - \mathcal{A}y_0^*)^2 dx \leq \delta^2(1 + \mathbb{E}|y_0^*|_{H^2(G)}^2). \quad (3.5.8)$$

Apply [Theorem 3.4.3](#) to the difference of the two solutions, using [\(3.5.7\)](#) and [\(3.5.8\)](#). If $\delta\sqrt{M} < 1$, its linear terminal-data term is absorbed into the displayed logarithmic bound after enlarging $\mathcal{C}$. If $\delta\sqrt{M} \geq 1$, the forward energy estimate and [\(3.5.7\)](#) give a bound by $\mathcal{C}M$, which is the same formula with $\ln^+(M^{-1/2}\delta^{-1}) = 0$. $\square$

3.6 Solution to inverse state problem with an internal measurement. II

In this section, assume additionally that $b^{jk} \in C^2(G)$.

This section focuses on the following problem:

- **Problem 3.5:** Can we determine $\tilde{y}$ in $(0, T) \times G$ uniquely from $\tilde{y}|_{(0,T) \times G_0}$?

For the situation when boundary conditions are unknown, i.e., [Problem 3.5](#), we can obtain the following quantitative unique continuation property for the equation [\(3.1.7\)](#).

Theorem 3.6.1. *For any $\hat{G} \subset\subset G$ with $G_0 \subset\subset \hat{G}$ and $\kappa \in (0, \frac{1}{2})$ being arbitrarily fixed, there exists a constant $\mathcal{C}$ and a subdomain $\tilde{G} \subset\subset G$ with $\hat{G} \subset\subset \tilde{G}$, such that for any solution $\tilde{y}$ to the equation [\(3.1.7\)](#), it holds that*

$$\begin{aligned} & \mathbb{E} \int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T} \int_{\hat{G}} (|\nabla \tilde{y}|^2 + \tilde{y}^2) dx dt \\ & \leq \mathcal{C} \left[\exp\left(\frac{\mathcal{C}}{\varepsilon}\right) \mathbb{E} \int_0^T \int_{G_0} (\tilde{y}^2 + |\nabla \tilde{y}|^2) dx dt + \varepsilon \mathbb{E} \int_0^T \int_{\tilde{G}} (\tilde{y}^2 + |\nabla \tilde{y}|^2) dx dt \right], \end{aligned} \quad (3.6.1)$$

for all $\varepsilon > 0$.

Remark 3.6.1. From the arbitrariness of $\hat{G} \subset\subset G$, [Theorem 3.6.1](#) provide a positive answer to [Problem 3.5](#). Moreover, on sets with a uniform bound in $L^2_{\mathbb{F}}(0, T; H^1(\tilde{G}))$, it provides the conditional continuous dependence of the solution in $(\frac{T}{2} - \kappa T, \frac{T}{2} + \kappa T) \times \hat{G}$ with respect to the measurement on $(0, T) \times G_0$.

Proof of Theorem 3.6.1. We divide the proof into four steps.

Step 1. In this step, we introduce a cut-off function to convert the equation [\(3.1.7\)](#) to an equation with the homogeneous boundary condition.

Note that since we obtain a local result, it is not necessary to assume any boundary conditions for $\tilde{y}$. Without loss of generality, we assume that the boundary ∂G of G is smooth. Otherwise, we can choose a domain G'' with smooth

boundary $\partial G''$ such that $\widehat{G} \subset \subset G'' \subset \subset G$ and study the local Carleman estimate on G'' .

Recall that $G_0 \subset \subset G$ is an open subset. Thanks to [Theorem 3.3.2](#), there exists a function $\psi \in C^4(G)$ such that

$$\psi > 0 \text{ in } G, \quad \psi = 0 \text{ on } \partial G, \quad |\nabla \psi| > 0 \text{ in } G \setminus G_0.$$

Since $\widehat{G} \subset \subset G'' \subset \subset G$, it is clear that we can fix N large enough such that

$$\widehat{G} \subset \left\{ x \in G \mid \psi(x) > \frac{4}{N} |\psi|_{L^\infty(G)} \right\}. \quad (3.6.2)$$

For $\kappa \in (0, \frac{1}{2})$, take $\iota = \frac{1}{\sqrt{2}}(\frac{1}{2} - \kappa)T > 0$. Then $\sqrt{2}\iota = (\frac{1}{2} - \kappa)T$. Choose a constant $c > 0$ satisfying

$$c\iota^2 < |\psi|_{L^\infty(G)} < 2c\iota^2. \quad (3.6.3)$$

Let $t_0 \in [\sqrt{2}\iota, T - \sqrt{2}\iota]$. Set

$$\phi(t, x) \triangleq \psi(x) - c(t - t_0)^2, \quad \alpha(t, x) \triangleq e^{\mu\phi}.$$

Put

$$\beta_k \triangleq \exp \left(\mu \left(\frac{k}{N} |\psi|_{L^\infty(G)} - \frac{c}{N} \iota^2 \right) \right), \quad k = 1, 2, 3, 4$$

and

$$Q_k \triangleq \{(t, x) \in [0, T] \times \overline{G} \mid \alpha(t, x) > \beta_k\}. \quad (3.6.4)$$

It is obvious that $Q_{k+1} \subset Q_k, k = 1, 2, 3$.

We claim that

$$\left(t_0 - \frac{\iota}{\sqrt{N}}, t_0 + \frac{\iota}{\sqrt{N}} \right) \times \widehat{G} \subset Q_4 \subset Q_1 \subset (t_0 - \sqrt{2}\iota, t_0 + \sqrt{2}\iota) \times \overline{G}. \quad (3.6.5)$$

In fact, thanks to [\(3.6.2\)](#), for all $(t, x) \in (t_0 - \frac{\iota}{\sqrt{N}}, t_0 + \frac{\iota}{\sqrt{N}}) \times \widehat{G}$, it holds that

$$(t - t_0)^2 \leq \frac{\iota^2}{N}, \quad \psi(x) > \frac{4}{N} |\psi|_{L^\infty(G)},$$

which implies that

$$\phi(t, x) = \psi(x) - c(t - t_0)^2 > \frac{4}{N} |\psi|_{L^\infty(G)} - \frac{c\iota^2}{N}.$$

Consequently, we have $\alpha(t, x) > \beta_4$ and $(t, x) \in Q_4$. On the other hand, for all $(t, x) \in Q_1$, it holds that $\frac{1}{N} |\psi|_{L^\infty(G)} - \frac{c\iota^2}{N} < \psi - c(t - t_0)^2$. From [\(3.6.3\)](#), we obtain $0 < 2c\iota^2 - c(t - t_0)^2$. Hence, it holds that $(t - t_0)^2 \leq 2\iota^2$, which implies $(t, x) \in (t_0 - \sqrt{2}\iota, t_0 + \sqrt{2}\iota) \times \overline{G}$ and [\(3.6.5\)](#) is proved.

Let $\eta \in C_0^\infty(Q_2)$ be such that $\eta \in [0, 1]$ and $\eta = 1$ in Q_3 . Let $z = \eta \tilde{y}$. Then

$$\begin{cases} dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt = \tilde{a}_1 \cdot \nabla z dt + \tilde{a}_2 z dt + f dt + \tilde{a}_3 z dW(t) & \text{in } Q_1 \\ z = \frac{\partial z}{\partial \nu} = 0 & \text{on } \partial Q_1, \end{cases} \quad (3.6.6)$$

where

$$f = \eta_t \tilde{y} - 2 \sum_{j,k=1}^n b^{jk} \eta_{x_j} \tilde{y}_{x_k} - \tilde{y} \sum_{j,k=1}^n (b^{jk} \eta_{x_j})_{x_k} - \tilde{y} \tilde{a}_1 \cdot \nabla \eta. \quad (3.6.7)$$

Clearly, f is supported in $Q_2 \setminus Q_3$.

Step 2. In this step, we apply [Theorem 3.2.2](#) to the equation (3.6.7) and handle the boundary terms.

Let us choose $u = z$ in [Theorem 3.2.2](#). Integrate (3.2.4) over $G \times (0, T)$. Noting that z is supported in Q_1 , choosing $\theta = e^\ell$ and $\ell = \lambda \alpha$, setting $w = \theta z$, after taking mathematical expectation in both sides, we have

$$\begin{aligned} & 2\mathbb{E} \int_{Q_1} \theta \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right] \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\ & + 2\mathbb{E} \int_{Q_1} \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n (2b^{jk} b^{j'k'} \ell_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk} b^{j'k'} \ell_{x_j} w_{x_{j'}} w_{x_{k'}}) + \Psi b^{jk} w_{x_j} w \right. \\ & \quad \left. - b^{jk} \left(\mathcal{A} \ell_{x_j} + \frac{\Psi_{x_j}}{2} \right) w^2 \right]_{x_k} dx dt + 2\mathbb{E} \int_{Q_1} \sum_{j,k=1}^n (b^{jk} w_{x_j} dw)_{x_k} dx \\ & = 2\mathbb{E} \int_{Q_1} \sum_{j,k=1}^n c^{jk} w_{x_j} w_{x_k} dt dx + \mathbb{E} \int_{Q_1} \mathcal{B} w^2 dt dx \\ & + 2\mathbb{E} \int_{Q_1} \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right]^2 dt dx - \mathbb{E} \int_{Q_1} \theta^2 \mathcal{A}(dz)^2 dx \\ & - \mathbb{E} \int_{Q_1} \theta^2 \sum_{j,k=1}^n b^{jk} (dz_{x_j} + \ell_{x_j} dz) (dz_{x_k} + \ell_{x_k} dz) dx. \end{aligned} \quad (3.6.8)$$

Next, we estimate terms in (3.6.8) one by one.

From (3.6.6), the first term in the left hand side of (3.6.8) satisfies that

$$\begin{aligned} & 2\mathbb{E} \int_{Q_1} \theta \left[- \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right] \left[dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] dx \\ & \leq \mathbb{E} \int_{Q_1} \left| - \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} + \mathcal{A}w \right|^2 dx dt + \mathbb{E} \int_{Q_1} \theta^2 (\tilde{a}_1 \cdot \nabla z + \tilde{a}_2 z + f)^2 dx dt. \end{aligned} \quad (3.6.9)$$

Thanks to Divergence Theorem, recalling that z is supported in Q_1 , we have

$$2\mathbb{E} \int_{Q_1} \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n (2b^{jk}b^{j'k'} \ell_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk}b^{j'k'} \ell_{x_j} w_{x_{j'}} w_{x_{k'}}) + \Psi b^{jk} w_{x_j} w - b^{jk} \left(\mathcal{A} \ell_{x_j} + \frac{\Psi_{x_j}}{2} \right) w^2 \right]_{x_k} dx dt = 0. \quad (3.6.10)$$

Combining (3.6.8)–(3.6.10), we have

$$\begin{aligned} & \mathbb{E} \int_{Q_1} \theta^2 (\tilde{a}_1 \cdot \nabla z + \tilde{a}_2 z + f)^2 dt dx \\ & \geq 2\mathbb{E} \int_{Q_1} \sum_{j,k=1}^n c^{jk} w_{x_j} w_{x_k} dt dx + \mathbb{E} \int_{Q_1} \mathcal{B} w^2 dx dt - \mathbb{E} \int_{Q_1} \theta^2 \mathcal{A} (dz)^2 dx \\ & \quad - \mathbb{E} \int_{Q_1} \theta^2 \sum_{j,k=1}^n b^{jk} [(dz_{x_j} + \ell_{x_j} dz)(dz_{x_k} + \ell_{x_k} dz)] dx. \end{aligned} \quad (3.6.11)$$

From (3.6.6) and (3.6.7), we have

$$\begin{aligned} & \mathbb{E} \int_{Q_1} \theta^2 (\tilde{a}_1 \cdot \nabla z + \tilde{a}_2 z + f)^2 dx dt \\ & \leq C\mathbb{E} \int_{Q_1} \theta^2 |\nabla z|^2 dx dt + C\mathbb{E} \int_{Q_1} \theta^2 z^2 dx dt + C\mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla \tilde{y}|^2 + \tilde{y}^2) dx dt. \end{aligned} \quad (3.6.12)$$

Step 3. In this step, we handle the terms in the right hand side of inequality (3.6.11). Choosing Ψ as (3.3.3), for $j, k = 1, 2, \dots, n$, it is straightforward to verify that

$$\ell_t = \lambda \alpha O(\mu), \quad \ell_{x_j} = \lambda \mu \alpha \psi_{x_j}, \quad \ell_{x_j x_k} = \lambda \mu^2 \alpha \psi_{x_j} \psi_{x_k} + \lambda \mu \alpha \psi_{x_j x_k}. \quad (3.6.13)$$

Similarly to the proof of Theorem 3.3.3, combining (3.2.5), (3.3.3) and (3.6.13), we get that

$$\begin{aligned} \sum_{j,k=1}^n c^{jk} w_{x_j} w_{x_k} &= 2\lambda \mu^2 \alpha \left(\sum_{j,k=1}^n b^{jk} \psi_{x_j} w_{x_k} \right)^2 \\ &+ \lambda \mu^2 \alpha \left(\sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k} \right) \left(\sum_{j,k=1}^n b^{jk} w_{x_j} w_{x_k} \right) + \lambda \alpha O(\mu) |\nabla w|^2. \end{aligned} \quad (3.6.14)$$

Thanks to (3.2.5), (3.3.3) and (3.6.13), we obtain that

$$\mathcal{A} = -\lambda^2 \mu^2 \alpha^2 \sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k} + \lambda \alpha O(\mu^2), \quad (3.6.15)$$

and that

$$\mathcal{B} = 2\lambda^3\mu^4\alpha^3\left(\sum_{j,k=1}^n b^{jk}\psi_{x_j}\psi_{x_k}\right)^2 + \lambda^3\alpha^3O(\mu^3) + \lambda^2\alpha^2O(\mu^4). \quad (3.6.16)$$

From (3.6.7), we find that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 \sum_{j,k=1}^n b^{jk} (dz_{x_j} + \ell_{x_j} dz) (dz_{x_k} + \ell_{x_k} dz) dx \\ & \leq \mathcal{C} \left[\mathbb{E} \int_Q \theta^2 (z^2 + |\nabla z|^2) dx dt + \lambda^2 \mu^2 \mathbb{E} \int_Q \theta^2 \alpha^2 |z|^2 dx dt \right]. \end{aligned} \quad (3.6.17)$$

Noting that $w = \theta z$ and $z = \theta^{-1}w$, we get that

$$\frac{1}{\mathcal{C}} \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \alpha^2 z^2) \leq |\nabla w|^2 + \lambda^2 \mu^2 \alpha^2 |w|^2 \leq \mathcal{C} \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \alpha^2 z^2).$$

Combining it with (3.6.11), (3.6.12) and (3.6.14)–(3.6.17), we obtain

$$\begin{aligned} & \mathcal{C} \mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla \tilde{y}|^2 + |\tilde{y}|^2) dx dt + \mathcal{C} \mathbb{E} \int_{Q_1} \theta^2 (|\nabla z|^2 + (\lambda^2 \mu^2 \alpha^2 + 1) z^2) dx dt \\ & + \mathcal{C} \mathbb{E} \int_{Q_0} \theta^2 (\lambda \mu^2 \alpha |\nabla z|^2 + \lambda^3 \mu^4 \alpha^3 |z|^2) dx dt \\ & \geq \mathbb{E} \int_{Q_1 \cup Q_0} \theta^2 (\lambda \mu^2 \alpha |\nabla z|^2 + \lambda^3 \mu^4 \alpha^3 |z|^2) dx dt \\ & + \mathbb{E} \int_{Q_1} \theta^2 [\lambda \alpha O(\mu) |\nabla z|^2 + (\lambda^3 \alpha^3 O(\mu^3) + \lambda^2 \alpha^2 O(\mu^4)) z^2] dx dt \end{aligned}$$

This implies that there exists a constant $\mu_0 > 1$, such that for all $\mu > \mu_0$, there exists a constant $\lambda_0 = \lambda_0(\mu)$, so that for any $\lambda > \lambda_0(\mu)$, it holds that

$$\begin{aligned} & \mathcal{C} \mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla \tilde{y}|^2 + |\tilde{y}|^2) dx dt + \mathcal{C} \mathbb{E} \int_{Q_0} \theta^2 (\lambda \mu^2 \alpha |\nabla z|^2 + \lambda^3 \mu^4 \alpha^3 |z|^2) dx dt \\ & \geq \mathbb{E} \int_{Q_1} \theta^2 (\lambda \mu^2 \alpha |\nabla z|^2 + \lambda^3 \mu^4 \alpha^3 |z|^2) dx dt \end{aligned} \quad (3.6.18)$$

Noting that $z = \tilde{y}$ in Q_3 and $Q_3 \subset Q_1$, for $\mu > \mu_0$ and $\lambda > \lambda_0$, we deduce from (3.6.18) that

$$\begin{aligned} & \mathcal{C} \mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla \tilde{y}|^2 + |\tilde{y}|^2) dx dt + \mathcal{C} \mathbb{E} \int_{Q_0} \theta^2 (\lambda \mu^2 \alpha |\nabla \tilde{y}|^2 + \lambda^3 \mu^4 \alpha^3 |\tilde{y}|^2) dx dt \\ & \geq \mathbb{E} \int_{Q_3} \theta^2 (\lambda \mu^2 \alpha |\nabla \tilde{y}|^2 + \lambda^3 \mu^4 \alpha^3 |\tilde{y}|^2) dx dt. \end{aligned} \quad (3.6.19)$$

From the definition of $Q_k, k = 1, 2, 3, 4$ (see (3.6.4)) and $\widehat{G}$, we see that

$$\lambda^3 \mu^4 \mathbb{E} \int_{Q_3} \alpha^3 \theta^2 \tilde{y}^2 dx dt + \lambda \mu^2 \mathbb{E} \int_{Q_3} \alpha \theta^2 |\nabla \tilde{y}|^2 dx dt$$

$$\geq \mathcal{C}e^{2\lambda\beta_4}\mathbb{E}\int_{t_0-\frac{\iota}{\sqrt{N}}}^{t_0+\frac{\iota}{\sqrt{N}}}\int_{\widehat{G}}(\lambda^3\tilde{y}^2+\lambda|\nabla\tilde{y}|^2)dxdt, \quad (3.6.20)$$

where we use (3.6.5) and recall that $\iota = \frac{1}{\sqrt{2}}(\frac{1}{2} - \kappa)T > 0$.

From the relationship between Q_2 and Q_3 , we find

$$\mathbb{E}\int_{Q_2\setminus Q_3}\theta^2(|\nabla\tilde{y}|^2+\tilde{y}^2)dxdt\leq e^{2\lambda\beta_3}\mathbb{E}\int_{Q_1}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt. \quad (3.6.21)$$

Combining (3.6.19)–(3.6.21), for $\mu > \mu_0$ and $\lambda > \lambda_0$, we obtain that

$$\begin{aligned} & e^{2\lambda\beta_4}\mathbb{E}\int_{t_0-\frac{\iota}{\sqrt{N}}}^{t_0+\frac{\iota}{\sqrt{N}}}\int_{\widehat{G}}(\lambda^3\tilde{y}^2+\lambda|\nabla\tilde{y}|^2)dxdt \\ & \leq \mathcal{C}\lambda^3\mu^4\mathbb{E}\int_{Q_0}\alpha^3\theta^2\tilde{y}^2dxdt + \mathcal{C}\lambda\mu^2\mathbb{E}\int_{Q_0}\alpha\theta^2|\nabla\tilde{y}|^2dxdt \\ & \quad + \mathcal{C}e^{2\lambda\beta_3}\mathbb{E}\int_{Q_1}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt, \end{aligned}$$

which yields that

$$\begin{aligned} & \mathbb{E}\int_{t_0-\frac{\iota}{\sqrt{N}}}^{t_0+\frac{\iota}{\sqrt{N}}}\int_{\widehat{G}}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt \\ & \leq \mathcal{C}e^{\mathcal{C}\lambda}\mathbb{E}\int_{Q_0}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt + \mathcal{C}e^{-2\lambda(\beta_4-\beta_3)}\mathbb{E}\int_{Q_1}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt. \end{aligned} \quad (3.6.22)$$

Put $\varepsilon_0 = \exp(-2\lambda_0(\beta_4 - \beta_3))$. Then one finds that for any $0 < \varepsilon < \varepsilon_0$, it holds that

$$\begin{aligned} & \mathbb{E}\int_{t_0-\frac{\iota}{\sqrt{N}}}^{t_0+\frac{\iota}{\sqrt{N}}}\int_{\widehat{G}}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt \\ & \leq \mathcal{C}\exp\left(\frac{\mathcal{C}}{\varepsilon}\right)\mathbb{E}\int_{Q_0}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt + \mathcal{C}\varepsilon\mathbb{E}\int_{Q_1}(\tilde{y}^2+|\nabla\tilde{y}|^2)dxdt. \end{aligned} \quad (3.6.23)$$

Clearly, (3.6.23) holds for $\varepsilon \geq \varepsilon_0$ after increasing $\mathcal{C}$, since the domain on the left is contained in Q_1 . Hence, it is true for all $\varepsilon > 0$.

Step 4. In this step, we handle the left hand side of (3.6.23) to end the proof.

Choose $\tilde{G} \subset\subset G$ containing $\widehat{G}$ and the compact set $\{x \in \overline{G} : \psi(x) \geq (|\psi|_{L^\infty(G)} - c\iota^2)/N\}$. By (3.6.3), this set lies in G and contains the spatial projection of every Q_1 used below.

Let $m \in \mathbb{N} \cup \{0\}$ such that

$$\sqrt{2}\iota + \frac{m\iota}{\sqrt{N}} \leq T - \sqrt{2}\iota \leq \sqrt{2}\iota + \frac{(m+1)\iota}{\sqrt{N}}.$$

Recalling that t_0 can be any element in $[\sqrt{2}\iota, T - \sqrt{2}\iota]$, we take

$$t_0 = \sqrt{2}\iota + \frac{k\iota}{\sqrt{N}}, \quad k = 0, 1, 2, \dots, m.$$

Then, it follows from (3.6.23) that

$$\begin{aligned} & \mathbb{E} \int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T} \int_{\widehat{G}} (\tilde{y}^2 + |\nabla \tilde{y}|^2) dx dt \\ & \leq C \exp\left(\frac{C}{\varepsilon}\right) \mathbb{E} \int_{Q_0} (\tilde{y}^2 + |\nabla \tilde{y}|^2) dx dt + C\varepsilon \mathbb{E} \int_{(0,T) \times \widehat{G}} (\tilde{y}^2 + |\nabla \tilde{y}|^2) dx dt. \end{aligned}$$

This completes the proof of Theorem 3.6.1. $\square$

3.7 Solution to inverse state problem with a boundary measurement

This section focuses on the following problem:

- **Problem 3.6:** Can we determine y in Q uniquely by the boundary measurements $y|_{(0,T) \times \Gamma_0}$ and $\frac{\partial y}{\partial \nu}|_{(0,T) \times \Gamma_0}$?

Assume in this section that ∂G is of class C^4 and $b^{jk} \in C^2(\overline{G})$. Consider the following stochastic parabolic equation:

$$\begin{cases} dy - \sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y + f) dt + a_3 y dW(t) & \text{in } Q, \\ y = g_1 & \text{on } (0, T) \times \Gamma_0, \\ \frac{\partial y}{\partial \nu} = g_2 & \text{on } (0, T) \times \Gamma_0, \end{cases} \quad (3.7.1)$$

where a_1, a_2, a_3 satisfy (3.1.10) and $f \in L^2_{\mathbb{F}}(0, T; L^2(G))$.

For the solution of (3.7.1), we have the following conditional stability.

Theorem 3.7.1. *For any given $\widehat{G} \subset\subset G$ and $\varepsilon \in (0, \frac{T}{2})$, there exists a constant $C > 0$ such that for some generic constant $\gamma \in (0, 1)$, it holds that*

$$\mathbb{E} \int_{\varepsilon}^{T-\varepsilon} \int_{\widehat{G}} (y^2 + |\nabla y|^2) dx dt \leq C |y|_{L^2_{\mathbb{F}}((0,T); H^1(G))}^{2(1-\gamma)} (\mathcal{K}(f, g_1, g_2))^{\gamma}, \quad (3.7.2)$$

where $y(\cdot)$ is the solution to (3.7.1) and

$$\begin{aligned} \mathcal{K}(f, g_1, g_2) &= |f|_{L^2_{\mathbb{F}}(0,T; L^2(G))}^2 + |g_1|_{L^2_{\mathbb{F}}(0,T; H^1(\Gamma_0))}^2 + |g_1, t|_{L^2_{\mathbb{F}}(0,T; L^2(\Gamma_0))}^2 \\ &\quad + |g_2|_{L^2_{\mathbb{F}}(0,T; L^2(\Gamma_0))}^2. \end{aligned} \quad (3.7.3)$$

Proof. Step 1. In this step, we introduce a cut-off function to localize the equation (3.7.1) to an equation with boundary condition.

Choose a bounded C^4 domain $\tilde{G}$ by extending G across a nonempty boundary patch whose closure is contained in Γ_0 . Then we obtain

$$G \subset \tilde{G}, \quad \overline{\partial G \cap \tilde{G}} \subset \Gamma_0, \quad \text{and } \partial G \setminus \Gamma_0 \subset \partial \tilde{G}.$$

Let $G_0 \subset \subset \tilde{G} \setminus G$ be an open subset. Thanks to Theorem 3.3.2, there exists a function $\psi \in C^4(\tilde{G})$ such that

$$\psi > 0 \text{ in } \tilde{G}, \quad \psi = 0 \text{ on } \partial \tilde{G}, \quad \text{and } |\nabla \psi| > 0 \text{ in } \tilde{G} \setminus G_0. \quad (3.7.4)$$

Let $\iota = \rho = \varepsilon/\sqrt{2}$. Choose t_0, N, c, ϕ, α and β_k, Q_k for $k = 1, 2, 3, 4$ as in the proof of Theorem 3.6.1, using $|\psi|_{L^\infty(\tilde{G})}$ in the definitions of c and β_k and defining Q_k on $[0, T] \times \tilde{G}$.

Next, we claim that

$$\partial Q_1 = \Sigma_0 \cup \Sigma_1, \quad (3.7.5)$$

where $\Sigma_0 \subset [0, T] \times \Gamma_0$ and $\Sigma_1 = \{(t, x) \in [0, T] \times G \mid \alpha(t, x) = \beta_1\}$.

In fact, for any $(t, x) \in \partial Q_1$, thanks to (3.6.4), we know $x \in \tilde{G}$ and $\alpha(t, x) \geq \beta_1$. If $x \in G$, we have $\alpha(t, x) = \beta_1$. If $x \in \partial G$, it must hold $x \in \Gamma_0$. Indeed, supposing that $x \in \partial G \setminus \Gamma_0$, from $\partial G \setminus \Gamma_0 \subset \partial \tilde{G}$ and (3.7.4), we get $\psi(x) = 0$. On the other hand, since $\alpha(t, x) \geq \beta_1$, we obtain

$$\psi(x) - c(t - t_0)^2 = -c(t - t_0)^2 \geq \frac{1}{N}|\psi|_{L^\infty(\tilde{G})} - \frac{c\rho^2}{N},$$

which implies

$$0 \leq c(t - t_0)^2 \leq \frac{1}{N}(c\rho^2 - |\psi|_{L^\infty(\tilde{G})}),$$

contrary to (3.6.3). Hence, (3.7.5) is proved.

Choose a smooth function $\chi : \mathbb{R} \rightarrow [0, 1]$ such that $\chi = 0$ on $(-\infty, \beta_2]$ and $\chi = 1$ on $[\beta_3, \infty)$, and set $\eta = \chi(\alpha)$. Let $z = \eta y$, then z solves

$$\begin{cases} dz - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt = (\langle a_1, \nabla z \rangle + a_2 z + \tilde{f} + \eta f) dt + a_3 z dW(t) & \text{in } Q_1, \\ z = \frac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma_1. \end{cases} \quad (3.7.6)$$

Here $\tilde{f} = - \sum_{j,k=1}^n (b_{x_j}^{jk} \eta_{x_k} y + b^{jk} \eta_{x_j x_k} y + 2b^{jk} \eta_{x_j} y_{x_k}) - \langle a_1, \nabla \eta \rangle y + \eta_t y$ and $\tilde{f}$ is supported in $Q_2 \setminus Q_3$.

Step 2. In this step, we apply Theorem 3.2.2 to the equation (3.7.6) to get the desired estimate (3.7.2).

Let us choose $u = z$ in [Theorem 3.2.2](#). Integrate [\(3.2.4\)](#) over $G \times (0, T)$. Noting that z is supported in Q_1 , choosing $\theta = e^\ell$ and $\ell = \lambda\alpha$, setting $w = \theta z$, after taking mathematical expectation in both sides, we have [\(3.6.8\)](#). The rest of the proof is similar to that of [Theorem 3.6.1](#) with some minor differences, which are stated below.

It is clear that $\overline{Q_2} \cap \overline{\Sigma_1} = \emptyset$. Hence, $\eta = 0$ on Σ_1 . Recall that $y = g_1$ on Σ_0 and $g_1 \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(\Gamma_0))) \cap L^2_{\mathbb{F}}(0, T; H^1(\Gamma_0))$. Then $z = \eta g_1$ on Σ_0 and $z = 0$ on Σ_1 , and it holds that

$$\begin{aligned} \mathbb{E} \int_{Q_1} \sum_{i,j=1}^n (b^{ij} w_{x_i} dw)_{x_j} dx &= \mathbb{E} \int_{\Sigma_0} \sum_{j,k=1}^n b^{jk} w_{x_j} \nu^k dw d\Gamma \\ &\leq C\lambda\mu \mathbb{E} \int_0^T \int_{\Gamma_0} \alpha \theta^2 (|g_2|^2 + |\nabla_{\Gamma} g_1|^2 + |g_{1,t}|^2 + \lambda^2 \mu^2 \alpha^2 |g_1|^2) d\Gamma dt, \end{aligned}$$

where ∇_{Γ} denotes the tangential gradient on Γ_0 .

Noting that $z = \frac{\partial z}{\partial \nu} = 0$ on Σ_1 , we find

$$\begin{aligned} \mathbb{E} \int_{Q_1} \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n (2b^{jk} b^{j'k'} \ell_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk} b^{j'k'} \ell_{x_j} w_{x_{j'}} w_{x_{k'}}) \right. \\ \left. + \Psi b^{jk} w_{x_j} w - b^{jk} \left(\mathcal{A} \ell_{x_j} + \frac{\Psi_{x_j}}{2} \right) w^2 \right]_{x_k} dx dt \\ \leq C\lambda\mu \mathbb{E} \int_0^T \int_{\Gamma_0} \alpha \theta^2 (|g_2|^2 + |\nabla_{\Gamma} g_1|^2 + \lambda^2 \mu^2 \alpha^2 |g_1|^2) d\Gamma dt. \end{aligned}$$

Similar to [\(3.6.22\)](#), there exists a $\mu_1 > 0$, such that for all $\mu > \mu_1$, there is a $\lambda_1(\mu) > 0$, such that for all $\lambda > \lambda_1$, it holds that

$$\begin{aligned} \mathbb{E} \int_{t_0 - \frac{\iota}{\sqrt{N}}}^{t_0 + \frac{\iota}{\sqrt{N}}} \int_{\widehat{G}} (y^2 + |\nabla y|^2) dx dt \\ \leq C e^{C\lambda} (|f|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2 + |g_1|_{L^2_{\mathbb{F}}(0,T;H^1(\Gamma_0))}^2 + |g_{1,t}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}^2 + |g_2|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}^2) \\ + C e^{-2\lambda(\beta_4 - \beta_3)} \mathbb{E} \int_{Q_1} (y^2 + |\nabla y|^2) dx dt. \end{aligned} \quad (3.7.7)$$

Let $m \in \mathbb{N}$ such that

$$\sqrt{2} \iota + \frac{m\iota}{\sqrt{N}} \leq T - \sqrt{2} \iota \leq \sqrt{2} \iota + \frac{(m+1)\iota}{\sqrt{N}}.$$

Recalling that t_0 can be any element in $[\sqrt{2}\iota, T - \sqrt{2}\iota]$, we take

$$t_0 = \sqrt{2}\iota + \frac{k\iota}{\sqrt{N}}, \quad k = 0, 1, 2, \dots, m.$$

Then, it follows from [\(3.7.7\)](#) that

$$\mathbb{E} \int_{\varepsilon}^{T-\varepsilon} \int_{\widehat{G}} (y^2 + |\nabla y|^2) dx dt$$

$$\begin{aligned}
&\leq \mathcal{C}e^{\mathcal{C}\lambda}(|f|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2 + |g_1|_{L^2_{\mathbb{F}}(0,T;H^1(\Gamma_0))}^2 + |g_{1,t}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}^2 + |g_2|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}^2) \\
&\quad + \mathcal{C}e^{-2\lambda(\beta_4-\beta_3)}\mathbb{E}\int_Q(y^2 + |\nabla y|^2)dxd t. \tag{3.7.8}
\end{aligned}$$

Minimize the right-hand side of (3.7.8) over $\lambda \geq \lambda_1$. If the minimizing value lies below λ_1 , use the bound of the left-hand side by $|y|_{L^2_{\mathbb{F}}(0,T;H^1(G))}^2$. By (3.7.3), we obtain

$$\mathbb{E}\int_{\varepsilon}^{T-\varepsilon}\int_{\widehat{G}}(y^2 + |\nabla y|^2)dxd t \leq \mathcal{C}|y|_{L^2_{\mathbb{F}}((0,T);H^1(G))}^{2(1-\gamma)}(\mathcal{K}(f, g_1, g_2))^{\gamma},$$

$$\text{with } \gamma = \frac{2(\beta_4-\beta_3)}{\mathcal{C}+2(\beta_4-\beta_3)}.$$

□

3.8 Reconstruct the unknown state with boundary measurements

This section focuses on the following problem:

- **Problem 3.7:** Find the unknown solution $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_{loc}(G))) \cap L^2_{\mathbb{F}}(0, T; H^2_{\Gamma_0}(G))$ to the equation (3.1.9) via $y|_{(0,T)\times\Gamma_0}$ and $\frac{\partial y}{\partial \nu}\Big|_{(0,T)\times\Gamma_0}$.

We retain the geometric assumptions of Section 3.7. To solve this problem, we use Tikhonov regularization method. We first construct a Tikhonov functional and prove the uniqueness of the minimizer. Then, we prove the convergence rate for the optimization problem.

Let

$$\mathcal{X} = L^2_{\mathbb{F}}(0, T; H^2(G)) \cap L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))).$$

Define $\mathcal{P} : \mathcal{X} \rightarrow L^2_{\mathbb{F}}(0, T; L^2(G))$ by

$$\begin{aligned}
(\mathcal{P}u)(t, x) &\triangleq u(t, x) - u(0, x) - \int_0^t \sum_{j,k=1}^n (b^{jk}(s, x)u_{x_j}(s, x))_{x_k} ds \\
&\quad - \int_0^t [\langle a_1(s, x), \nabla u(s, x) \rangle + a_2(s, x)u(s, x)] ds \\
&\quad - \int_0^t a_3(s, x)u(s, x)dW(s), \quad \mathbb{P}\text{-a.s.}, \quad \forall u \in \mathcal{X}, \text{ a.e. } (t, x) \in Q.
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{H} &\triangleq \{u \in \mathcal{X} \mid \mathcal{P}u \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G))), \\
&\quad u|_{\Gamma_0} = g_1, \partial_{\nu}u|_{\Gamma_0} = g_2\}.
\end{aligned}$$

Assume that the measured boundary data admit an extension in $\mathcal{H}$, so that $\mathcal{H} \neq \emptyset$. If $u \in \mathcal{X}$ solves (3.7.1), then $\mathcal{P}u(t) = \int_0^t f(s) ds$ and $u \in \mathcal{H}$.

Denote $\mathbf{I}f(t) = \int_0^t f(s)ds$ for $t \in [0, T]$. Given a function $F \in \mathcal{H}$, we construct the Tikhonov functional as follows:

$$\mathcal{J}_\tau(u) = |\mathcal{P}u - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \tau |u - F|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2, \quad (3.8.1)$$

where $u \in \mathcal{H}$ and $\tau \in (0, 1)$.

We have the following result.

Proposition 3.8.1. *For $\tau \in (0, 1)$, there exists a unique minimizer $u_\tau \in \mathcal{H}$ of the functional $\mathcal{J}_\tau(\cdot)$.*

Proof. The uniqueness of the minimizer follows from the strong convexity of the functional $\mathcal{J}_\tau(\cdot)$. We only need to show the existence of a minimizer.

Let

$$\begin{aligned} \mathcal{H}_0 \triangleq \{u \in \mathcal{X} \mid \mathcal{P}u \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G))), \\ u|_{\Gamma_0} = \partial_\nu u|_{\Gamma_0} = 0\}. \end{aligned}$$

Fix $\tau \in (0, 1)$. Define the inner products as follows:

$$\langle \varphi, \psi \rangle_{\mathcal{H}_0} = \langle \mathcal{P}\varphi, \mathcal{P}\psi \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \tau \langle \varphi, \psi \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))} \quad (3.8.2)$$

where $\varphi, \psi \in \mathcal{H}_0$. This inner product makes $\mathcal{H}_0$ a Hilbert space. Indeed, let (u_m) be Cauchy in its norm. Then u_m converges in $L^2_{\mathbb{F}}(0, T; H^2(G))$ and $\mathcal{P}u_m$ converges in $L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))$. Let V_m be the sum of the three integrals subtracted in the definition of $\mathcal{P}$ and $\mathcal{P}u_m$. The deterministic integral estimate and the Itô maximal inequality show that (V_m) is Cauchy in $L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G)))$. Since $u_m(t) = u_m(0) + V_m(t)$,

$$u_m(0) = \frac{1}{T} \int_0^T [u_m(t) - V_m(t)] dt$$

is Cauchy in $L^2_{\mathbb{F}_0}(\Omega; L^2(G))$. Thus the limit belongs to $\mathcal{X}$ and satisfies the defining identity for $\mathcal{P}$. The continuity of the spatial trace maps preserves the zero boundary data, proving completeness.

For $u \in \mathcal{H}$, let $v = u - F$. Then $v \in \mathcal{H}_0$. Consider the following functional

$$\overline{\mathcal{J}}_\tau(v) = |\mathcal{P}v + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \tau |v|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2,$$

where $v \in \mathcal{H}_0$ and $\tau \in (0, 1)$. Clearly, u_τ minimizes $\mathcal{J}_\tau(u)$ if and only if $v_\tau = u_\tau - F$ is the minimizer of the functional $\overline{\mathcal{J}}_\tau(v)$.

By the variational principle, the minimizer v_τ of the functional $\overline{\mathcal{J}}_\tau(v)$ satisfy the following equation

$$\langle \mathcal{P}v_\tau + \mathcal{P}F - \mathbf{I}f, \mathcal{P}h \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \tau \langle v_\tau, h \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))} = 0, \quad \forall h \in \mathcal{H}_0. \quad (3.8.3)$$

Conversely, the solution v_τ of equation (3.8.3) is the minimizer of the functional $\overline{\mathcal{J}}_\tau(v)$. In fact, if v_τ solves (3.8.3), then for each $v \in \mathcal{H}_0$, it holds that

$$\begin{aligned}
& \overline{\mathcal{J}}_\tau(v) - \overline{\mathcal{J}}_\tau(v_\tau) \\
&= |\mathcal{P}v + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \tau |v|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\
&\quad - |\mathcal{P}v_\tau + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 - \tau |v_\tau|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\
&= |\mathcal{P}v_\tau + \mathcal{P}F - \mathbf{I}f + \mathcal{P}(v - v_\tau)|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 \\
&\quad + \tau |v_\tau + (v - v_\tau)|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 - |\mathcal{P}v_\tau + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 \\
&\quad - \tau |v_\tau|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\
&= 2\langle \mathcal{P}v_\tau + \mathcal{P}F - \mathbf{I}f, \mathcal{P}(v - v_\tau) \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} \\
&\quad + 2\tau \langle v_\tau, v - v_\tau \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))} + |\mathcal{P}(v - v_\tau)|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 \\
&\quad + \tau |v - v_\tau|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\
&= |\mathcal{P}(v - v_\tau)|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \tau |v - v_\tau|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\
&\geq 0,
\end{aligned}$$

which means v_τ is a minimizer of the functional $\overline{\mathcal{J}}_\tau(v)$.

From (3.8.2), equation (3.8.3) is equivalent to

$$\langle v_\tau, h \rangle_{\mathcal{H}_0} = \langle \mathbf{I}f - \mathcal{P}F, \mathcal{P}h \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}, \quad \forall h \in \mathcal{H}_0. \quad (3.8.4)$$

Thanks to Cauchy-Schwarz inequality and (3.8.2), for all $h \in \mathcal{H}_0$, we have

$$\begin{aligned}
& |\langle \mathbf{I}f - \mathcal{P}F, \mathcal{P}h \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}| \\
&\leq \mathcal{C}(|\mathcal{P}F|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + |f|_{L^2_{\mathbb{F}}(0, T; L^2(G))})|h|_{\mathcal{H}_0}.
\end{aligned}$$

By the Riesz representation theorem, there exists $w_\tau \in \mathcal{H}_0$ such that, for every $h \in \mathcal{H}_0$,

$$\langle w_\tau, h \rangle_{\mathcal{H}_0} = \langle \mathbf{I}f - \mathcal{P}F, \mathcal{P}h \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}.$$

Hence w_τ solves (3.8.4), which means w_τ is the minimizer of the functional $\overline{\mathcal{J}}_\tau(\cdot)$. $\square$

Assume that $y^* \in \mathcal{X}$ is the true solution to (3.7.1) with the exact data

$$\begin{cases} f^* \in L^2_{\mathbb{F}}(0, T; L^2(G)), \\ y^*|_{\Gamma_0} = g_1^* \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(\Gamma_0))) \cap L^2_{\mathbb{F}}(0, T; H^1(\Gamma_0)), \\ \partial_\nu y^*|_{\Gamma_0} = g_2^* \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0)). \end{cases}$$

From [Theorem 3.7.1](#), the solution y^* is unique. Let $\mathcal{H}^*$ be the affine space obtained from $\mathcal{H}$ by replacing g_1, g_2 with g_1^*, g_2^* . Choose $F^* \in \mathcal{H}^*$, for example $F^* = y^*$, satisfying

$$|F^*|_{L_{\mathbb{F}}^2(0,T;H^2(G))} \leq \mathcal{C}|y^*|_{L_{\mathbb{F}}^2(0,T;H^2(G))}. \quad (3.8.5)$$

Assume that the measurement satisfies that

$$|f^* - f|_{L_{\mathbb{F}}^2(0,T;L^2(G))} \leq \delta, \quad |g_1^* - g_1|_{L_{\mathbb{F}}^2(0,T;H^1(\Gamma_0))} \leq \delta, \quad |g_2^* - g_2|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma_0))} \leq \delta, \quad (3.8.6)$$

and

$$|\mathcal{P}F^* - \mathcal{P}F|_{L_{\mathbb{F}}^2(\Omega;H^1(0,T;L^2(G)))} + |F^* - F|_{L_{\mathbb{F}}^2(0,T;H^2(G))} \leq \delta. \quad (3.8.7)$$

We now establish the estimate for the error between the minimizer y_τ for the functional $\mathcal{J}_\tau(\cdot)$ and the exact solution y^* .

Theorem 3.8.2. *Assume (3.8.6), (3.8.7), and that $\alpha \in (0, 1]$ is a constant. For any given $\widehat{G} \subset \subset G$ and $\varepsilon \in (0, \frac{T}{2})$, let $\gamma \in (0, 1)$ as stated in [Theorem 3.7.1](#). Then there exists a constant $\mathcal{C} > 0$ such that for all $\delta \in (0, 1)$,*

$$|y_\tau - y^*|_{L_{\mathbb{F}}^2(\varepsilon, T-\varepsilon; H^1(\widehat{G}))} \leq \mathcal{C}(1 + |y^*|_{L_{\mathbb{F}}^2(0,T;H^2(G))})\delta^{\alpha\gamma},$$

where y_τ is the minimizer of the functional (3.8.1) with the regularization parameter $\tau = \delta^{2\alpha}$.

Proof. Let $v^* = y^* - F^*$. Then $v^* \in \mathcal{H}_0$ and $\mathcal{P}v^* = \mathbf{I}f^* - \mathcal{P}F^*$. Hence, for all $h \in \mathcal{H}_0$, it holds that

$$\langle \mathcal{P}v^* + \mathcal{P}F^* - \mathbf{I}f^*, \mathcal{P}h \rangle_{L_{\mathbb{F}}^2(\Omega;H^1(0,T;L^2(G)))} = 0. \quad (3.8.8)$$

Subtracting (3.8.8) from (3.8.3) and denoting $\tilde{v}_\tau = v^* - v_\tau$, $\tilde{f} = \mathbf{I}f^* - \mathbf{I}f$ and $\tilde{F} = F^* - F$, for all $h \in \mathcal{H}_0$, we have that

$$\begin{aligned} & \langle \mathcal{P}\tilde{v}_\tau, \mathcal{P}h \rangle_{L_{\mathbb{F}}^2(\Omega;H^1(0,T;L^2(G)))} + \tau \langle \tilde{v}_\tau, h \rangle_{L_{\mathbb{F}}^2(0,T;H^2(G))} \\ &= \langle \tilde{f} - \mathcal{P}\tilde{F}, \mathcal{P}h \rangle_{L_{\mathbb{F}}^2(\Omega;H^1(0,T;L^2(G)))} + \tau \langle v^*, h \rangle_{L_{\mathbb{F}}^2(0,T;H^2(G))}. \end{aligned} \quad (3.8.9)$$

By choosing $h = \tilde{v}_\tau$ in (3.8.9), we get that

$$\begin{aligned} & |\mathcal{P}\tilde{v}_\tau|_{L_{\mathbb{F}}^2(\Omega;H^1(0,T;L^2(G)))}^2 + \tau |\tilde{v}_\tau|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 \\ & \leq |\tilde{f} - \mathcal{P}\tilde{F}|_{L_{\mathbb{F}}^2(\Omega;H^1(0,T;L^2(G)))}^2 + \tau |v^*|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2. \end{aligned} \quad (3.8.10)$$

Since $\tau = \delta^{2\alpha}$ and $\delta < 1$, it holds that $\delta^2 \leq \tau$. Combining this with (3.8.6), (3.8.7) and (3.8.10), we obtain that

$$|\tilde{v}_\tau|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 \leq \mathcal{C}(1 + |v^*|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2) \quad (3.8.11)$$

and that

$$|\mathcal{P}\tilde{v}_\tau|_{L^2_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))}^2 \leq \mathcal{C}(1 + |v^*|_{L^2_{\mathbb{F}}(0,T;H^2(G))})^2 \delta^{2\alpha}. \quad (3.8.12)$$

Let $w_\tau = \tilde{v}_\tau(1 + |v^*|_{L^2_{\mathbb{F}}(0,T;H^2(G))})^{-1}$. From (3.8.11), (3.8.12) and Theorem 3.7.1, applied with zero boundary data and forcing $\partial_t \mathcal{P}w_\tau$, we obtain, for $\delta \in (0, 1)$,

$$|w_\tau|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(\widehat{G}))} \leq \mathcal{C}\delta^{\alpha\gamma}.$$

This, together with (3.8.5), implies that

$$\begin{aligned} |\tilde{v}_\tau|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(\widehat{G}))} &\leq \mathcal{C}(1 + |v^*|_{L^2_{\mathbb{F}}(0,T;H^2(G))})\delta^{\alpha\gamma} \\ &\leq \mathcal{C}(1 + |y^*|_{L^2_{\mathbb{F}}(0,T;H^2(G))})\delta^{\alpha\gamma}. \end{aligned} \quad (3.8.13)$$

Since $\tilde{v}_\tau = v^* - v_\tau = y^* - F^* - y_\tau + F$, from (3.8.7), we have

$$\begin{aligned} |\tilde{v}_\tau|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(\widehat{G}))} &\geq |y_\tau - y^*|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(\widehat{G}))} - |F^* - F|_{L^2_{\mathbb{F}}(0,T;H^2(G))} \\ &\geq |y_\tau - y^*|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(\widehat{G}))} - \delta. \end{aligned} \quad (3.8.14)$$

Noting that $\delta^{\alpha\gamma} \geq \delta$, combining (3.8.13) and (3.8.14), we complete the proof. $\square$

3.9 Solution to the inverse source problem with a terminal measurement

This section focuses on the following problem:

- **Problem 3.8:** For any $t_0 \in (0, T)$, does $y(T)$ uniquely determine $(y(t_0), g)$?

We give a positive answer to it by establishing the following result for the conditional stability:

Theorem 3.9.1. *Let $g(t, x) = g_1(t)g_2(x)$, where $g_1 \in L^2_{\mathbb{F}}(0, T)$ and $g_2 \in H^1(G)$ is nonzero. For any $t_0 \in (0, T]$, there exist constants $\mathcal{C} = \mathcal{C}(t_0, g_2) > 0$ and $\gamma = \gamma(t_0, g_2) \in (0, 1)$ such that*

$$\begin{aligned} &|y(t_0)|_{L^2_{\mathcal{F}_{t_0}}(\Omega; L^2(G))} + |g_1|_{L^2_{\mathbb{F}}(t_0, T)} \\ &\leq \mathcal{C} \left[|y|_{L^2_{\mathbb{F}}(0, T; L^2(G))}^{1-\gamma} |y(T)|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))}^\gamma + |y(T)|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))} \right], \end{aligned} \quad (3.9.1)$$

where $y(\cdot)$ is the solution to (3.1.11). The constants may depend on $|g_2|_{H^1(G)}$ and $|g_2|_{L^2(G)}^{-1}$.

We first establish an estimate which retains the quadratic variation as a signed quadratic form. It will also be used in Section 3.11. Let $H = L^2(G)$, with inner product $(\cdot, \cdot)_H$, and let A be the positive self-adjoint Dirichlet realization of $-\operatorname{div}(b\nabla)$ in H . Thus $D(A^{1/2}) = H_0^1(G)$ and

$$|A^{1/2}v|_H^2 = \int_G b\nabla v \cdot \nabla v \, dx, \quad v \in H_0^1(G).$$

Lemma 3.9.2. *Let $f, g \in L^2_{\mathbb{F}}(0, T; H)$ and let*

$$z \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H)) \cap L^2_{\mathbb{F}}(0, T; H^1_0(G))$$

solve $dz + Az dt = f dt + g dW(t)$ in the variational sense. For $s, \lambda \geq 1$, set

$$\ell(t) = se^{\lambda t}, \quad \theta(t) = e^{\ell(t)}, \quad p(t) = s\lambda e^{\lambda t}, \quad B(t) = (A - p(t))(A + p(t))^{-1}.$$

Then

$$\begin{aligned} & \frac{\lambda}{9} \mathbb{E} \int_0^T \theta^2 (|z|_H^2 + p^{-1} |A^{1/2} z|_H^2) dt - \mathbb{E} \int_0^T \theta^2 (Bg, g)_H dt \\ & \leq \theta^2(0) \mathbb{E} |z(0)|_H^2 + \theta^2(T) \mathbb{E} |z(T)|_H^2 + \mathbb{E} \int_0^T \frac{\theta^2}{p} |f|_H^2 dt. \end{aligned} \quad (3.9.2)$$

Moreover, for each nonzero $q \in H$, there is a number $P_q > 0$ such that

$$-((A - p)(A + p)^{-1} q, q)_H \geq \frac{3}{4} |q|_H^2, \quad p \geq P_q. \quad (3.9.3)$$

For $q \in H^1(G)$, the threshold P_q can be chosen in terms of $|q|_{H^1(G)}/|q|_H$ and the fixed operator A .

Proof. Put $w = \theta z$. Since $p' = \lambda p$, we have $B' = -2p'A(A + p)^{-2}$ and $|B|_{\mathcal{L}(H)} \leq 1$. Applying Itô's formula to finite spectral projections gives

$$\begin{aligned} 2(Bw, dw + (A - p)w dt)_H &= d(Bw, w)_H + 2p'(A(A + p)^{-2}w, w)_H dt \\ &\quad + 2|(A - p)(A + p)^{-1/2}w|_H^2 dt - (B\theta g, \theta g)_H dt. \end{aligned}$$

Take expectations, substitute $dw + (A - p)w dt = \theta f dt + \theta g dW(t)$, and use

$$2(Bw, \theta f)_H \leq |(A - p)(A + p)^{-1/2}w|_H^2 + |(A + p)^{-1/2}\theta f|_H^2.$$

For every spectral value $\mu > 0$, writing $x = \mu/p$ and using $p \geq \lambda$, we obtain

$$\frac{2\lambda p\mu/(\mu + p)^2 + (\mu - p)^2/(\mu + p)}{\lambda(1 + \mu/p)} \geq \frac{2x}{(1 + x)^3} + \frac{(x - 1)^2}{(1 + x)^2} \geq \frac{1}{9}.$$

Indeed, the first term is at least $4/27$ for $1/2 \leq x \leq 2$, and the second is at least $1/9$ outside that interval. Since $|(A + p)^{-1/2}f|_H^2 \leq p^{-1}|f|_H^2$, integration yields (3.9.2). Passing to the limit in the spectral projections is justified by the stated energy regularity and the boundedness of B . Finally,

$$-((A - p)(A + p)^{-1} q, q)_H = |q|_H^2 - 2(A(A + p)^{-1} q, q)_H \longrightarrow |q|_H^2$$

as $p \rightarrow \infty$, monotonically, by the spectral theorem. This proves (3.9.3). The convergence is uniform on compact subsets of H ; applying the compact embedding $H^1(G) \hookrightarrow H$ to $q/|q|_H$ gives the final assertion. $\square$

Proof of Theorem 3.9.1. Fix $r = t_0/2$ and choose $0 < t_1 < t_2 < r$. Let $\rho \in C^\infty(\mathbb{R})$ satisfy $0 \leq \rho \leq 1$ and

$$\rho(t) = \begin{cases} 0, & t \leq t_1, \\ 1, & t \geq t_2. \end{cases} \quad (3.9.4)$$

Set $z = \rho y$. Then $z(0) = 0$, $z(T) = y(T)$, and

$$dz + Az dt = (a_1 \cdot \nabla z + a_2 z + \rho_t y) dt + (a_3 z + \rho g_1 g_2) dW(t). \quad (3.9.5)$$

Use the weights in Theorem 3.9.2. For $s\lambda \geq P_{g_2}$, (3.9.3) and $|B|_{\mathcal{L}(H)} \leq 1$ imply

$$-(B(a_3 z + \rho g_1 g_2), a_3 z + \rho g_1 g_2)_H \geq \frac{1}{2} \rho^2 |g_1|^2 |g_2|_H^2 - 5 |a_3 z|_H^2. \quad (3.9.6)$$

Also,

$$|a_1 \cdot \nabla z + a_2 z + \rho_t y|_H^2 \leq \mathcal{C}(|\nabla z|_H^2 + |z|_H^2 + |\rho_t y|_H^2).$$

Choose λ sufficiently large to absorb the first two terms, after division by p , and the last term in (3.9.6) into the left-hand side of (3.9.2). Fix this λ and put $s_* = \max\{1, P_{g_2}/\lambda\}$. For every $s \geq s_*$, we obtain

$$\mathbb{E} \int_0^T \theta^2 (|z|_H^2 + \rho^2 |g_1|^2) dt \leq \mathcal{C} \theta^2(T) D^2 + \mathcal{C} \theta^2(t_2) M^2, \quad (3.9.7)$$

where $M = |y|_{L_{\mathbb{F}}^2(0,T;H)}$ and $D = |y(T)|_{L_{\mathcal{F}_T}^2(\Omega;H)}$. The constant is independent of s . Since $\rho = 1$ on $[r, T]$, it follows that

$$\mathbb{E} \int_r^T (|y|_H^2 + |g_1|^2) dt \leq \mathcal{C} e^{-as} M^2 + \mathcal{C} e^{bs} D^2, \quad (3.9.8)$$

where $a = 2(e^{\lambda r} - e^{\lambda t_2}) > 0$ and $b = 2(e^{\lambda T} - e^{\lambda r}) > 0$.

The forward energy estimate for (3.1.11) gives, for $r \leq t \leq t_0$,

$$\mathbb{E} |y(t_0)|_H^2 \leq \mathcal{C} \mathbb{E} |y(t)|_H^2 + \mathcal{C} \mathbb{E} \int_t^{t_0} |g_1|^2 d\tau. \quad (3.9.9)$$

Average (3.9.9) over $t \in [r, t_0]$ and apply (3.9.8). Shifting s by the fixed lower bound s_* yields, for every $s \geq 0$,

$$\mathbb{E} |y(t_0)|_H^2 + \mathbb{E} \int_{t_0}^T |g_1|^2 dt \leq \mathcal{C} e^{-as} M^2 + \mathcal{C} e^{bs} D^2. \quad (3.9.10)$$

If $0 < D < M$, take $s = 2 \log(M/D)/(a+b)$; if $D \geq M$, take $s = 0$; and if $D = 0$, let $s \rightarrow \infty$. With $\gamma = a/(a+b) \in (0, 1)$, these choices give

$$|y(t_0)|_{L_{\mathcal{F}_{t_0}}^2(\Omega;H)} + |g_1|_{L_{\mathbb{F}}^2(t_0,T)} \leq \mathcal{C} (M^{1-\gamma} D^\gamma + D).$$

This proves (3.9.1). □

3.10 Solution to the inverse source problem with a boundary measurement

This section focuses on the following problem:

- **Problem 3.9:** For a given R and a prescribed initial datum, determine the source function $h(\cdot)$ by means of the measurement $\frac{\partial y}{\partial \nu} \Big|_{\Sigma}$.

This problem is answered by the following uniqueness result.

Theorem 3.10.1. *Assume that $b_{x_1}^{jk} = 0$ for $j, k = 1, \dots, n$. Let $h, \hat{h} \in L^2_{\mathbb{F}}(0, T; H^1(G'))$ and $R \in C^{1,3}(\overline{Q})$ satisfy*

$$|R(t, x)| \neq 0 \quad \text{for all } (t, x) \in [0, T] \times \overline{G}. \quad (3.10.1)$$

Let $a_3 \in L^\infty_{\mathbb{F}}(0, T; W^{2,\infty}(G))$ and let $y, \hat{y} \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H^2(G) \cap H^1_0(G)))$ solve (3.1.12) with source and noise pairs $(hR, a_3 y)$ and $(\hat{h}R, a_3 \hat{y})$, respectively, and the same prescribed initial datum $y(0) = \hat{y}(0) = y_0$. If

$$\frac{\partial y}{\partial \nu} = \frac{\partial \hat{y}}{\partial \nu} \quad \text{on } (0, T) \times \partial G, \quad \mathbb{P}\text{-a.s.},$$

then

$$h(t, x') = \hat{h}(t, x') \quad \text{for a.e. } (t, x') \in (0, T) \times G', \quad \mathbb{P}\text{-a.s.}$$

Proof. Set $v = y - \hat{y}$, $z = v/R$ and $u = z_{x_1}$. Then $z = \partial_\nu z = 0$ on Σ , $z(0) = 0$, and $u = 0$ on Σ . Define

$$c_1 = a_1 + 2R^{-1}b\nabla R, \quad c_0 = a_2 + R^{-1}(\operatorname{div}(b\nabla R) - R_t + a_1 \cdot \nabla R). \quad (3.10.2)$$

Since $b_{x_1} = 0$, we have $\partial_{x_1} \operatorname{div}(b\nabla R) = \operatorname{div}(b\nabla R_{x_1})$. Hence $c_1, c_0, (c_1)_{x_1}, (c_0)_{x_1}$ are bounded under the stated assumptions. The transformed equation is

$$dz + Az dt = (c_1 \cdot \nabla z + c_0 z + h - \hat{h}) dt + a_3 z dW(t). \quad (3.10.3)$$

Differentiate (3.10.3) weakly in x_1 . The source disappears because h and $\hat{h}$ are independent of x_1 , and $b_{x_1} = 0$ gives

$$\begin{aligned} du + Au dt &= (c_1 \cdot \nabla u + c_0 u + (c_1)_{x_1} \cdot \nabla z + (c_0)_{x_1} z) dt \\ &\quad + (a_3 u + (a_3)_{x_1} z) dW(t), \quad u(0) = 0. \end{aligned} \quad (3.10.4)$$

As $z(t, 0, x') = 0$, we have

$$z(t, x_1, x') = Vu(t, x_1, x') \triangleq \int_0^{x_1} u(t, \eta, x') d\eta. \quad (3.10.5)$$

The Cauchy–Schwarz inequality and the zero trace of u on $x_1 = 0$ yield

$$|Vu|_H \leq l|u|_H, \quad \nabla Vu = V\nabla u, \quad |\nabla Vu|_H \leq l|\nabla u|_H. \quad (3.10.6)$$

Consequently, the drift in (3.10.4), apart from Au , has H -norm at most $\mathcal{C}(|u|_H + |\nabla u|_H)$, and its noise coefficient has H -norm at most $\mathcal{C}|u|_H$. Applying the variational Itô formula to $|u|_H^2$, using uniform ellipticity and Young's inequality, and taking expectations, we obtain

$$\mathbb{E}|u(t)|_H^2 + c\mathbb{E} \int_0^t |\nabla u|_H^2 d\tau \leq \mathcal{C}\mathbb{E} \int_0^t |u|_H^2 d\tau, \quad (3.10.7)$$

where $c > 0$. By Gronwall's inequality, $u = 0$, and (3.10.5) gives $z = v = 0$. Equation (3.10.3) then implies $h = \hat{h}$ a.e., as claimed. $\square$

3.11 Solution to the inverse source problem with a boundary measurement and a terminal measurement

This section focuses on the following problem:

- **Problem 3.10:** For a given R , determine the source function $h(\cdot)$ and $g(\cdot, \cdot)$ by means of the measurement $\frac{\partial y}{\partial \nu}|_\Sigma$ and $y(T)$.

This problem is answered by the following uniqueness result.

Theorem 3.11.1. *Assume that $b_{x_1}^{jk} = 0$ for $j, k = 1, \dots, n$.*

Let $h \in L^2_{\mathbb{F}}(0, T; H^1(G'))$ and $R \in C^{1,3}(\overline{Q})$ satisfy

$$|R(t, x)| \neq 0 \quad \text{for all } (t, x) \in [0, T] \times \overline{G}. \quad (3.11.1)$$

Assume that $g = g_1(t)g_2(x) \in L^2_{\mathbb{F}}(0, T; H^1(G))$, where $g_1 \in L^2_{\mathbb{F}}(0, T)$, $g_2 \in H^1(G)$ and $R = R_1(t)R_2(x)$. If $y \in L^2_{\mathbb{F}}(0, T; H^2(G) \cap H^1_0(G)) \cap L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G)))$ satisfies (3.1.12) and

$$\frac{\partial y}{\partial \nu} = 0 \quad \text{on } (0, T) \times \partial G, \quad y(T) = 0 \quad \text{in } G, \quad \mathbb{P}\text{-a.s.},$$

then

$$\begin{aligned} h(t, x') &= 0 \quad \text{for a.e. } (t, x') \in (0, T) \times G', \quad \mathbb{P}\text{-a.s.}, \\ g(t, x) &= 0 \quad \text{for a.e. } (t, x) \in (0, T) \times G, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Proof. Set $z = y/R$, $u = z_{x_1}$ and $q = (g_2/R_2)_{x_1} \in H$. Use c_1, c_0 from (3.10.2) and V from (3.10.5). The boundary conditions imply $z = \partial_\nu z = u = 0$ on Σ and

$$z = Vu. \quad (3.11.2)$$

The assumed regularity gives

$$u \in L^2_{\mathbb{F}}(0, T; H^1_0(G)) \cap L^2_{\mathbb{F}}(\Omega; C([0, T]; H)), \quad u(T) = 0.$$

Differentiating the equation for z weakly in x_1 yields

$$du + Au dt = \mathcal{L}u dt + \frac{g_1}{R_1} q dW(t), \quad (3.11.3)$$

where

$$\mathcal{L}u = c_1 \cdot \nabla u + c_0 u + (c_1)_{x_1} \cdot \nabla V u + (c_0)_{x_1} V u, \quad |\mathcal{L}u|_H \leq \mathcal{C}(|u|_H + |\nabla u|_H). \quad (3.11.4)$$

The last bound follows from (3.10.6). In particular, for every time-dependent weight θ and every $w \in L^2(0, T; H_0^1(G))$,

$$\int_0^T \theta^2 |Vw|_H^2 dt \leq l^2 \int_0^T \theta^2 |w|_H^2 dt, \quad (3.11.5)$$

$$\int_0^T \theta^2 |\nabla V w|_H^2 dt \leq l^2 \int_0^T \theta^2 |\nabla w|_H^2 dt. \quad (3.11.6)$$

Fix $\varepsilon \in (0, T)$ and choose $0 < t_1 < t_2 < \varepsilon$. Let $\chi \in C^\infty(\mathbb{R})$ satisfy $0 \leq \chi \leq 1$ and

$$\chi(t) = \begin{cases} 0, & t \leq t_1, \\ 1, & t \geq t_2. \end{cases} \quad (3.11.7)$$

Then $w = \chi u$ satisfies $w(0) = w(T) = 0$ and

$$dw + Aw dt = (\mathcal{L}w + \chi_t u) dt + \chi \frac{g_1}{R_1} q dW(t).$$

Apply Theorem 3.9.2. If $q = 0$, its noise term vanishes. If $q \neq 0$, choose $s\lambda \geq P_q$ so that (3.9.3) makes that term nonnegative on the left-hand side. By (3.11.4) and uniform ellipticity, a fixed sufficiently large λ absorbs the contribution of $\mathcal{L}w$. Hence, for all sufficiently large s ,

$$\mathbb{E} \int_0^T \theta^2 |w|_H^2 dt \leq C \mathbb{E} \int_{t_1}^{t_2} \frac{\theta^2}{p} \chi_t^2 |u|_H^2 dt, \quad (3.11.8)$$

with a constant independent of s . Since $p \geq 1$ and θ is increasing,

$$\mathbb{E} \int_\varepsilon^T |u|_H^2 dt \leq C e^{-2s(e^{\lambda\varepsilon} - e^{\lambda t_2})} \mathbb{E} \int_{t_1}^{t_2} \chi_t^2 |u|_H^2 dt. \quad (3.11.9)$$

Let $s \rightarrow \infty$. Then $u = 0$ on $(\varepsilon, T) \times G$, and (3.11.2) gives $z = y = 0$ there. Equation (3.1.12) therefore implies, in H ,

$$\int_\varepsilon^t h(\tau, x') R(\tau, x) d\tau + \int_\varepsilon^t g(\tau, x) dW(\tau) = 0, \quad t \in [\varepsilon, T], \quad \mathbb{P}\text{-a.s.} \quad (3.11.10)$$

Testing against a countable dense subset of H gives scalar continuous local martingales that are also absolutely continuous, so their quadratic variations vanish. Thus $g = 0$ a.e. on $(\varepsilon, T) \times G$, $\mathbb{P}$ -a.s. Equation (3.11.10) then gives $hR = 0$ a.e. and, by (3.11.1), $h = 0$ a.e. on $(\varepsilon, T) \times G'$, $\mathbb{P}$ -a.s. Since $\varepsilon > 0$ is arbitrary, the proof is complete. $\square$

3.12 Further comments

To the best of our knowledge, [53] is the first work employing Carleman estimate in studying inverse problems for stochastic parabolic equations. Subsequently, there have been numerous studies that have also addressed inverse problems for stochastic parabolic equations using Carleman estimates (e.g., [20, 25, 29, 44, 46, 47, 53, 57, 64, 73, 74]). Some of these works are presented in this chapter. Theorem 3.2.2 was established in [68]. The proof of Theorem 3.3.1 above uses the improved Carleman estimate in [51, Theorem 1.1], which avoids imposing a spatial-gradient condition on the nonlinear noise difference. Theorem 3.4.1 is an extension of [53, Theorem 1.3]. Theorem 3.4.6 is taken from [20]. Theorem 3.9.1 concerns the terminal inverse source problem studied in [78, Theorem 1.1]; here Theorem 3.9.2 yields an estimate with an $L^2(G)$ terminal observation. The main contents of Sections 3.7 and 3.8 are borrowed from [22], while the main content of Section 3.5 is derived from [20].

There are many open problems related to inverse problems for stochastic parabolic equations. Some of them require new ideas and further development.

- *Inverse coefficient problems for stochastic parabolic equations*

Carleman estimates are widely employed in studying inverse coefficient problems for deterministic parabolic equations (e.g., [75]). However, there is currently no relevant work on stochastic parabolic equations. The main difficulty lies in that, by the classical way to utilize Carleman estimate to solve inverse coefficient problem, one needs to take derivative of the solution with respect to the time variable to reduce an inverse coefficient problem to an inverse state problem. Such a method is not applicable to solutions of stochastic parabolic equations.

- *Inverse state and source problems for semilinear stochastic parabolic equations*

For a general semilinear stochastic parabolic equation (3.1.2), can we investigate its inverse source problem with boundary measurements?

- *Inverse geometric shape of the domain problems*

As we explained in Section 1.2, in many practical problems, equations may involve unknown boundaries, such as a change of phase, a reaction front, or an unknown population. Can the unknown boundaries be determined through measurements of the solutions? This constitutes an important class of inverse problems, extensively studied in deterministic partial differential equations [8–10, 23, 70]. As far as we know, the only result for stochastic parabolic equations is [48]. Compared with the deterministic counterpart, there are still many open problems in this area.

- *Inverse the boundary transfer coefficient problems*

In practical applications, the equation (3.1.2) may be with Robin boundary conditions, specifically $\frac{\partial y}{\partial \nu} + \gamma(t, x)y = \varphi$ on Σ . In this case, the boundary transfer coefficient γ might also be unknown. Is it possible to determine both the state and the boundary transfer coefficient using terminal measurements or internal measurements simultaneously?

- *Inverse problems with many measurements*

In this chapter, when discussing boundary measurements, we typically use Dirichlet data or Neumann data. See Problems 3.6, 3.7 and 3.9. If we know the Neumann data corresponding to all Dirichlet data, that is, if we know the so-called lateral Dirichlet-to-Neumann map, can we obtain better results for determining the state and source term?

- *Inverse problems with non-local measurements*

Due to inevitable noise, pointwise measurements can be highly sensitive. In engineering applications, it is a nature way to take an (weighted) average values of various measurement points. For example, measurements h may take the following two non-local forms:

$$\int_D w_1(x)y(t, x)dx = h(t), \quad t \in (0, T),$$

or

$$\int_0^T w_2(t)y(t, x)dt = h(x), \quad x \in \Gamma,$$

where w_1, w_2 are the weight function. For these types of non-local measurements, due to less information being given, the inverse problem becomes more challenging.

- *Efficiency algorithm for the construction of unknowns*

In Sections 3.5 and 3.8, we discuss reconstructing the solution y from certain measurements using the Tikhonov regularization approach. The reconstruction problem is reframed as determining the minimizer of the Tikhonov functional defined by (3.5.2) and (3.8.1). The same approach can be extended to investigate the reconstruction of the unknown state through the internal measurement and the unknown source by the internal/terminal measurement. We encourage interested readers to explore this.

On the other hand, many algorithms feasible for deterministic problems encounter essential difficulties when applied to stochastic problems. For instance, when solving stochastic problems using the conjugate gradient method, the dual system is a backward stochastic parabolic equation, which is highly challenging to solve numerically. Indeed, although there are some progresses (e.g., [56, 67]), the problem of solving backward stochastic partial

differential equations remains far from well understood. In recent work, [22] applies kernel-based learning theory to obtain numerical solutions for Problem 3.7. The article [20] employs conjugate gradient methods and Picard-type algorithms for numerical solutions of Problem 3.4. Further efficiency algorithm remains to be done.

- *Brownian motion depends on spatial variable*

This chapter uses a scalar Brownian motion $W(t)$; its noise coefficients may already depend on the spatial variable. For finitely many independent Brownian motions, the weighted identities extend by summing the quadratic-variation terms over the noise components, with the same deterministic weights. For a Q -Wiener process [15], passing from finite sums to infinitely many components requires suitable covariance and spatial regularity assumptions; for example, the time-dependent identities require summability of the squared $H_0^1(G)$ norms of the noise coefficients. Extending the inverse results to more general spatial covariance operators remains a separate problem.

Chapter 4

Inverse problems for stochastic hyperbolic equations

Abstract This chapter focuses on inverse problems for stochastic hyperbolic equations. We investigate the determination of unknown initial data and source terms from partial observations, including boundary, terminal, and internal measurements. For the problems considered, we address uniqueness, stability, and reconstruction. Our analysis is based on Carleman estimates derived from a fundamental pointwise weighted identity, and the chapter concludes with several related open problems.

Throughout this chapter, let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$. Write $\mathbb{F}$ for the progressive σ -field with respect to $\mathbf{F}$.

Let $T > 0$, and $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a C^2 boundary Γ . Let Γ_0 be a subset of Γ . Denote by $\mathbf{L}(G)$ all Lebesgue measurable subsets of G . Put

$$Q = (0, T) \times G, \quad \Sigma = (0, T) \times \Gamma, \quad \Sigma_0 = (0, T) \times \Gamma_0.$$

Also, unless other stated, we denote by $\nu(x) = (\nu^1(x), \nu^2(x), \dots, \nu^n(x))$ the unit outward normal vector of Γ at $x \in \Gamma$.

Let $(b^{jk})_{1 \leq j, k \leq n} \in C^1(\overline{G}; \mathbb{R}^{n \times n})$ satisfying that $b^{jk} = b^{kj}$ for all $j, k = 1, \dots, n$, and for some constant $s_0 > 0$,

$$\sum_{j,k=1}^n b^{jk} \xi_j \xi_k \geq s_0 |\xi|^2, \quad \forall (x, \xi) \triangleq (x, \xi_1, \dots, \xi_n) \in G \times \mathbb{R}^n. \quad (4.0.1)$$

4.1 Formulation of the problems

Let $F : [0, T] \times \Omega \times G \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $K : [0, T] \times \Omega \times G \times \mathbb{R} \rightarrow \mathbb{R}$ be two functions such that, for each $(\eta, \varrho, \zeta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, $F(\cdot, \cdot, \cdot, \eta, \varrho, \zeta) :$

$[0, T] \times \Omega \times G \rightarrow \mathbb{R}$ and $K(\cdot, \cdot, \cdot, \eta) : [0, T] \times \Omega \times G \rightarrow \mathbb{R}$ are $\mathbb{F} \times \mathbf{L}(G)$ -measurable; and for a.e. $(t, \omega, x) \in [0, T] \times \Omega \times G$ and any $(\eta_i, \varrho_i, \zeta_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ ($i = 1, 2$),

$$\begin{cases} |F(t, \omega, x, \eta_1, \varrho_1, \zeta_1) - F(t, \omega, x, \eta_2, \varrho_2, \zeta_2)| \\ \leq L(|\eta_1 - \eta_2| + |\varrho_1 - \varrho_2| + |\zeta_1 - \zeta_2|_{\mathbb{R}^n}), \\ |K(t, \omega, x, \eta_1) - K(t, \omega, x, \eta_2)| \leq L|\eta_1 - \eta_2|, \\ |F(\cdot, \cdot, \cdot, 0, 0, 0)| \in L^2_{\mathbb{F}}(0, T; L^2(G)), \quad |K(\cdot, \cdot, \cdot, 0)| \in L^2_{\mathbb{F}}(0, T; H^1_0(G)) \end{cases} \quad (4.1.1)$$

for some constant $L > 0$.

Consider the following semilinear stochastic hyperbolic equation:

$$\begin{cases} dy_t - \sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt = F(y, y_t, \nabla y) dt + K(y) dW(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } G, \end{cases} \quad (4.1.2)$$

where $(y_0, y_1) \in L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G))$ are unknown random variables. The wave operator on the energy space $H^1_0(G) \times L^2(G)$ generates a strongly continuous group. By (4.1.1), its drift and diffusion maps are globally Lipschitz on this space, with square-integrable inhomogeneous terms. Picard iteration for the corresponding stochastic convolution therefore gives a unique mild solution of (4.1.2) $y \in \mathbb{H}_T \triangleq L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G))) \cap L^2_{\mathbb{F}}(\Omega; C^1([0, T]; L^2(G)))$. Moreover,

$$\begin{aligned} & |y|_{L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_0(G)))} + |y_t|_{L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G)))} \\ & \leq e^{C(L^2+1)} (|(y_0, y_1)|_{L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G))} + |F(\cdot, \cdot, 0, 0, 0)|_{L^2_{\mathbb{F}}(0, T; L^2(G))} \\ & \quad + |K(\cdot, \cdot, 0)|_{L^2_{\mathbb{F}}(0, T; L^2(G))}). \end{aligned} \quad (4.1.3)$$

To simplify notations, we will suppress the time variable t ($\in [0, T]$), the sample point ω ($\in \Omega$), and/or the space variable x ($\in G$) in the functions if there is no risk of confusion.

In this section, we formulate two main classes of inverse problems: inverse state problems and inverse source problems for (4.1.2), aiming to determine the initial data or source terms by measuring partial information of the solution. The characteristics of the inverse problems vary significantly depending on the type of measurement. For each inverse problem, we consider its uniqueness, stability, and reconstruction.

4.1.1 Inverse state problems

The most classical inverse problem is the inverse state problem, that is, to determine the unknown initial data of the equation through measurements. Taking partial boundary observation as an example, our question is whether the initial

Our objective is to reconstruct the solution z to the equation (4.1.5) with observed data from the lateral boundary. More precisely, consider the following problem:

Problem 4.2 (Reconstruct the state with a boundary measurement). *Find a function z solving (4.1.5) with $\frac{\partial z}{\partial \nu} = h$ on Σ_0 for a given $h \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$.*

Above inverse problems are global; that is, we are expected to obtain all the information about the unknowns. As a cost, we need to measure enough information. For instance, to solve Problem 4.1, we require a boundary condition and the value of $\frac{\partial y}{\partial \nu}$ on Σ_0 for the time T being large enough and Γ_0 fulfilling some geometric conditions (see Section 4.3 for more details).

It is natural to ask another kind of question, say, if the ability for the measurement is limited, what kind of information for the solution can be derived? In what follows, we formulate one of such problems, in which the boundary condition of the equation is unknown.

Consider the following stochastic hyperbolic equation:

$$\sigma d\tilde{z}_t - \Delta \tilde{z} dt = (\tilde{b}_1 \tilde{z}_t + \tilde{b}_2 \cdot \nabla \tilde{z} + \tilde{b}_3 \tilde{z}) dt + \tilde{b}_4 \tilde{z} dW(t) \quad \text{in } Q, \quad (4.1.7)$$

where, $\sigma \in C^1(\overline{Q})$ is positive,

$$\begin{aligned} \tilde{b}_1 &\in L^\infty_{\mathbb{F}}(0, T; L^\infty_{\text{loc}}(G)), & \tilde{b}_2 &\in L^\infty_{\mathbb{F}}(0, T; L^\infty_{\text{loc}}(G; \mathbb{R}^n)), \\ \tilde{b}_3 &\in L^\infty_{\mathbb{F}}(0, T; L^n_{\text{loc}}(G)), & \tilde{b}_4 &\in L^\infty_{\mathbb{F}}(0, T; L^\infty_{\text{loc}}(G)). \end{aligned} \quad (4.1.8)$$

We call $\tilde{z} \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1_{\text{loc}}(G))) \cap L^2_{\mathbb{F}}(\Omega; C^1([0, T]; L^2_{\text{loc}}(G)))$ a solution to the equation (4.1.7) if for any $t \in (0, T]$, nonempty open subset $\hat{G} \subset\subset G$ and $\eta \in H^1_0(\hat{G})$, it holds that

$$\begin{aligned} &\int_{\hat{G}} \sigma(t, x) \tilde{z}_t(t, x) \eta(x) dx - \int_{\hat{G}} \sigma(0, x) \tilde{z}_t(0, x) \eta(x) dx \\ &- \int_0^t \int_{\hat{G}} \sigma_t(s, x) \tilde{z}_t(s, x) \eta(x) dx ds \\ &= \int_0^t \int_{\hat{G}} [-\nabla \tilde{z} \cdot \nabla \eta + (\tilde{b}_1 \tilde{z}_t + \tilde{b}_2 \cdot \nabla \tilde{z} + \tilde{b}_3 \tilde{z}) \eta(x)] dx ds \\ &+ \int_0^t \int_{\hat{G}} \tilde{b}_4 \tilde{z} \eta(x) dx dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (4.1.9)$$

Put S as a C^2 -hypersurface in $\mathbb{R}^n$ with $S \subset\subset G$. Let $x_0 \in S \setminus \partial S$. We assume that S divides the open ball $B_\rho(x_0) \subset G$, centered at x_0 and with radius ρ , into two parts $\mathcal{D}_\rho^+$ and $\mathcal{D}_\rho^-$. In this case, for any $x \in S$, we denote by $\nu(x)$ the unit normal vector of S at x inward to $\mathcal{D}_\rho^+$. Let $\tilde{z}$ be a solution to the equation (4.1.7), and $\varepsilon \in (0, T/2)$. Consider the following local inverse problem, which is addressed in Section 4.5:

Problem 4.3 (Inverse state problem with the internal measurement). *Can $\tilde{z}$ in $\mathcal{D}_\rho^+ \times (\varepsilon, T - \varepsilon)$ be uniquely determined by the values of $\tilde{z}$ in $\mathcal{D}_\rho^- \times (0, T)$?*

Remark 4.1.1. [Problem 4.3](#) investigates whether a solution to [\(4.1.7\)](#) on a single side of S uniquely determines the solution on the opposite side. We remark that the number ε depends on both $\sigma(\cdot)$ and ρ , and this number cannot be zero. In fact, due to the finite propagation speed of solutions [\(4.1.7\)](#), one needs some time to observe the solutions.

Remark 4.1.2. Unlike the stochastic parabolic equation, due to the finite propagation speed of the solution to a stochastic hyperbolic equation, we typically cannot infer the global properties of the solution through local observations.

4.1.2 Inverse source problems

By utilizing partial information of the solution, we can not only determine the state but also identify the source term in the equation. In particular, combining boundary measurements with terminal observations allows for the unique determination of the source term in the diffusion term.

Define a map

$$\begin{aligned} \mathcal{M}_2 : L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G)) \times L^2_{\mathbb{F}}(0, T; L^2(G)) \\ \rightarrow L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathcal{F}_T}(\Omega; H^1(G)) \end{aligned}$$

as follows:

$$\mathcal{M}_2(z_0, z_1, g) = \left(\frac{\partial z}{\partial \nu} \Big|_{\Sigma_0}, z(T) \right),$$

where z is the corresponding solution to [\(4.1.5\)](#).

This gives rise to the following inverse problem, which is discussed in [Section 4.6](#):

Problem 4.4 (Inverse source problem. I). *Does $\mathcal{M}_2(z_0, z_1, g)$ uniquely determine (z_0, z_1, g) ?*

Noting that the random force $\int_0^t g dW$ is assumed to cause the random vibration starting from some initial state (z_0, z_1) . The objective of the inverse source problem ([Problem 4.4](#)) is to determine the unknown random force intensity g , the unknown initial displacement z_0 and the initial velocity z_1 from the (partial) boundary observation $\frac{\partial z}{\partial \nu} \Big|_{\Sigma_0}$ and the measurement on the terminal displacement $z(T)$.

In [Problem 4.4](#), we consider the problem of determining the source term in the diffusion term based on the measurement $\left(\frac{\partial z}{\partial \nu} \Big|_{\Sigma_0}, z(T) \right)$. It was highlighted in [Remark 1.1.2](#) that these measurements are insufficient for determining the source term in the drift component. Then it naturally raises the question: with additional measurements or more prior information on the source terms, can we ascertain both unknown nonhomogeneous terms in the stochastic hyperbolic equation simultaneously? It turns out that such a simultaneous determination is indeed

achievable by imposing an assumption akin to the equations for the nonhomogeneous terms.

Denote $x = (x_1, x') \in \mathbb{R}^n$, where $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Let $G = (0, l) \times G'$, where $l > 0$ and $G' \subset \mathbb{R}^{n-1}$ be a bounded domain with a C^2 boundary. Consider the following stochastic hyperbolic equation:

$$\begin{cases} dz_t - \Delta z dt = (b_1 z_t + b_2 \cdot \nabla z + b_3 z + hR)dt + (b_4 z + g)dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \end{cases} \quad (4.1.10)$$

where $b_1, b_3, b_4 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G))$, $b_2 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G; \mathbb{R}^n))$, $h \in L^2_{\mathbb{F}}(0, T; H^1(G'))$, $g \in L^2_{\mathbb{F}}(0, T; H^1(G))$ and $R \in C^3(\overline{Q})$. Note that R is a deterministic function and h does not depend on x_1 .

We propose an inverse source problem that aims to simultaneously identify the two unknown nonhomogeneous terms using both boundary measurements and terminal measurements:

Problem 4.5 (Inverse source problem. II). *For a given R , determine the source function h and g by means of the measurement $\left(\frac{\partial z}{\partial \nu}\Big|_{\Sigma}, \frac{\partial z_{x_1}}{\partial \nu}\Big|_{\Sigma}, z(T), z_t(T)\right)$.*

We provide the answer to this problem in [Section 4.7](#).

Remark 4.1.3. It is also quite interesting to consider [Problem 4.5](#), when replacing Δz in [\(4.1.10\)](#) with the general coefficient $\sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k}$. However, in that case,

constructing a suitable Carleman weight function to ensure that the Carleman estimate in [Theorem 4.7.3](#) holds remains an open problem. In fact, the main difficulty in constructing a Carleman weight function lies in choosing an appropriate form so that the non-positive terms can be absorbed during the estimation.

In the parabolic case (see [Theorem 3.4.5](#)), it suffices to select $\psi \in C^\infty(\mathbb{R})$ such that $\psi_t \geq 1$. This time-dependent Carleman weight applies to general variable principal coefficients. The inverse source result in [Theorem 3.11.1](#) additionally assumes $b_{x_1}^{jk} = 0$, so that differentiating in the source direction does not introduce principal-part commutator terms.

However, in the hyperbolic case, the form of the weight function is much more complicated (see [\(4.7.3\)](#)). In particular, a special parameter β must be introduced for the first spatial component to absorb the non-positive terms. Such a form of weight function is difficult to apply to general second-order operators with variable coefficients, which is the reason why the hyperbolic case is more challenging.

The rest of this chapter is organized as follows. In [Subsection 4.2.1](#), we present some preliminaries, including a fundamental pointwise weighted identity for second order stochastic hyperbolic-like operators, which is used to derive Carleman estimates for stochastic hyperbolic equations, a hidden regularity property of solutions to stochastic hyperbolic equations and an energy estimate for stochastic hyperbolic equations. Sections [4.3–4.4](#) are devoted to studying Problems [4.1–4.2](#), respectively. At last, in [Section 4.8](#), we give some comments for the content in this chapter and some related open problems.

4.2 Some preliminaries

To obtain Carleman estimates for stochastic hyperbolic equations, this section presents a fundamental weighted identity for second order hyperbolic-like operators. Furthermore, we establish the hidden regularity of the stochastic hyperbolic equation ([Theorem 4.2.3](#)), as well as energy estimates ([Theorem 4.2.5](#)).

4.2.1 A fundamental weighted identity

In this subsection, we prove the following result.

Theorem 4.2.1. *Let $\phi \in C^2((0, T) \times \mathbb{R}^n)$, $b^{jk} = b^{kj} \in C^2(\mathbb{R}^n)$ for $j, k = 1, 2, \dots, n$, and $\ell \in C^3((0, T) \times \mathbb{R}^n)$, $\Psi \in C^2((0, T) \times \mathbb{R}^n)$. Let u be an $H^2(G)$ -valued, $\mathbf{F}$ -adapted process such that u_t is an $L^2(G)$ -valued Itô process. Set $\theta = e^\ell$ and $w = \theta u$. Then, for a.e. $x \in G$ and $\mathbb{P}$ -a.s. $\omega \in \Omega$,*

$$\begin{aligned}
& \theta \left(-2\phi \ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right) \left[\phi du_t - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right] \\
& + \sum_{j,k=1}^n \left[\sum_{j',k'=1}^n \left(2b^{jk} b^{j'k'} \ell_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk} b^{j'k'} \ell_{x_j} w_{x_{j'}} w_{x_{k'}} \right) \right. \\
& \quad \left. - 2\phi b^{jk} \ell_t w_{x_j} w_t + \phi b^{jk} \ell_{x_j} w_t^2 + \Psi b^{jk} w_{x_j} w - \left(\mathcal{A} \ell_{x_j} + \frac{\Psi_{x_j}}{2} \right) b^{jk} w^2 \right]_{x_k} dt \\
& + d \left\{ \phi \sum_{j,k=1}^n b^{jk} \ell_t w_{x_j} w_{x_k} - 2\phi \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} w_t + \phi^2 \ell_t w_t^2 - \phi \Psi w_t w \right. \\
& \quad \left. + \left[\phi \mathcal{A} \ell_t + \frac{1}{2} (\phi \Psi)_t \right] w^2 \right\} \\
& = \left[(\phi^2 \ell_t)_t + \sum_{j,k=1}^n (\phi b^{jk} \ell_{x_j})_{x_k} - \phi \Psi \right] w_t^2 dt \\
& \quad - 2 \sum_{j,k=1}^n [(\phi b^{jk} \ell_{x_k})_t + b^{jk} (\phi \ell_t)_{x_k}] w_{x_j} w_t dt + \sum_{j,k=1}^n c^{jk} w_{x_j} w_{x_k} dt + \mathcal{B} w^2 dt \\
& \quad + \left(-2\phi \ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right)^2 dt + \phi^2 \theta^2 \ell_t (du_t)^2, \tag{4.2.1}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A} & \triangleq \phi(\ell_t^2 - \ell_{tt}) - \sum_{j,k=1}^n (b^{jk} \ell_{x_j} \ell_{x_k} - b_{x_k}^{jk} \ell_{x_j} - b^{jk} \ell_{x_j x_k}) - \Psi, \\
\mathcal{B} & \triangleq \mathcal{A} \Psi + (\phi \mathcal{A} \ell_t)_t - \sum_{j,k=1}^n (\mathcal{A} b^{jk} \ell_{x_j})_{x_k} + \frac{1}{2} \left[(\phi \Psi)_{tt} - \sum_{j,k=1}^n (b^{jk} \Psi_{x_j})_{x_k} \right],
\end{aligned}$$

$$c^{jk} \triangleq (\phi b^{jk} \ell_t)_t + \sum_{j',k'=1}^n \left[2b^{jk'} (b^{j'k} \ell_{x_{j'}})_{x_{k'}} - (b^{jk} b^{j'k'} \ell_{x_{j'}})_{x_{k'}} \right] + \Psi b^{jk}. \quad (4.2.2)$$

Proof. By $\theta = e^\ell$, $w = \theta u$, we derive that

$$\theta du_t = dw_t - 2\ell_t w_t dt + (\ell_t^2 - \ell_{tt}) w_t dt. \quad (4.2.3)$$

Let

$$\begin{cases} I = -2\phi \ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w, \\ I_2 = \phi dw_t - \sum_{j,k=1}^n (b^{jk} w_{x_j})_{x_k} dt + \mathcal{A} w dt. \end{cases} \quad (4.2.4)$$

From (3.2.6), (4.2.3) and (4.2.4), we obtain

$$\theta I \left[\phi du_t - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right] = I^2 dt + II_2. \quad (4.2.5)$$

Let us compute the terms in II_2 one by one.

Thanks to Itô's formula, we obtain that

$$-2\phi^2 \ell_t w_t dw_t = -d(\phi^2 \ell_t w_t^2) + \phi^2 \ell_t (dw_t)^2 + (\phi^2 \ell_t)_t w_t^2 dt, \quad (4.2.6)$$

$$\begin{aligned} & 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} \phi dw_t \\ &= d \left(2\phi \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} w_t \right) - 2 \sum_{j,k=1}^n (\phi b^{jk} \ell_{x_j})_t w_t w_{x_k} dt \\ & \quad - \sum_{j,k=1}^n (\phi b^{jk} \ell_{x_j} w_t^2)_{x_k} dt + \sum_{j,k=1}^n (\phi b^{jk} \ell_{x_j})_{x_k} w_t^2 dt, \end{aligned} \quad (4.2.7)$$

and that

$$\Psi w \phi dw_t = d(\phi \Psi w w_t) - \phi \Psi w_t^2 dt - \frac{1}{2} d[(\phi \Psi)_t w^2] + \frac{1}{2} (\phi \Psi)_{tt} w^2 dt. \quad (4.2.8)$$

By direct computations, we get that

$$\begin{aligned} & 2 \sum_{j,k=1}^n \phi \ell_t w_t (b^{jk} w_{x_j})_{x_k} dt \\ &= \sum_{j,k=1}^n \left[2(\phi \ell_t b^{jk} w_t w_{x_j})_{x_k} - 2b^{jk} (\phi \ell_t)_{x_k} w_t w_{x_j} - (b^{jk} \phi \ell_t w_{x_k} w_{x_j})_t \right] dt \end{aligned}$$

$$+ (b^{jk}\phi\ell_t)_t w_{x_k} w_{x_j} \Big] dt, \quad (4.2.9)$$

$$\begin{aligned} & 2 \sum_{j,k=1}^n \sum_{j',k'=1}^n b^{jk} \ell_{x_j} w_{x_k} (-b^{j'k'} w_{x_{j'}})_{x_k} dt \\ &= \sum_{j,k=1}^n \sum_{j',k'=1}^n \left[-2(b^{jk} b^{j'k'} \ell_{x_j} w_{x_k} w_{x_{j'}})_{x_k} + 2(b^{jk} \ell_{x_j})_{x_k} b^{j'k'} w_{x_k} w_{x_{j'}} \right. \\ & \quad \left. + (b^{jk} b^{j'k'} \ell_{x_j} w_{x_k} w_{x_{j'}})_{x_k} - (b^{jk} b^{j'k'} \ell_{x_j})_{x_k} w_{x_k} w_{x_{j'}} \right] dt, \end{aligned} \quad (4.2.10)$$

and

$$\begin{aligned} & \sum_{j,k=1}^n \Psi w (-b^{jk} w_{x_j})_{x_k} dt \\ &= \sum_{j,k=1}^n \left[- (b^{jk} \Psi w w_{x_j})_{x_k} + b^{jk} \Psi w_{x_k} w_{x_j} + \frac{1}{2} (b^{jk} \Psi_{x_k} w^2)_{x_j} - \frac{1}{2} (b^{jk} \Psi_{x_k})_{x_j} w^2 \right] dt. \end{aligned} \quad (4.2.11)$$

Similarly, from (4.2.4) and Itô's formula, we have

$$\begin{aligned} I\mathcal{A}w dt &= \sum_{j,k=1}^n (b^{jk} \ell_{x_j} \mathcal{A}w^2)_{x_k} dt - d(\phi \ell_t \mathcal{A}w^2) + (\phi \mathcal{A} \ell_t)_t w^2 dt \\ & \quad - \sum_{j,k=1}^n (b^{jk} \ell_{x_j} \mathcal{A})_{x_k} w^2 dt + \Psi \mathcal{A}w^2 dt. \end{aligned} \quad (4.2.12)$$

Combining (4.2.5)–(4.2.12), we get (4.2.1). $\square$

4.2.2 Hidden regularity property of solutions to stochastic hyperbolic equations

In this subsection, we derive a hidden regularity property of solutions to the equation (4.1.2).

Lemma 4.2.2. [50, Lemma 3.1] *There exists a vector field $\xi = (\xi_1, \dots, \xi_n) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $\xi = \nu$ on Γ .*

Proof. Since the boundary Γ is of class C^2 , for any fixed point $x_0 \in \Gamma$, there exists an open neighborhood V of x_0 in $\mathbb{R}^n$ and a function $\varphi : V \rightarrow \mathbb{R}$ of class C^2 such that the gradient $\nabla\varphi(x)$ is non-zero for all x in V and $\varphi(x) = 0$ for x in $V \cap \Gamma$. If necessary, we can consider $-\varphi$ instead of φ to ensure that $\nu(x_0) \cdot \nabla\varphi(x_0) > 0$. By choosing V sufficiently small, we can also assume that $V \cap \Gamma$ is a connected set. The function $\psi : V \rightarrow \mathbb{R}^n$ defined by unit vector $\eta = \nabla\varphi/|\nabla\varphi|$ is of class C^1 and equals the normal vector ν on $V \cap \Gamma$.

Given that Γ is compact, it can be covered by a finite number of neighborhoods $V_1, \dots, V_m$ of this type. Let $\eta_1, \dots, \eta_m$ denote the corresponding functions. We then have the following conditions:

$$\Gamma \subset \bigcup_{j=1}^m V_j, \quad \eta_j = \nu \text{ on } V_j \cap \Gamma, \text{ for } j = 1, \dots, m.$$

We select an open subset $V_0 \Subset G$ such that $\overline{G} \subset \bigcup_{j=0}^m V_j$ and define $\eta_0 : V_0 \rightarrow \mathbb{R}^n$ by setting $\eta_0(x) = 0$ for $x \in V_0$. Let $\alpha_0, \dots, \alpha_m$ be a C^2 partition of unity corresponding to the covering $V_0, \dots, V_m$ of G . Then, the vector field $\zeta = \sum_{j=0}^m \alpha_j \eta_j$ agrees with ν on Γ . Choose the partition on an open neighborhood of $\overline{G}$ and multiply ζ by a compactly supported cutoff equal to one near $\overline{G}$. Extending by zero gives the required field on $\mathbb{R}^n$. $\square$

Proposition 4.2.3. *For any solution to the equation (4.1.2), we have*

$$\begin{aligned} & \left| \frac{\partial y}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))} \\ & \leq e^{C(L^2+1)} \left(|(y_0, y_1)|_{L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G))} + |F(\cdot, \cdot, 0, 0, 0)|_{L^2_{\mathbb{F}}(0,T;L^2(G))} \right. \\ & \quad \left. + |K(\cdot, \cdot, 0)|_{L^2_{\mathbb{F}}(0,T;L^2(G))} \right). \end{aligned} \quad (4.2.13)$$

Proof. For any $\rho = (\rho^1, \dots, \rho^n) \in C^1(\mathbb{R}^{1+n}; \mathbb{R}^n)$, by a direct computation, we obtain

$$\begin{aligned} & \sum_{j,k=1}^n \left(2\rho \cdot \nabla y b^{jk} y_{x_j} - \sum_{i=1}^n \rho^k b^{ij} y_{x_i} y_{x_j} \right)_{x_k} \\ & = 2 \sum_{j,k=1}^n \left(\rho \cdot \nabla y (b^{jk} y_{x_j})_{x_k} + \sum_{i=1}^n \rho_{x_k}^i b^{jk} y_{x_i} y_{x_j} \right) - \sum_{i,j,k=1}^n (\rho^k b^{ij})_{x_k} y_{x_i} y_{x_j}. \end{aligned} \quad (4.2.14)$$

Thanks to Itô's formula, it holds that

$$\begin{aligned} 2\rho \cdot \nabla y dy_t &= 2d(y_t \rho \cdot \nabla y) - 2y_t \rho_t \cdot \nabla y dt - 2\rho \cdot \nabla y_t y_t dt \\ &= 2d(y_t \rho \cdot \nabla y) - 2y_t \rho_t \cdot \nabla y dt - \sum_{j=1}^n (\rho^j y_t^2)_{x_j} dt + \sum_{j=1}^n \rho_{x_j}^j y_t^2 dt. \end{aligned} \quad (4.2.15)$$

From (4.2.14) and (4.2.15), we get

$$\begin{aligned} & \sum_{k=1}^n \left(2\rho \cdot \nabla y \sum_{j=1}^n b^{jk} y_{x_j} + \rho^k y_t^2 - \sum_{i,j=1}^n \rho^k b^{ij} y_{x_i} y_{x_j} \right)_{x_k} dt \\ & = 2\rho \cdot \nabla y \left[\sum_{j,k=1}^n (b^{jk} y_{x_j})_{x_k} dt - dy_t \right] + 2d(y_t \rho \cdot \nabla y) - 2y_t \rho_t \cdot \nabla y dt \end{aligned}$$

$$+ \sum_{j=1}^n \rho_{x_j}^j y_t^2 dt + \sum_{i,j,k=1}^n (2b^{jk} \rho_{x_k}^i y_{x_j} y_{x_i} - (b^{ij} \rho^k)_{x_k} y_{x_j} y_{x_i}) dt. \quad (4.2.16)$$

Since $\Gamma \in C^2$, by [Theorem 4.2.2](#), there exists a vector field $\zeta \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $\zeta = \nu$ on Γ . Letting $\rho = \zeta$, integrating (4.2.16) in Q , and taking expectation on Ω , recalling (4.1.2), we deduce that

$$\begin{aligned} & \mathbb{E} \int_{\Sigma} \sum_{k=1}^n \left(2\rho \cdot \nabla y \sum_{j=1}^n b^{jk} y_{x_j} + \rho^k y_t^2 - \sum_{i,j=1}^n \rho^k b^{ij} y_{x_i} y_{x_j} \right) \nu^k d\Gamma dt \\ &= 2\mathbb{E} \int_G y_t(T) \rho(T) \cdot \nabla y(T) dx - 2\mathbb{E} \int_G y_1 \rho(0) \cdot \nabla y_0 dx \\ &+ \mathbb{E} \int_Q \left[-2\rho \cdot \nabla y F(y, y_t, \nabla y) - 2y_t \rho_t \cdot \nabla y + \sum_{j=1}^n \rho_{x_j}^j y_t^2 \right. \\ &\quad \left. + \sum_{i,j,k=1}^n (2b^{jk} \rho_{x_k}^i y_{x_j} y_{x_i} - (b^{ij} \rho^k)_{x_k} y_{x_j} y_{x_i}) \right] dx dt. \end{aligned} \quad (4.2.17)$$

Noting that $y = 0$ and $\rho = \nu$ on Γ , we have

$$\begin{aligned} & \mathbb{E} \int_{\Sigma} \sum_{k=1}^n \left(2\rho \cdot \nabla y \sum_{j=1}^n b^{jk} y_{x_j} + \rho^k y_t^2 - \sum_{i,j=1}^n \rho^k b^{ij} y_{x_i} y_{x_j} \right) \nu^k d\Gamma dt \\ &= \mathbb{E} \int_{\Sigma} \sum_{j,k=1}^n b^{jk} \nu^j \nu^k \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \geq s_0 \mathbb{E} \int_{\Sigma} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt. \end{aligned} \quad (4.2.18)$$

Combining (4.1.3), (4.2.17) and (4.2.18), the desired estimate (4.2.13) holds. $\square$

Let

$$\begin{cases} r_1 \triangleq |b_1|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(G))} + |b_2|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(G;\mathbb{R}^n))} + |b_4|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(G))}, \\ r_2 \triangleq |b_3|_{L_{\mathbb{F}}^{\infty}(0,T;L^p(G))}. \end{cases} \quad (4.2.19)$$

Repeating the multiplier argument for [Theorem 4.2.3](#), using

$$\|b_3 z\|_{L^2(G)} \leq C \|b_3\|_{L^p(G)} \|z\|_{H_0^1(G)},$$

gives the following result.

Corollary 4.2.4. *For any solution to the equation (4.1.5), it holds that*

$$\begin{aligned} \left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma_0))} &\leq e^{C(r_1^2 + r_2^2 + 1)} \left(|(z_0, z_1)|_{L_{\mathcal{F}_0}^2(\Omega; H_0^1(G) \times L^2(G))} \right. \\ &\quad \left. + |f|_{L_{\mathbb{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathbb{F}}^2(0,T;L^2(G))} \right). \end{aligned} \quad (4.2.20)$$

4.2.3 An energy estimate for stochastic hyperbolic equations

In this subsection, we derive an energy estimate for (4.1.5).

Proposition 4.2.5. *For any solution z to (4.1.5) and $s, t \in [0, T]$, it holds that*

$$\begin{aligned} & \mathbb{E} \int_G (|z_t(t)|^2 + |\nabla z(t)|^2) dx \\ & \leq C e^{C(r_1^2 + r_2^{1/(2-n/p)+1})T} \left[\mathbb{E} \int_G (|z_t(s)|^2 + |\nabla z(s)|^2) dx + \mathbb{E} \int_Q (f^2 + g^2) dx dt \right]. \end{aligned} \quad (4.2.21)$$

The constant C may depend on G, T and (b^{jk}) , but not on r_1, r_2, f, g or z .

Proof. Put $\alpha = n/p$, with $\alpha = 0$ when $p = \infty$, and $a = 1 + r_2^{1/(2-\alpha)}$. Define

$$E(t) = \mathbb{E} \int_G \left[z_t^2 + \sum_{j,k=1}^n b^{jk} z_{x_j} z_{x_k} + a^2 z^2 \right] dx.$$

The Gagliardo–Nirenberg inequality gives

$$\|z\|_{L^{2p/(p-2)}(G)} \leq C \|\nabla z\|_{L^2(G)}^\alpha \|z\|_{L^2(G)}^{1-\alpha}.$$

This includes $p = n > 2$ by the Sobolev inequality and $p = \infty$ by the identity in $L^2(G)$. Thus Hölder's inequality and ellipticity yield

$$\left| \mathbb{E} \int_G b_3 z z_t dx \right| \leq C r_2 a^{-(1-\alpha)} E(t) \leq C a E(t), \quad a^2 \left| \mathbb{E} \int_G z z_t dx \right| \leq C a E(t).$$

Itô's formula and integration by parts give, for a.e. t ,

$$E'(t) = \mathbb{E} \int_G [2z_t(b_1 z_t + b_2 \cdot \nabla z + b_3 z + f) + (b_4 z + g)^2 + 2a^2 z z_t] dx.$$

Consequently,

$$|E'(t)| \leq C(r_1^2 + a + 1)E(t) + C \mathbb{E} \int_G (f^2 + g^2) dx.$$

Forward or backward Gronwall's inequality therefore implies

$$E(t) \leq C e^{C(r_1^2 + a + 1)T} \left[E(s) + \mathbb{E} \int_Q (f^2 + g^2) dx dt \right].$$

By ellipticity and Poincaré's inequality, $E(t)$ controls the energy on the left of (4.2.21), while $E(s) \leq C(1 + a^2) \mathbb{E} \int_G (|z_t(s)|^2 + |\nabla z(s)|^2) dx$. Absorbing this polynomial factor into the exponential proves the assertion. The calculation for energy solutions follows by the usual Galerkin approximation. $\square$

4.3 Solution to inverse state problem with the boundary measurement

This section focuses on the following problem:

- **Problem 4.1:** Does there exist a constant $\mathcal{C} > 0$ such that for any initial data $(y_0, y_1), (\hat{y}_0, \hat{y}_1) \in L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G))$, the inequality (4.1.4) holds?

Prior to unveiling the solution to this problem, certain assumptions regarding $(b^{jk})_{1 \leq j, k \leq n}$, T , and Γ_0 are required.

Condition 4.3.1. There exists a positive function $\psi(\cdot) \in C^2(\overline{G})$ satisfying the following:

- (1) For some constant $\mu_0 > 0$ and any $(x, \xi) \in \overline{G} \times \mathbb{R}^n$, it holds that

$$\sum_{j,k=1}^n \left\{ \sum_{j',k'=1}^n \left[2b^{jk'}(b^{j'k}\psi_{x_{j'}})_{x_{k'}} - b^{jk}_{x_{k'}}b^{j'k'}\psi_{x_{j'}} \right] \right\} \xi_j \xi_k \geq \mu_0 \sum_{j,k=1}^n b^{jk} \xi_j \xi_k.$$

- (2) There is no critical point of $\psi(\cdot)$ in $\overline{G}$, i.e.,

$$\min_{x \in \overline{G}} |\nabla \psi(x)| > 0.$$

Remark 4.3.1. Condition 4.3.1 can be viewed as a form of pseudo-convex condition (e.g., [34, Section 28.3]). Further exploration on this subject can be found in [52]. It is worth noting that if $(b^{jk})_{1 \leq j, k \leq n}$ is the identity matrix, then $\psi(x) = |x - x_0|^2$ satisfies Condition 4.3.1, where x_0 is any point belongs to $\mathbb{R}^n \setminus \overline{G}$. If $c \in C^1(\overline{G})$ has no zeros, put $a = c^2$ and $q = 2(x - x_0)$. For $b^{jk} = a\delta_{jk}$ and $\psi(x) = |x - x_0|^2$, the quadratic form in Condition 4.3.1 is

$$a[4a|\xi|^2 + 2(\nabla a \cdot \xi)(q \cdot \xi) - (\nabla a \cdot q)|\xi|^2].$$

Consequently, this choice of ψ satisfies Condition 4.3.1 if and only if there is a constant $\mu_1 > 0$ such that

$$4a|\xi|^2 + 2(\nabla a \cdot \xi)(q \cdot \xi) - (\nabla a \cdot q)|\xi|^2 \geq \mu_1|\xi|^2, \quad (x, \xi) \in \overline{G} \times \mathbb{R}^n.$$

One can verify that if $\psi(\cdot)$ satisfies Condition 4.3.1, then for any constants $a \geq 1$ and $b \in \mathbb{R}$ such that $a\psi + b > 0$ on $\overline{G}$, the function $\tilde{\psi} = a\psi + b$ also satisfies this condition with μ_0 replaced by $a\mu_0$. Therefore, we can choose ψ , μ_0 , $c_0 > 0$, $c_1 > 0$ and T such that the following condition holds:

Condition 4.3.2.

$$\begin{cases} \frac{1}{4} \sum_{j,k=1}^n b^{jk}(x) \psi_{x_j}(x) \psi_{x_k}(x) \geq R_1^2 \triangleq \max_{x \in \overline{G}} \psi(x) \geq R_0^2 \triangleq \min_{x \in \overline{G}} \psi(x), & \forall x \in \overline{G}, \\ T > T_0 \triangleq 2R_1, \quad \left(\frac{2R_1}{T}\right)^2 < c_1 < \frac{2R_1}{T}, \quad \mu_0 - 4c_1 - c_0 > 0. \end{cases}$$

Put

$$\Gamma_0 \triangleq \left\{ x \in \Gamma \mid \sum_{j,k=1}^n b^{jk}(x) \psi_{x_j}(x) \nu^k(x) > 0 \right\}. \quad (4.3.1)$$

We have the following positive answer to [Problem 4.1](#).

Theorem 4.3.1. *Assume [Condition 4.3.1](#) and [Condition 4.3.2](#) are satisfied. Then there exists a constant $\mathcal{C} > 0$ such that for any initial data $(y_0, y_1), (\hat{y}_0, \hat{y}_1) \in L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G))$, the inequality [\(4.1.4\)](#) holds for Γ_0 given by [\(4.3.1\)](#).*

Remark 4.3.2. [Theorem 4.3.1](#) shows that the state $y(t)$ of [\(4.1.2\)](#) (for $t \in [0, T]$) can be uniquely determined by the measurement $\frac{\partial y}{\partial \nu} \Big|_{\Sigma_0}$, $\mathbb{P}$ -a.s., and continuously depends on it. Therefore, it provides an affirmative answer to the inverse state problem of the system [\(4.1.2\)](#).

In what follows, for $\lambda \in \mathbb{R}$ and $r \in \mathbb{N}$, we use $O(\lambda^r)$ to denote a function of order λ^r for large λ .

First, we derive a boundary observability estimate for the equation [\(4.1.5\)](#).

Theorem 4.3.2. *Assume [Condition 4.3.1](#) and [Condition 4.3.2](#) are satisfied. For any solution z to the equation [\(4.1.5\)](#), it holds that*

$$\begin{aligned} & |(z_0, z_1)|_{L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G))} \\ & \leq e^{\mathcal{C}(r_1^2 + r_2^{\frac{1}{3/2-n/p}} + 1)} \left(\left| \frac{\partial z}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))} + |f|_{L^2_{\mathbb{F}}(0, T; L^2(G))} + |g|_{L^2_{\mathbb{F}}(0, T; L^2(G))} \right). \end{aligned} \quad (4.3.2)$$

Proof. The proof is divided into four steps.

Step 1. In this step, we introduce a cut-off function and reduce the equation [\(4.1.5\)](#) to an equation with both initial and final data equal zero.

Put

$$\ell(t, x) = \lambda \left[\psi(x) - c_1 \left(t - \frac{T}{2} \right)^2 \right], \quad (t, x) \in Q.$$

From [Condition 4.3.2](#), there exists $\varepsilon_1 \in (0, 1/2)$ such that for all $(t, x) \in \left[\left(0, \frac{T}{2} - \varepsilon_1 T\right) \cup \left(\frac{T}{2} + \varepsilon_1 T, T\right) \right] \times G$, it holds that

$$\ell(t, x) \leq \lambda [R_1^2 - c_1(\varepsilon_1 T)^2] < 0. \quad (4.3.3)$$

Thanks to [Condition 4.3.2](#) again, noting that

$$\ell\left(\frac{T}{2}, x\right) = \lambda \psi(x) \geq \lambda R_0^2, \quad \forall x \in G,$$

there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that

$$\ell(t, x) \geq \frac{\lambda R_0^2}{2}, \quad \forall (t, x) \in \left(\frac{T}{2} - \varepsilon_0 T, \frac{T}{2} + \varepsilon_0 T \right) \times G. \quad (4.3.4)$$

Choose a cut-off function $\chi \in C_0^\infty[0, T]$ satisfying

$$\chi(t) = 1, \quad \forall t \in \left(\frac{T}{2} - \varepsilon_1 T, \frac{T}{2} + \varepsilon_1 T\right). \quad (4.3.5)$$

Put $u = \chi z$. Then, u solves the following equation:

$$\begin{cases} du_t - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt = (b_1 u_t + b_2 \cdot \nabla u + b_3 u + \tilde{f}) dt \\ \quad \quad \quad + (b_4 u + \chi g) dW(t) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u(T) = 0, \quad u_t(0) = u_t(T) = 0 & \text{in } G, \end{cases} \quad (4.3.6)$$

where $\tilde{f} = \chi f + \chi_{tt} z + 2\chi_t z_t - b_1 \chi_t z$.

Step 2. We choose the multiplier and estimate its coefficients. Write $b = (b^{jk})$, $V = b \nabla \psi$, $k = \nabla \psi \cdot b \nabla \psi$, $q(t) = -c_1(2t - T)$ and $D = q^2 - k$. Then

$$\ell_t = \lambda q, \quad \ell_{tt} = -2c_1 \lambda, \quad \nabla \ell = \lambda \nabla \psi, \quad \nabla \ell_t = 0. \quad (4.3.7)$$

Set $d = -2c_1 + \nabla \cdot V - c_0$. For $\varepsilon > 0$, choose a smooth function d_ε on a neighborhood of $\overline{G}$ with $|d_\varepsilon - d|_{C(\overline{G})} \leq \varepsilon$, and put

$$\Psi = \lambda d_\varepsilon. \quad (4.3.8)$$

Put $e = d_\varepsilon - d$, $w = \theta u$, $\mathcal{A}_0 = \lambda^2 D$ and $r = 4c_1 + c_0 - e$. Define

$$I = -2\ell_t w_t + 2b \nabla \ell \cdot \nabla w + \Psi w. \quad (4.3.9)$$

The conjugated equation has the exact decomposition

$$\begin{aligned} \theta [du_t - \nabla \cdot (b \nabla u) dt] &= I dt + dw_t - \nabla \cdot (b \nabla w) dt \\ &\quad + \mathcal{A}_0 w dt + \lambda r w dt. \end{aligned} \quad (4.3.10)$$

Multiply (4.3.10) by I and integrate by parts as in the proof of [Theorem 4.2.1](#). The resulting integrated identity has $\mathcal{A}_0$ in place of $\mathcal{A}$ and the additional term $\lambda r I w$ on its right-hand side. Only first derivatives of $b, \mathcal{A}_0$, second derivatives of ℓ , and second derivatives of the smooth function Ψ occur; thus these product rules apply with $b \in C^1(\overline{G})$ and $\psi \in C^2(\overline{G})$.

The coefficient of w_t^2 is

$$[\ell_{tt} + \nabla \cdot (b \nabla \ell) - \Psi] w_t^2 = \lambda(c_0 - e) w_t^2, \quad (4.3.11)$$

and the mixed term vanishes:

$$[(b \nabla \ell)_t + b \nabla \ell_t] \cdot \nabla w w_t = 0. \quad (4.3.12)$$

Writing c^{jk} for the spatial coefficients in the integrated identity, [Condition 4.3.1](#) gives

$$\sum_{j,k=1}^n c^{jk} w_{x_j} w_{x_k} \geq \lambda(\mu_0 - 4c_1 - c_0 - \varepsilon) \sum_{j,k=1}^n b^{jk} w_{x_j} w_{x_k}. \quad (4.3.13)$$

The zeroth-order coefficient is

$$\begin{aligned} \mathcal{B}_0 &= \mathcal{A}_0 \Psi + (\mathcal{A}_0 \ell_t)_t - \nabla \cdot (\mathcal{A}_0 b \nabla \ell) - \frac{1}{2} \nabla \cdot (b \nabla \Psi) \\ &= \lambda^3 [(4c_1 + c_0)k + V \cdot \nabla k - (8c_1 + c_0)q^2 + De] - \frac{\lambda}{2} \nabla \cdot (b \nabla d_\varepsilon). \end{aligned} \quad (4.3.14)$$

From [Condition 4.3.1](#), we get that

$$\begin{aligned} & \sum_{j,k=1}^n \sum_{j',k'=1}^n b^{jk} \psi_{x_j} (b^{j'k'} \psi_{x_{j'}} \psi_{x_{k'}})_{x_k} \\ &= \sum_{j,k=1}^n \sum_{j',k'=1}^n (b^{jk} b_{x_k}^{j'k'} \psi_{x_j} \psi_{x_{j'}} \psi_{x_{k'}} + 2b^{jk} b^{j'k'} \psi_{x_j} \psi_{x_k x_{j'}} \psi_{x_{k'}}) \\ &= \sum_{j,k=1}^n \sum_{j',k'=1}^n (b^{jk} b_{x_k}^{j'k'} \psi_{x_j} \psi_{x_{j'}} \psi_{x_{k'}} - 2b^{j'k'} b_{x_{j'}}^{jk} \psi_{x_j} \psi_{x_k} \psi_{x_{k'}} \\ & \quad + 2b^{j'k'} (b^{jk} \psi_{x_k})_{x_{j'}} \psi_{x_j} \psi_{x_{k'}}) \\ &= \sum_{j,k=1}^n \left\{ \sum_{j',k'=1}^n [2b^{jk'} (b^{j'k} \psi_{x_{j'}})_{x_{k'}} - b_{x_{k'}}^{jk} b^{j'k'} \psi_{x_{j'}}] \right\} \psi_{x_j} \psi_{x_k} \\ &\geq \mu_0 \sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k}. \end{aligned} \quad (4.3.15)$$

At $e = 0$, [\(4.3.15\)](#) and [Condition 4.3.2](#) give

$$(4c_1 + c_0)k + V \cdot \nabla k - (8c_1 + c_0)q^2 \geq 2(4c_1 + c_0)(4R_1^2 - c_1^2 T^2) > 0.$$

We first fix ε small enough to preserve this lower bound and the positive coefficients in [\(4.3.11\)](#) and [\(4.3.13\)](#), and then fix d_ε . The last term in [\(4.3.14\)](#) is now $O(\lambda)$, so there is $\lambda_1 > 0$ such that

$$\mathcal{B}_0 \geq C\lambda^3 \quad \text{for all } \lambda \geq \lambda_1. \quad (4.3.16)$$

The remaining term in [\(4.3.10\)](#) satisfies

$$|\lambda r I w| \leq \frac{1}{4} I^2 + C\lambda^2 w^2. \quad (4.3.17)$$

Increasing λ_1 absorbs the last term into [\(4.3.16\)](#). All choices in this step depend only on the fixed principal coefficients, weight, and geometric constants. The

calculation is first performed for regular solutions and then extended by energy approximation; [Theorem 4.2.4](#) gives convergence of the observed normal traces.

Step 3. We integrate the multiplier identity for [\(4.3.6\)](#).

Integrating the identity obtained from [\(4.3.10\)](#) in Q , taking mathematical expectation, and using [\(4.3.11\)–\(4.3.13\)](#), [\(4.3.16\)](#) and [\(4.3.17\)](#), we note that

$$\begin{aligned} \mathbb{E} \int_0^T \int_G d \left[\sum_{j,k=1}^n b^{jk} \ell_t w_{x_j} w_{x_k} - 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} w_t + \ell_t w_t^2 - \Psi w_t w \right. \\ \left. + \left(\mathcal{A}_0 \ell_t + \frac{1}{2} \Psi_t \right) w^2 \right] dx = 0, \end{aligned} \quad (4.3.18)$$

where the time cutoff gives $w(0) = w_t(0) = w(T) = w_t(T) = 0$. The spatial boundary terms containing w or w_t vanish because $u = 0$ on Σ . We obtain

$$\begin{aligned} \mathbb{E} \int_Q \theta \left\{ \left(-2\ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right) \left[du_t - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right] \right\} dx \\ + \lambda \mathbb{E} \int_{\Sigma} \sum_{j,k=1}^n \sum_{j',k'=1}^n (2b^{jk} b^{j'k'} \psi_{x_j} w_{x_j} w_{x_{k'}} - b^{jk} b^{j'k'} \psi_{x_j} w_{x_{j'}} w_{x_{k'}}) \nu_k d\Gamma dt \\ - \mathbb{E} \int_Q \theta^2 \ell_t |du_t|^2 dx \\ \geq \mathcal{C} \mathbb{E} \int_Q \theta^2 \left[(\lambda u_t^2 + \lambda |\nabla u|^2) + \lambda^3 u^2 \right] dx dt \\ + \frac{3}{4} \mathbb{E} \int_Q \left(-2\ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right)^2 dx dt. \end{aligned} \quad (4.3.19)$$

Noting that $u = 0$ on Σ , $\mathbb{P}$ -a.s., from [\(4.3.1\)](#), we deduce that

$$\begin{aligned} \mathbb{E} \int_{\Sigma} \sum_{j,k=1}^n \sum_{j',k'=1}^n \left(2b^{jk} b^{j'k'} \psi_{x_j} w_{x_j} w_{x_{k'}} - b^{jk} b^{j'k'} \psi_{x_j} w_{x_{j'}} w_{x_{k'}} \right) \nu^k d\Gamma dt \\ = \mathbb{E} \int_{\Sigma} \left(\sum_{j,k=1}^n b^{jk} \nu^j \nu^k \right) \left(\sum_{j',k'=1}^n b^{j'k'} \psi_{x_{j'}} \nu^{k'} \right) \theta^2 \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt \\ \leq \mathbb{E} \int_{\Sigma_0} \left(\sum_{j,k=1}^n b^{jk} \nu^j \nu^k \right) \left(\sum_{j',k'=1}^n b^{j'k'} \psi_{x_{j'}} \nu^{k'} \right) \theta^2 \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt. \end{aligned} \quad (4.3.20)$$

Thanks to [\(4.3.6\)](#), it holds that

$$\begin{aligned} \mathbb{E} \int_Q \theta \left\{ \left(-2\ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right) \left[du_t - \sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} dt \right] \right\} dx \\ \leq \mathcal{C} \left[\mathbb{E} \int_Q \theta^2 (|\tilde{f}|^2 + |g|^2) dx dt + |b_1|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(G))}^2 \mathbb{E} \int_Q \theta^2 u_t^2 dx dt \right] \end{aligned}$$

$$\begin{aligned}
& + |b_2|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(G,\mathbb{R}^n))}^2 \mathbb{E} \int_Q \theta^2 |\nabla u|^2 dxdt + \mathbb{E} \int_Q \theta^2 b_3^2 u^2 dxdt \\
& + |b_4|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(G))}^2 \mathbb{E} \int_Q \theta^2 u^2 dxdt \Big] \\
& + \frac{1}{2} \mathbb{E} \int_Q \left(-2\ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right)^2 dxdt, \tag{4.3.21}
\end{aligned}$$

and

$$-\mathbb{E} \int_Q \theta^2 \ell_t |du_t|^2 dx \leq C\lambda |b_4|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(G))}^2 \mathbb{E} \int_Q \theta^2 u^2 dxdt + C\lambda \mathbb{E} \int_Q \theta^2 |g|^2 dxdt. \tag{4.3.22}$$

Step 4. In this step, we handle the terms in (4.3.21) and (4.3.22).

Choosing $s = 2p/(p-2)$, we have $\frac{2}{p} + \frac{2}{s} = 1$, which, combined with Hölder's inequality and (4.2.19), implies that

$$|b_3 \theta u|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 \leq r_2^2 |\theta u|_{L_{\mathbb{F}}^2(0,T;L^s(G))}^2. \tag{4.3.23}$$

Since $\frac{1}{s} = \frac{1}{2} - \frac{1}{p}$, by Sobolev's inequality, we get

$$|\theta u|_{L_{\mathbb{F}}^2(0,T;L^s(G))}^2 \leq C |\theta u|_{L_{\mathbb{F}}^2(0,T;H^{n/p}(G))}^2. \tag{4.3.24}$$

For any $\rho \in L_{\mathcal{F}_T}^2(\Omega; H^1(\mathbb{R}^n))$, by Hölder's inequality again, we obtain

$$\begin{aligned}
|\rho|_{L_{\mathcal{F}_T}^2(\Omega; H^{n/p}(\mathbb{R}^n))}^2 &= \mathbb{E} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{n/p} |\hat{\rho}(\xi)|^{2n/p} |\hat{\rho}(\xi)|^{2(1-n/p)} d\xi \\
&\leq |\rho|_{L_{\mathcal{F}_T}^2(\Omega; H^1(\mathbb{R}^n))}^{2n/p} |\rho|_{L_{\mathcal{F}_T}^2(\Omega; L^2(\mathbb{R}^n))}^{2(1-n/p)}. \tag{4.3.25}
\end{aligned}$$

Hence, it holds that

$$|\tilde{\rho}|_{L_{\mathcal{F}_T}^2(\Omega; H^{n/p}(G))}^2 \leq C |\tilde{\rho}|_{L_{\mathcal{F}_T}^2(\Omega; H_0^1(G))}^{2n/p} |\tilde{\rho}|_{L_{\mathcal{F}_T}^2(\Omega; L^2(G))}^{2(1-n/p)}, \quad \forall \tilde{\rho} \in L_{\mathcal{F}_T}^2(\Omega; H_0^1(G)), \tag{4.3.26}$$

and

$$|\bar{\rho}|_{L_{\mathbb{F}}^2(0,T;H^{n/p}(G))}^2 \leq C |\bar{\rho}|_{L_{\mathbb{F}}^2(0,T;H_0^1(G))}^{2n/p} |\bar{\rho}|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^{2(1-n/p)}, \quad \forall \bar{\rho} \in L_{\mathbb{F}}^2(0,T;H_0^1(G)). \tag{4.3.27}$$

For $n < p < \infty$, combining (4.3.23)–(4.3.27) and Young's inequality, for every $\varepsilon > 0$ we obtain

$$\begin{aligned}
& |b_3 \theta u|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 \\
& \leq C r_2^2 |\theta u|_{L_{\mathbb{F}}^2(0,T;H_0^1(G))}^{2n/p} |\theta u|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^{2(1-n/p)}
\end{aligned}$$

$$\leq \varepsilon \lambda |\theta u|_{L^2_{\mathbb{F}}(0,T;H^1_0(G))}^2 + \mathcal{C}(\varepsilon) r_2^{2p/(p-n)} \lambda^{-n/(p-n)} |\theta u|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2. \quad (4.3.28)$$

For $p = n > 2$, use instead $\|b_3 \theta u\|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2 \leq C r_2^2 \|\theta u\|_{L^2_{\mathbb{F}}(0,T;H^1_0(G))}^2$; for $p = \infty$, use $r_2^2 \|\theta u\|_{L^2_{\mathbb{F}}(0,T;L^2(G))}^2$. These bounds give the same parameter choice below, with $n/p = 0$ when $p = \infty$. Choosing

$$\lambda_2 = \max \left\{ \lambda_1, \mathcal{C} \left(r_1^2 + r_2^{\frac{1}{3/2-n/p}} + 1 \right) \right\},$$

from (4.3.19)–(4.3.22) and (4.3.28), for all $\lambda \geq \lambda_2$, it holds that

$$\begin{aligned} & C \lambda \mathbb{E} \int_{\Sigma_0} \theta^2 \left(\sum_{j,k=1}^n b^{jk} \nu^j \nu^k \right) \left(\sum_{j',k'=1}^n b^{j'k'} \psi_{x_{j'}} \nu^{k'} \right) \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt \\ & + C \mathbb{E} \int_Q \theta^2 (|\tilde{f}|^2 + \lambda |g|^2) dx dt \\ & \geq \mathbb{E} \int_Q \theta^2 \left(\lambda u_t^2 + \lambda |\nabla u|^2 + \lambda^3 u^2 \right) dx dt. \end{aligned} \quad (4.3.29)$$

Noting that (4.3.5) and $u = \chi z$, by (4.3.29), we get

$$\begin{aligned} & C \lambda \mathbb{E} \int_{\Sigma_0} \theta^2 \left(\sum_{j,k=1}^n b^{jk} \nu^j \nu^k \right) \left(\sum_{j',k'=1}^n b^{j'k'} \psi_{x_{j'}} \nu^{k'} \right) \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \\ & + C(1 + r_1^2) \mathbb{E} \int_{(0, \frac{T}{2} - \varepsilon_1 T) \cup (\frac{T}{2} + \varepsilon_1 T, T)} \int_G \theta^2 (z_t^2 + |\nabla z|^2 + z^2) dx dt \\ & + C \mathbb{E} \int_Q \theta^2 (|f|^2 + \lambda |g|^2) dx dt \\ & \geq \mathbb{E} \int_{\frac{T}{2} - \varepsilon_0 T}^{\frac{T}{2} + \varepsilon_0 T} \int_G \theta^2 \left(\lambda z_t^2 + \lambda |\nabla z|^2 + \lambda^3 z^2 \right) dx dt. \end{aligned} \quad (4.3.30)$$

From (4.2.21), (4.3.3) and (4.3.4), we obtain that

$$\begin{aligned} & \mathbb{E} \int_{(0, \frac{T}{2} - \varepsilon_1 T) \cup (\frac{T}{2} + \varepsilon_1 T, T)} \int_G \theta^2 (z_t^2 + |\nabla z|^2 + z^2) dx dt \\ & \leq C e^{\lambda[R_1^2 - c_1(\varepsilon_1 T)^2]} \mathbb{E} \int_{(0, \frac{T}{2} - \varepsilon_1 T) \cup (\frac{T}{2} + \varepsilon_1 T, T)} \int_G (z_t^2 + |\nabla z|^2) dx dt \\ & \leq C e^{C(r_1^2 + r_2^{\frac{1}{2-n/p}} + 1)T} e^{\lambda[R_1^2 - c_1(\varepsilon_1 T)^2]} \mathbb{E} \int_G (z_1^2 + |\nabla z_0|^2) dx \\ & + C e^{C\lambda} \mathbb{E} \int_Q (|f|^2 + |g|^2) dx dt, \end{aligned} \quad (4.3.31)$$

and that

$$\mathbb{E} \int_{\frac{T}{2} - \varepsilon_0 T}^{\frac{T}{2} + \varepsilon_0 T} \int_G \theta^2 \left(\lambda z_t^2 + \lambda |\nabla z|^2 + \lambda^3 z^2 \right) dx dt$$

$$\begin{aligned}
&\geq \mathcal{C}e^{\frac{R_0^2}{2}\lambda}\mathbb{E}\int_{\frac{T}{2}-\varepsilon_0T}^{\frac{T}{2}+\varepsilon_0T}\int_G\left(z_t^2+|\nabla z|^2\right)dxdt \\
&\geq \mathcal{C}e^{\frac{R_0^2}{2}\lambda}e^{-\mathcal{C}\left(r_1^2+r_2^{\frac{1}{2-n/p}+1}\right)T}\mathbb{E}\int_G\left(z_1^2+|\nabla z_0|^2\right)dx \\
&\quad -\mathcal{C}e^{\frac{R_0^2}{2}\lambda}\mathbb{E}\int_Q\left(|f|^2+|g|^2\right)dxdt.
\end{aligned} \tag{4.3.32}$$

Combining (4.3.30)–(4.3.32), noting that $R_1^2 - c_1(\varepsilon_1 T)^2 < 0$, there exists a

$$\lambda_3 = \mathcal{C}\left(r_1^2 + r_2^{\frac{1}{3/2-n/p}+1}\right) \geq \lambda_2$$

such that for all $\lambda \geq \lambda_3$, it holds that

$$\begin{aligned}
&\mathcal{C}\lambda\mathbb{E}\int_{\Sigma_0}\theta^2\left(\sum_{j,k=1}^nb^{jk}\nu_j\nu_k\right)\left(\sum_{j',k'=1}^nb^{j'k'}\psi_{x_{j'}}\nu_{k'}\right)\left|\frac{\partial z}{\partial\nu}\right|^2d\Gamma dt \\
&\quad +\mathcal{C}e^{C\lambda}\mathbb{E}\int_Q\left(|f|^2+|g|^2\right)dxdt \\
&\geq \mathbb{E}\int_G\left(z_1^2+|\nabla z_0|^2\right)dx.
\end{aligned} \tag{4.3.33}$$

Taking $\lambda = \lambda_3$, we deduce that

$$\mathcal{C}e^{C\lambda_3}\left[\mathbb{E}\int_{\Sigma_0}\left|\frac{\partial z}{\partial\nu}\right|^2d\Gamma dt + \mathbb{E}\int_Q\left(|f|^2+|g|^2\right)dxdt\right] \geq \mathbb{E}\int_G\left(z_1^2+|\nabla z_0|^2\right)dx,$$

which gives the desired inequality (4.3.2). $\square$

We are in a position to prove [Theorem 4.3.1](#).

Proof of Theorem 4.3.1. Let $\hat{y}$ solve (4.1.2) with the initial datum $(\hat{y}_0, \hat{y}_1)$. Put $z = y - \hat{y}$. Then z fulfills (4.1.5) with $f = g = 0$, $z_0 = y_0 - \hat{y}_0$, $z_1 = y_1 - \hat{y}_1$, and the following bounded progressively measurable coefficients:

$$\begin{aligned}
b_1 &= \begin{cases} \frac{F(y, y_t, \nabla y) - F(y, \hat{y}_t, \nabla y)}{y_t - \hat{y}_t}, & y_t \neq \hat{y}_t, \\ 0, & y_t = \hat{y}_t, \end{cases} \\
b_2 &= \begin{cases} \frac{F(\hat{y}, \hat{y}_t, \nabla y) - F(\hat{y}, \hat{y}_t, \nabla \hat{y})}{|\nabla z|^2} \nabla z, & |\nabla z| \neq 0, \\ 0, & |\nabla z| = 0, \end{cases} \\
b_3 &= \begin{cases} \frac{F(y, \hat{y}_t, \nabla y) - F(\hat{y}, \hat{y}_t, \nabla y)}{y - \hat{y}}, & y \neq \hat{y}, \\ 0, & y = \hat{y}, \end{cases}
\end{aligned}$$

$$b_4 = \begin{cases} \frac{K(y) - K(\hat{y})}{y - \hat{y}}, & y \neq \hat{y}, \\ 0, & y = \hat{y}. \end{cases}$$

Indeed, the three differences defining b_1, b_3, b_2 telescope to $F(y, y_t, \nabla y) - F(\hat{y}, \hat{y}_t, \nabla \hat{y})$, and (4.1.1) bounds each coefficient by L . The proof is completed directly by (4.3.2). $\square$

4.4 Reconstruct the unknown state with a boundary measurement

This section focuses on the following problem:

- **Problem 4.2:** Find a function z with $\frac{\partial z}{\partial \nu} = h$ on Σ_0 for a given $h \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$, where z solves (4.1.5) with $g = 0$.

Recall that the reconstruction problem means to reconstruct the solution z from some observation or measurement.

Similar to Section 3.5, to solve Problem 4.2, we use Tikhonov regularization strategy. To begin with, we give the following immediate corollary of Theorems 4.2.5 and 4.3.2.

Consider the following stochastic hyperbolic equation:

$$\begin{cases} dz_t - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt = (b_1 z_t + b_2 \cdot \nabla z + b_3 z + f) dt + b_4 z dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \end{cases} \quad (4.4.1)$$

where coefficients b_j ($1 \leq j \leq 4$) satisfy (4.1.6).

Proposition 4.4.1. Assume Condition 4.3.1 and Condition 4.3.2 are satisfied. For any solution z to the equation (4.4.1), it holds that

$$\begin{aligned} & |(z, z_t)|_{L^2_{\mathbb{F}}(0, T; H^1(G)) \times L^2_{\mathbb{F}}(0, T; L^2(G))} \\ & \leq e^{C(r_1^2 + r_2^{\frac{1}{3/2 - n/p} + 1})} \left(\left| \frac{\partial z}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))} + |f|_{L^2_{\mathbb{F}}(0, T; L^2(G))} \right), \end{aligned} \quad (4.4.2)$$

where $\Gamma_0 \subset \Gamma$ is defined in (4.3.1).

Denote $\mathcal{H} = L^2_{\mathbb{F}}(0, T; H^2(G)) \cap L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))$. For $u \in \mathcal{H}$, set

$$\begin{aligned} V_u(t) &= u_t(t) - \int_0^t \left[\sum_{j,k=1}^n (b^{jk} u_{x_j})_{x_k} + b_1 u_t + b_2 \cdot \nabla u + b_3 u \right] ds - \int_0^t b_4 u dW(s), \\ \mathcal{D} &= \{u \in \mathcal{H} : V_u \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))\}. \end{aligned}$$

Define $\mathcal{P}u = V_u - V_u(0)$ for $u \in \mathcal{D}$. Here $V_u(0)$ is a well-defined trace. Moreover, u_t has a continuous $L^2(G)$ -valued version on this domain and $V_u(0) = u_t(0)$. Put

$$\mathcal{U} = \{u \in \mathcal{D} : u|_{\Sigma} = 0, \quad \partial_{\nu}u|_{\Sigma_0} = h\}.$$

We assume that h admits such a lift, so that $\mathcal{U} \neq \emptyset$. In particular, this holds for the exact observation of a solution belonging to $\mathcal{H}$. For such a solution, $\mathcal{P}u(t) = \int_0^t f(s)ds$.

Denote $\mathbf{I}f(t) = \int_0^t f(s)ds$ for $t \in [0, T]$. Fixing a function $F \in \mathcal{U}$, we construct the Tikhonov functional as follows:

$$\mathcal{J}_{\gamma}(u) = |\mathcal{P}u - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma|u - F|_{\mathcal{H}}^2, \quad (4.4.3)$$

where $u \in \mathcal{U}$ and $\gamma \in (0, 1)$.

We have the following result regarding the existence of the minimizer.

Proposition 4.4.2. *For $\gamma \in (0, 1)$, there exists a unique minimizer $u_{\gamma} \in \mathcal{U}$ of the functional $\mathcal{J}_{\gamma}(u)$.*

Proof. Let

$$\mathcal{U}_0 \triangleq \{u \in \mathcal{D} \mid \mathcal{P}u \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G))), u|_{\Sigma} = 0, \partial_{\nu}u|_{\Sigma_0} = 0\}.$$

Fix $\gamma \in (0, 1)$. We define the inner product as follows:

$$\langle \varphi, \psi \rangle_{\mathcal{U}_0} = \langle \mathcal{P}\varphi, \mathcal{P}\psi \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle \varphi, \psi \rangle_{\mathcal{H}}, \quad (4.4.4)$$

where $\varphi, \psi \in \mathcal{U}_0$. The space $\mathcal{U}_0$ is complete for this inner product. Indeed, if $u_m \rightarrow u$ in $\mathcal{H}$ and $\mathcal{P}u_m \rightarrow q$ in $L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))$, then $V_{u_m} \rightarrow V_u$ in $L^2_{\mathbb{F}}(0, T; L^2(G))$. The time-constant functions $V_{u_m}(0) = V_{u_m} - \mathcal{P}u_m$ therefore converge in $L^2(\Omega; L^2(G))$ to a time-constant function c . It follows that $V_u = q + c$ belongs to H^1 in time, $q(0) = 0$, and $\mathcal{P}u = q$. The boundary traces pass to the limit by the continuity of the $H^2(G)$ trace maps. Thus the limit remains in $\mathcal{U}_0$.

For $u \in \mathcal{U}$, let $v = u - F$. Then $v \in \mathcal{U}_0$. Consider the following functional

$$\overline{\mathcal{J}}_{\gamma}(v) = |\mathcal{P}v + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma|v|_{\mathcal{H}}^2,$$

where $v \in \mathcal{U}_0$ and $\gamma \in (0, 1)$. Clearly, u_{γ} minimizes $\mathcal{J}_{\gamma}(u)$ if and only if $v_{\gamma} = u_{\gamma} - F$ is the minimizer of the functional $\overline{\mathcal{J}}_{\gamma}(v)$.

By the variational principle, the minimizer v_{γ} of the functional $\overline{\mathcal{J}}_{\gamma}(v)$ satisfy the following equation

$$\langle \mathcal{P}v_{\gamma} + \mathcal{P}F - \mathbf{I}f, \mathcal{P}\rho \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle v_{\gamma}, \rho \rangle_{\mathcal{H}} = 0, \quad \forall \rho \in \mathcal{U}_0. \quad (4.4.5)$$

Conversely, the solution v_{γ} of equation (4.4.5) is the minimizer of the functional $\overline{\mathcal{J}}_{\gamma}(v)$. In fact, if v_{γ} solves (4.4.5), then for each $v \in \mathcal{U}_0$, it holds that

$$\overline{\mathcal{J}}_{\gamma}(v) - \overline{\mathcal{J}}_{\gamma}(v_{\gamma})$$

$$\begin{aligned}
&= |\mathcal{P}v + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}^2 + \gamma|v|_{\mathcal{H}}^2 \\
&\quad - |\mathcal{P}v_{\gamma} + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}^2 - \gamma|v_{\gamma}|_{\mathcal{H}}^2 \\
&= |\mathcal{P}v_{\gamma} + \mathcal{P}F - \mathbf{I}f + \mathcal{P}(v - v_{\gamma})|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}^2 \\
&\quad + \gamma|v_{\gamma} + (v - v_{\gamma})|_{\mathcal{H}}^2 - |\mathcal{P}v_{\gamma} + \mathcal{P}F - \mathbf{I}f|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}^2 - \gamma|v_{\gamma}|_{\mathcal{H}}^2 \\
&= 2\langle \mathcal{P}v_{\gamma} + \mathcal{P}F - \mathbf{I}f, \mathcal{P}(v - v_{\gamma}) \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))} \\
&\quad + 2\gamma\langle v_{\gamma}, v - v_{\gamma} \rangle_{\mathcal{H}} + |\mathcal{P}(v - v_{\gamma})|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}^2 + \gamma|v - v_{\gamma}|_{\mathcal{H}}^2 \\
&= |\mathcal{P}(v - v_{\gamma})|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}^2 + \gamma|v - v_{\gamma}|_{\mathcal{H}}^2 \\
&\geq 0,
\end{aligned}$$

which means v_{γ} is a minimizer of the functional $\overline{\mathcal{J}}_{\gamma}(v)$.

From (4.4.4), equation (4.4.5) is equivalent to

$$\langle v_{\gamma}, \rho \rangle_{\mathcal{U}_0} = \langle \mathbf{I}f - \mathcal{P}F, \mathcal{P}\rho \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}, \quad \forall \rho \in \mathcal{U}_0. \quad (4.4.6)$$

Thanks to Cauchy-Schwarz inequality and (4.4.4), for all $\rho \in \mathcal{U}_0$, we have

$$\begin{aligned}
&|\langle \mathbf{I}f - \mathcal{P}F, \mathcal{P}\rho \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}| \\
&\leq C(|\mathcal{P}F|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))} + |f|_{L^2_{\mathbb{F}}(0,T; L^2(G))})|\rho|_{\mathcal{U}_0}.
\end{aligned}$$

By the Riesz representation theorem, there exists a unique $w_{\gamma} \in \mathcal{U}_0$ such that, for every $\rho \in \mathcal{U}_0$,

$$\langle w_{\gamma}, \rho \rangle_{\mathcal{U}_0} = \langle \mathbf{I}f - \mathcal{P}F, \mathcal{P}\rho \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))}.$$

Hence w_{γ} solves (4.4.6), which means w_{γ} is the minimizer of the functional $\overline{\mathcal{J}}_{\gamma}(\cdot)$. $\square$

Assume that z^* is the true solution to the equation (4.4.1) with the exact data

$$f^* \in L^2_{\mathbb{F}}(0, T; L^2(G)), \quad z^*|_{\Gamma} = 0, \quad \partial_{\nu} z^*|_{\Gamma_0} = h^* \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0)).$$

From Theorem 4.4.1, the solution z^* is unique. Assume $z^* \in \mathcal{D}$. Define $\mathcal{U}^* = \{u \in \mathcal{D} : u|_{\Sigma} = 0, \partial_{\nu} u|_{\Sigma_0} = h^*\}$. There exists $F^* \in \mathcal{U}^*$ (for instance $F^* = z^*$) such that

$$|F^*|_{\mathcal{H}} \leq C|z^*|_{\mathcal{H}}. \quad (4.4.7)$$

We assume that

$$|f^* - f|_{L^2_{\mathbb{F}}(0,T; L^2(G))} \leq \delta, \quad |h^* - h|_{L^2_{\mathbb{F}}(0,T; L^2(\Gamma_0))} \leq \delta, \quad (4.4.8)$$

and

$$|\mathcal{P}F^* - \mathcal{P}F|_{L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G)))} + |F^* - F|_{\mathcal{H}} \leq \delta. \quad (4.4.9)$$

Set $\mathcal{X} = L^2_{\mathbb{F}}(0, T; H^1(G)) \cap L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))$, with its natural Hilbert norm. We now establish the estimate for the error between the minimizer z_{γ} for the functional $\mathcal{J}_{\gamma}(\cdot)$ and the exact solution z^* .

Theorem 4.4.3. Assume (4.4.8), (4.4.9), and Conditions 4.3.1 and 4.3.2 hold. Then there exists a constant $\mathcal{C} > 0$ such that for all $0 < \delta < 1$,

$$|z_\gamma - z^*|_{\mathcal{X}} \leq \mathcal{C}(1 + \|z^*\|_{\mathcal{H}})\delta$$

where z_γ is the minimizer of the functional (4.4.3) with the regularization parameter $\gamma = \delta^2$.

Proof. Let $v^* = z^* - F^*$. Then $v^* \in \mathcal{U}_0$ and $\mathcal{P}v^* = \mathbf{I}f^* - \mathcal{P}F^*$. Hence, for all $\rho \in \mathcal{U}_0$, it holds that

$$\langle \mathcal{P}v^* + \mathcal{P}F^* - \mathbf{I}f^*, \mathcal{P}\rho \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} = 0. \quad (4.4.10)$$

Subtracting (4.4.10) from (4.4.5) and denoting $\tilde{v}_\gamma = v^* - v_\gamma$, $\tilde{f} = \mathbf{I}f^* - \mathbf{I}f$ and $\tilde{F} = F^* - F$, for all $\rho \in \mathcal{U}_0$, we obtain

$$\begin{aligned} & \langle \mathcal{P}\tilde{v}_\gamma, \mathcal{P}\rho \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle \tilde{v}_\gamma, \rho \rangle_{\mathcal{H}} \\ &= \langle \tilde{f} - \mathcal{P}\tilde{F}, \mathcal{P}\rho \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle v^*, \rho \rangle_{\mathcal{H}}. \end{aligned} \quad (4.4.11)$$

Choose $\rho = \tilde{v}_\gamma$ in (4.4.11). Then we have

$$|\mathcal{P}\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma |\tilde{v}_\gamma|_{\mathcal{H}}^2 \leq |\tilde{f} - \mathcal{P}\tilde{F}|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma |v^*|_{\mathcal{H}}^2. \quad (4.4.12)$$

Since $\gamma = \delta^2$, from (4.4.8), (4.4.9) and (4.4.12), we obtain

$$|\tilde{v}_\gamma|_{\mathcal{H}}^2 \leq \mathcal{C}(1 + |v^*|_{\mathcal{H}}^2), \quad (4.4.13)$$

and

$$|\mathcal{P}\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 \leq \mathcal{C}(1 + |v^*|_{\mathcal{H}}^2)\delta^2. \quad (4.4.14)$$

The embedding $L^2(0, T; H^2(G) \cap H_0^1(G)) \cap H^1(0, T; L^2(G)) \hookrightarrow C([0, T]; H_0^1(G))$, together with the continuous velocity supplied by the graph domain, places every element of $\mathcal{U}_0$ in $\mathbb{H}_T$. Since $\tilde{v}_\gamma \in \mathcal{U}_0$, it solves (4.4.1) with zero observed normal derivative and forcing $(\mathcal{P}\tilde{v}_\gamma)_t$. By Theorem 4.4.1 and (4.4.14),

$$\|\tilde{v}_\gamma\|_{\mathcal{X}} \leq C\|\mathcal{P}\tilde{v}_\gamma\|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} \leq C(1 + \|z^*\|_{\mathcal{H}})\delta. \quad (4.4.15)$$

Here we also used (4.4.7). Because $z^* - z_\gamma = \tilde{v}_\gamma + F^* - F$ and the embedding $\mathcal{H} \hookrightarrow \mathcal{X}$ is continuous, (4.4.9) gives

$$\|z^* - z_\gamma\|_{\mathcal{X}} \leq \|\tilde{v}_\gamma\|_{\mathcal{X}} + C\|F^* - F\|_{\mathcal{H}} \leq C(1 + \|z^*\|_{\mathcal{H}})\delta. \quad (4.4.16)$$

□

4.5 Solution to inverse state problem with the internal measurement

This section focuses on the following problem:

- **Problem 4.3:** Can $\tilde{z}$ in $\mathcal{D}_\rho^+ \times (\varepsilon, T - \varepsilon)$ be uniquely determined by the values of $\tilde{z}$ in $\mathcal{D}_\rho^- \times (0, T)$, where $\tilde{z}$ solves (4.1.7)?

To this end, we need the following definition.

Definition 4.5.1. Assume $x_0 \in S$ and K is a nonnegative number. Given the hypersurface S , it is said to satisfy the outer paraboloid condition with K at x_0 if there exists a neighborhood $\mathcal{V}$ of x_0 and a paraboloid $\mathcal{P}$ tangential to S at x_0 such that $\mathcal{P} \cap \mathcal{V} \subset \overline{\mathcal{D}_\rho^-}$ with $\mathcal{P}$ being congruent to $x_1 = -K \sum_{j=2}^n x_j^2$.

The following theorem gives a positive answer to Problem 4.3.

Theorem 4.5.1. Assume additionally that $\tilde{b}_3 \in L_{\mathbb{F}}^\infty(0, T; L_{\text{loc}}^p(G))$ for some $p \geq \max\{n, 2\}$, with $p > 2$ when $n = 2$. Let $x_0 \in S \setminus \partial S$ satisfy $\frac{\partial \sigma(T/2, x_0)}{\partial \nu} < 0$, and S satisfy the outer paraboloid condition with

$$K < \frac{-\frac{\partial \sigma}{\partial \nu}(T/2, x_0)}{4(|\sigma|_{L^\infty(B_\rho(T/2, x_0))} + 1)}. \quad (4.5.1)$$

If $z \in L_{\mathbb{F}}^2(\Omega; C([0, T]; H_{\text{loc}}^1(G))) \cap L_{\mathbb{F}}^2(\Omega; C^1([0, T]; L_{\text{loc}}^2(G)))$ is a solution to the equation (4.1.7) satisfying that

$$z = \frac{\partial z}{\partial \nu} = 0 \quad \text{on } (0, T) \times S, \quad \mathbb{P}\text{-a.s.}, \quad (4.5.2)$$

then for a sufficiently small neighborhood $\mathcal{V}$ of x_0 and some $\varepsilon \in (0, T/2)$, it holds that

$$z = 0 \quad \text{in } (\mathcal{V} \cap \mathcal{D}_\rho^+) \times (\varepsilon, T - \varepsilon), \quad \mathbb{P}\text{-a.s.} \quad (4.5.3)$$

The normal trace in (4.5.2) is understood in the weak Green formula sense.

Remark 4.5.1. Noting that in Theorem 4.5.1, we assume that $\frac{\partial \sigma(T/2, x_0)}{\partial \nu} < 0$, which can be regarded as a kind of pseudoconvex conditions (e.g., [34, Chapter XXVII]). This assumption is reasonable. In fact, even in the deterministic setting, without such an assumption, it may lead to the involved (local) uniqueness failing (e.g., [1]).

We first give the weight function which will be used to establish a Carleman estimate for the equation (4.1.7).

Without loss of generality, we assume that $x_0 = 0 \in S \setminus \partial S$ and $\nu(0) = (1, 0, \dots, 0)$. Since S is C^2 , for some $r > 0$ and $\gamma(\cdot) \in C^2(\mathbb{R}^{n-1})$, we can parameterize S in a neighborhood of the origin by

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 = \gamma(x_2, \dots, x_n), |x_2|^2 + \dots + |x_n|^2 < r\}. \quad (4.5.4)$$

We put

$$\begin{cases} B_r\left(\frac{T}{2}, 0\right) = \left\{ (t, x) \in \mathbb{R}^{n+1} \mid |x|^2 + \left(t - \frac{T}{2}\right)^2 < r^2 \right\}, \\ B_r(0) = \{x \in \mathbb{R}^n \mid |x| < r\}. \end{cases} \quad (4.5.5)$$

Then, it follows from (4.5.1) that

$$\begin{cases} -\alpha_0 \triangleq \frac{\partial \sigma}{\partial \nu}\left(\frac{T}{2}, 0\right) < 0, \\ K < \frac{\alpha_0}{4(|\sigma|_{L^\infty(B_r(T/2, 0))} + 1)}. \end{cases} \quad (4.5.6)$$

Thanks to Definition 4.5.1 and (4.5.4), we get

$$-K \sum_{j=2}^n x_j^2 \leq \gamma(x_2, \dots, x_n), \quad \text{if } \sum_{j=2}^n x_j^2 < r. \quad (4.5.7)$$

Denote

$$\mathcal{D}_r^- = \{x \in B_r(0) \mid x_1 < \gamma(x_2, \dots, x_n)\}, \quad \mathcal{D}_r^+ = B_r(0) \setminus \overline{\mathcal{D}_r^-}.$$

Since $\sigma \in C^1(\overline{Q})$, for any $\alpha \in (0, \alpha_0)$, we can find a sufficiently small $\delta_0 > 0$ such that $\delta_0 < \min\{1, r^2\}$ and

$$\sigma_{x_1}(t, x) < -\alpha, \quad \text{if } |x|^2 + \left(t - \frac{T}{2}\right)^2 \leq \delta_0. \quad (4.5.8)$$

Put

$$M_0 = |\sigma|_{L^\infty(B_r(T/2, 0))}, \quad M_1 = \max\{|\sigma|_{C^1(B_r(T/2, 0))}, 1\}. \quad (4.5.9)$$

From (4.5.6), we can choose $\alpha > 0$ and $h > 0$ such that

$$K < \frac{1}{2h} < \frac{\alpha}{4(M_0 + 1)}. \quad (4.5.10)$$

Put $\beta_0 = [4(M_0 + 1)^2]^{-1}$ and $\kappa = 2 + (M_0 + 1)^{-1}$. For $\tau \in (0, 1)$, define the weight function

$$\varphi(t, x) = hx_1 + \frac{1}{2} \sum_{j=2}^n x_j^2 + \frac{\beta_0}{2} \left(t - \frac{T}{2}\right)^2 + \frac{1}{2}\tau. \quad (4.5.11)$$

For any positive number μ with $\tau \geq \mu > \frac{\tau}{2}$, put

$$Q_\mu = \left\{ (t, x) \in \mathbb{R}^{n+1} \mid x_1 > \gamma(x_2, x_3, \dots, x_n), \sum_{j=2}^n x_j^2 < \delta_0, \varphi(t, x) < \mu \right\}. \quad (4.5.12)$$

The set Q_μ is nonempty: since $\gamma(0) = 0$, the point $(T/2, \varepsilon, 0, \dots, 0)$ belongs to Q_μ when $0 < \varepsilon < (\mu - \tau/2)/h$ is sufficiently small. Moreover, $x_1 > \gamma(x') \geq -K|x'|^2$ gives

$$\varphi(t, x) > \left(\frac{1}{2} - hK\right) |x'|^2 + \frac{\beta_0}{2}(t - T/2)^2 + \frac{\tau}{2} \geq \frac{\tau}{2} \quad ((t, x) \in Q_\mu).$$

Next, we show that there exists $\tau_1 > 0$ such that for all $\tau \leq \tau_1$ and $\mu \leq \tau$, we have

$$|x|^2 + \left(t - \frac{T}{2}\right)^2 \leq \delta_0, \quad \forall (t, x) \in Q_\mu. \quad (4.5.13)$$

Since $\varphi < \mu \leq \tau$, from (4.5.11), we get

$$x_1 < -\frac{1}{2h} \sum_{j=2}^n x_j^2 - \frac{\beta_0}{2h} \left(t - \frac{T}{2}\right)^2 + \frac{1}{2h} \tau \leq \frac{\tau}{2h}. \quad (4.5.14)$$

By (4.5.14) and

$$-K \sum_{j=2}^n x_j^2 \leq \gamma(x_2, \dots, x_n) < x_1,$$

we get

$$\sum_{j=2}^n x_j^2 \leq \frac{\tau}{1 - 2Kh}, \quad (4.5.15)$$

which implies

$$-x_1 \leq \frac{K\tau}{1 - 2Kh}. \quad (4.5.16)$$

From (4.5.14) and (4.5.16), we deduce that

$$|x_1| \leq \max \left\{ \frac{K}{1 - 2Kh}, \frac{1}{2h} \right\} \tau. \quad (4.5.17)$$

Combining (4.5.15) and (4.5.17), noting that $\varphi < \tau$, we obtain that

$$-\frac{Kh\tau}{1 - 2Kh} + \frac{1}{2} \sum_{j=2}^n x_j^2 + \frac{\beta_0}{2} \left(t - \frac{T}{2}\right)^2 + \frac{\tau}{2} \leq \varphi < \tau,$$

which implies

$$\left(t - \frac{T}{2}\right)^2 \leq \frac{\tau}{\beta_0(1 - 2Kh)}. \quad (4.5.18)$$

From (4.5.17) and (4.5.18), for $(t, x) \in Q_\mu$, we have

$$|x|^2 + \left(t - \frac{T}{2}\right)^2 \leq \mu_0(\tau) \triangleq \left| \max \left\{ \frac{K}{1 - 2Kh}, \frac{1}{2h} \right\} \right|^2 \tau^2 + \frac{(1 + \beta_0^{-1})\tau}{1 - 2Kh}. \quad (4.5.19)$$

Choosing $\tau_1 > 0$ such that $\mu_0(\tau_1) \leq \delta_0$, for all $\tau \leq \tau_1$, we obtain (4.5.13).

Theorem 4.5.2. *Let φ, β_0, κ be as above. Let u be an $H_{\text{loc}}^2(G)$ -valued adapted process such that u_t is an $L_{\text{loc}}^2(G)$ -valued Itô process. Put*

$$\begin{aligned} \ell &= s\varphi^{-\lambda}, \quad \theta = e^\ell, \quad w = \theta u, \quad a = s\lambda\varphi^{-\lambda-1}, \quad \Psi = \Delta\ell - \sigma\ell_{tt} + \kappa a, \\ I &= -2\sigma\ell_t w_t + 2\nabla\ell \cdot \nabla w + \Psi w. \end{aligned}$$

There exists $\tau_0 > 0$ such that, for each $0 < \tau \leq \tau_0$, there are positive constants C, λ_0, s_0 for which

$$\begin{aligned} & \mathbb{E} \int_{Q_\tau} \theta I(\sigma du_t - \Delta u dt) dx \\ & \geq C \mathbb{E} \int_{Q_\tau} \left[a(|\nabla w|^2 + w_t^2) + s^3 \lambda^4 \varphi^{-3\lambda-4} w^2 \right] dx dt + \mathbb{E} \int_{Q_\tau} I^2 dx dt \\ & \quad + \mathbb{E} \int_{Q_\tau} \sigma^2 \theta^2 \ell_t (du_t)^2 dx \end{aligned} \quad (4.5.20)$$

for all $\lambda \geq \lambda_0, s \geq s_0$, and u compactly supported in Q_τ . The same estimate holds when u and its first derivatives have zero traces on ∂Q_τ .

Proof. Write $r = \varphi_t = \beta_0(t-T/2)$, $q = \nabla\varphi = (h, x')$, $H = D_x^2\varphi = \text{diag}(0, 1, \dots, 1)$, and $D = \sigma r^2 - |q|^2$. Direct differentiation gives

$$\begin{aligned} \ell_t &= -ar, \quad \nabla\ell = -aq, \quad \ell_{tt} = \frac{a(\lambda+1)}{\varphi} r^2 - a\beta_0, \\ D_x^2\ell &= \frac{a(\lambda+1)}{\varphi} q \otimes q - aH, \quad \nabla\ell_t = \frac{a(\lambda+1)}{\varphi} r q. \end{aligned} \quad (4.5.21)$$

Multiply the conjugated equation by I and integrate by parts on Q_τ . In the term containing $\sigma\Psi w dw_t$ and the spatial term containing $-\Psi w \Delta w$, we leave the first derivatives of $\sigma\Psi$ and Ψ on the coefficients. The resulting identity is

$$\begin{aligned} \mathbb{E} \int_{Q_\tau} \theta I(\sigma du_t - \Delta u dt) dx &= \mathbb{E} \int_{Q_\tau} (\mathcal{Q} + \mathcal{B}_0 w^2 + \mathcal{R} + I^2) dx dt \\ &\quad + \mathbb{E} \int_{Q_\tau} \sigma^2 \theta^2 \ell_t (du_t)^2 dx, \end{aligned} \quad (4.5.22)$$

where $\mathcal{R} = -(\sigma\Psi)_t w w_t + w \nabla\Psi \cdot \nabla w$ and

$$\mathcal{B}_0 = \mathcal{A}\Psi + (\sigma\mathcal{A}\ell_t)_t - \nabla \cdot (\mathcal{A}\nabla\ell).$$

The boundary terms vanish by the support or zero-trace assumptions. These product rules involve only first derivatives of σ and Ψ . The quadratic form in $w_t, \nabla w$ is

$$\begin{aligned} \mathcal{Q} &= [(\sigma^2\ell_t)_t + \nabla \cdot (\sigma\nabla\ell) - \sigma\Psi] w_t^2 - 2[(\sigma\nabla\ell)_t + \nabla(\sigma\ell_t)] \cdot \nabla w w_t \\ &\quad + [(\sigma\ell_t)_t - \Delta\ell + \Psi] |\nabla w|^2 + 2D_x^2\ell[\nabla w, \nabla w]. \end{aligned} \quad (4.5.23)$$

Set $J = \sigma r w_t - q \cdot \nabla w$. Substitution in (4.5.23) gives the exact expression

$$\begin{aligned} \mathcal{Q} = & \frac{2a(\lambda+1)}{\varphi} J^2 - 2a\sigma_t J w_t \\ & + a[(-2\sigma^2\beta_0 - q \cdot \nabla\sigma - \sigma\kappa)w_t^2 + (\kappa - \sigma_t r)|\nabla w|^2 \\ & - 2H[\nabla w, \nabla w] + 2r\nabla\sigma \cdot \nabla w w_t]. \end{aligned}$$

By (4.5.10), $h\alpha > 2(M_0 + 1)$, whereas $2M_0^2\beta_0 + \kappa M_0 < 2M_0 + 3/2$ and $\kappa - 2 > 0$. After decreasing τ_0 so that $Q_\tau \subset (0, T) \times G$, the bounds (4.5.8) and (4.5.19) therefore make the coefficients of w_t^2 and $|\nabla w|^2$ strictly positive, after absorbing the term containing $r\nabla\sigma$. The remaining mixed term satisfies

$$2|\sigma_t J w_t| \leq \frac{\lambda+1}{\varphi} J^2 + \frac{\varphi\sigma_t^2}{\lambda+1} w_t^2.$$

Hence, for sufficiently large λ_0 ,

$$\mathcal{Q} \geq Ca(|\nabla w|^2 + w_t^2). \quad (4.5.24)$$

Substituting $\mathcal{A} = a^2 D - \kappa a$ gives

$$\begin{aligned} \mathcal{B}_0 = & a^3 \left[\frac{2(\lambda+1)}{\varphi} D^2 + \kappa D + q \cdot \nabla D - \sigma r D_t - \sigma_t r D \right] \\ & + \kappa a^2 \left[\sigma_t r - \kappa - \frac{\lambda+1}{\varphi} D \right]. \end{aligned}$$

We first shrink τ_0 so that $|D| \geq h^2/2$ and $\tau/2 < \varphi < \tau < 1$ on Q_τ . We then increase λ_0 to absorb the other cubic terms into $2a^3(\lambda+1)D^2/\varphi$, and increase s_0 to absorb the quadratic terms, whose ratio to this leading term is bounded by C/a . Thus

$$\mathcal{B}_0 \geq Ca^3\lambda/\varphi = Cs^3\lambda^4\varphi^{-3\lambda-4}. \quad (4.5.25)$$

The C^1 regularity of σ and (4.5.21) give

$$|(\sigma\Psi)_t| + |\nabla\Psi| \leq Ca\lambda^2\varphi^{-2}.$$

For any $\eta > 0$, it follows that

$$|\mathcal{R}| \leq \eta a(w_t^2 + |\nabla w|^2) + C(\eta)a\lambda^4\varphi^{-4}w^2.$$

Fix η small enough to absorb the first term by (4.5.24). The ratio of the last term to $a^3\lambda/\varphi$ is $C(\eta)\lambda\varphi^{2\lambda-1}/s^2$, which is bounded by $C(\eta)/s^2$ for $\lambda \geq 1$ and $\varphi \leq \tau_0 < 1$. Increasing s_0 once more and using (4.5.22), (4.5.24) and (4.5.25) proves (4.5.20). The identity is first applied to smooth processes and then extended by approximation at fixed s, λ ; its coefficients require only $\sigma \in C^1(\overline{Q})$. $\square$

With the help of the Carleman estimate, we are in a position to prove [Theorem 4.5.1](#).

Proof of Theorem 4.5.1. Use the coordinates and weights above and fix a sufficiently small $\tau > 0$. Choose $0 < \tilde{\tau} < \tau/8$, and set

$$Q_1 = Q_\tau, \quad Q_{k+1} = Q_{\tau-k\tilde{\tau}}, \quad k = 1, 2, 3. \quad (4.5.26)$$

By (4.5.15) and (4.5.19), the artificial boundary $|x'|^2 = \delta_0$ does not meet $\overline{Q_1}$ after decreasing τ . Choose a smooth function $\chi = \chi(\varphi)$ with $0 \leq \chi \leq 1$, $\chi = 1$ for $\varphi \leq \tau - 2\tilde{\tau}$ and $\chi = 0$ for $\varphi \geq \tau - \tilde{\tau}$. Set $u = \chi z$. The zero Cauchy data on S ensure that u and its first derivatives have zero boundary traces on ∂Q_1 . It satisfies

$$\sigma du_t - \Delta u dt = (\tilde{b}_1 u_t + \tilde{b}_2 \cdot \nabla u + \tilde{b}_3 u + f) dt + \tilde{b}_4 u dW(t), \quad (4.5.27)$$

where

$$f = \sigma \chi_{tt} z + 2\sigma \chi_t z_t - 2\nabla \chi \cdot \nabla z - z \Delta \chi - \tilde{b}_1 \chi_t z - \tilde{b}_2 \cdot \nabla \chi z.$$

The function f is supported in $Q_2 \setminus \overline{Q_3}$ and $|f|^2 \leq C(|z_t|^2 + |\nabla z|^2 + z^2)$ there. Apply (4.5.20) to u , initially by smooth approximation. More precisely, extend u by zero across S ; its zero Cauchy traces produce no surface terms in the weak equation. Divide the weak equation by σ before spatial mollification, which preserves adaptedness. The principal commutators $[\sigma^{-1}, J_\varepsilon] \Delta u$ are first-order Friedrichs commutators acting on ∇u ; they converge in L^2 locally under the C^1 regularity of σ . The lower-order and noise terms also converge by approximation. The mollified zero extension can cross S , so integrate the weighted identity on a slightly larger neighborhood where the same coefficient bounds hold, and then pass to the limit to obtain the estimate on Q_1 , for each fixed s, λ . The expectation of the Itô integral is zero, and Young's inequality bounds the drift multiplier by $\frac{1}{2} \mathbb{E} \int I^2 + C \mathbb{E} \int \theta^2(|u_t|^2 + |\nabla u|^2 + |\tilde{b}_3 u|^2 + |f|^2)$. Moreover,

$$\left| \mathbb{E} \int_{Q_1} \sigma^2 \theta^2 \ell_t (du_t)^2 dx \right| = \left| \mathbb{E} \int_{Q_1} \ell_t \tilde{b}_4^2 w^2 dx dt \right| \leq C \mathbb{E} \int_{Q_1} a w^2 dx dt.$$

Sobolev's inequality, applied to the zero extension of w , gives $\|\tilde{b}_3 w\|_{L^2(G)} \leq C \|w\|_{H^1(G)}$ locally under the stated condition on p . Use also $\theta^2(|u_t|^2 + |\nabla u|^2) \leq C(|w_t|^2 + |\nabla w|^2 + a^2 w^2)$. All these lower-order terms are absorbed for large λ, s , so that

$$\begin{aligned} & \mathbb{E} \int_{Q_1} [a(|\nabla w|^2 + w_t^2) + a^3 \lambda \varphi^{-1} w^2] dx dt \\ & \leq C \mathbb{E} \int_{Q_2 \setminus \overline{Q_3}} \theta^2(|z_t|^2 + |\nabla z|^2 + z^2) dx dt. \end{aligned} \quad (4.5.28)$$

The combined expression on the left, including its zeroth-order term, controls $C\theta^2[a(|\nabla u|^2 + u_t^2) + a^3\lambda\varphi^{-1}u^2]$. Since $u = z$ on Q_3 , fixing a sufficiently large λ gives, for all large s ,

$$\begin{aligned} & \mathbb{E} \int_{Q_4} (|\nabla z|^2 + z_t^2 + z^2) dx dt \\ & \leq C \exp \left\{ 2s \left[(\tau - 2\tilde{\tau})^{-\lambda} - (\tau - 3\tilde{\tau})^{-\lambda} \right] \right\} \mathbb{E} \int_{Q_1} (|z_t|^2 + |\nabla z|^2 + z^2) dx dt. \end{aligned} \quad (4.5.29)$$

The final integral is independent of s . Letting $s \rightarrow \infty$ proves $z = 0$ in Q_4 . This set contains $(\mathcal{V} \cap \mathcal{D}_\rho^+) \times (\varepsilon, T - \varepsilon)$ for a sufficiently small neighborhood $\mathcal{V}$ of x_0 and some $\varepsilon \in (0, T/2)$. $\square$

4.6 Solution to inverse source problem with the boundary measurement and the final time measurement. I

This section focuses on the following problem:

- **Problem 4.4:** Does $\mathcal{M}_2(z_0, z_1, g)$ uniquely determine (z_0, z_1, g) ?

Since the random source $g(\cdot)$ appears in the right hand side of (4.3.2), we cannot use Theorem 4.3.2 directly to solve Problem 4.4. To address this, we must develop a new Carleman estimate for (4.1.5) in which the source term g and the initial data are bounded by the measurement. To this end, for ψ satisfying Condition 4.3.1, we assume the following condition:

Condition 4.6.1.

$$\begin{cases} \mu_0 - 4c_1 - c_0 > 0, \\ \frac{\mu_0}{(8c_1 + c_0)} \sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k} > 4c_1^2 T^2 > \sum_{j,k=1}^n b^{jk} \psi_{x_j} \psi_{x_k}. \end{cases}$$

Given that Γ_0 is defined in (4.3.1), the following is a positive answer to Problem 4.4.

Theorem 4.6.1. *Let Conditions 4.3.1 and 4.6.1 hold. If the solution $z \in \mathbb{H}_T$ to (4.1.5) satisfies that $z(T) = 0$ in G , $\mathbb{P}$ -a.s., then*

$$\begin{aligned} & |(z_0, z_1)|_{L_{\mathcal{F}_0}^2(\Omega; H_0^1(G) \times L^2(G))} + |\sqrt{T - t}g|_{L_{\mathbb{F}}^2(0, T; L^2(G))} \\ & \leq C e^{C(r_1^2 + r_2^{\frac{1}{3/2 - n/p} + 1})} \left(\left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))} + |f|_{L_{\mathbb{F}}^2(0, T; L^2(G))} \right), \end{aligned}$$

where r_1 and r_2 are defined in (4.2.19).

Similar to the proof in [Theorem 4.3.2](#), [Theorem 4.6.1](#) can be directly derived by the following global Carleman estimate for [\(4.1.5\)](#).

Theorem 4.6.2. *Assume [Conditions 4.3.1](#) and [4.6.1](#) are satisfied. Then, there exists a constant $\tilde{\lambda} > 0$ such that for any $\lambda \geq \tilde{\lambda}$ and each solution $z \in \mathbb{H}_T$ to the equation [\(4.1.5\)](#) satisfying $z(T) = 0$ in G , $\mathbb{P}$ -a.s., it holds, with $\theta = \exp\{\lambda[\psi(x) - c_1(t - T)^2]\}$, that*

$$\begin{aligned} & \mathbb{E} \int_G \theta^2(0) (\lambda |z_1|^2 + \lambda |\nabla z_0|^2 + \lambda^3 |z_0|^2) dx + \lambda \mathbb{E} \int_Q (T - t) \theta^2 g^2 dx dt \\ & \leq C \left(\lambda \mathbb{E} \int_{\Sigma_0} \theta^2 \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt + \mathbb{E} \int_Q \theta^2 f^2 dx dt \right). \end{aligned} \quad (4.6.1)$$

Proof. The proof is divided into three steps.

Step 1. In this step, we give the weight function of the Carleman estimate and compute the order of the terms in the equality [\(4.2.1\)](#) with respect to λ .

We choose

$$\theta = e^\ell, \quad \ell = \lambda[\psi(x) - c_1(t - T)^2],$$

where $\lambda > 0$ is a parameter, $\psi(\cdot)$ fulfills [Conditions 4.3.1](#) and [4.6.1](#), and c_1 is the constant in [Condition 4.6.1](#).

We use the multiplier construction in the proof of [Theorem 4.3.2](#). Put $V = b\nabla\psi$, $k = \nabla\psi \cdot b\nabla\psi$ and $d = -2c_1 + \nabla \cdot V - c_0$. Choose a smooth d_ε with $|d_\varepsilon - d|_{C(\overline{G})} \leq \varepsilon$, and set

$$\Psi = \lambda d_\varepsilon. \quad (4.6.2)$$

Here ε is fixed sufficiently small below. Let $e = d_\varepsilon - d$, $w = \theta z$ and $q(t) = -2c_1(t - T)$. Then

$$\ell_t = \lambda q, \quad \ell_{tt} = -2c_1 \lambda, \quad \nabla \ell = \lambda \nabla \psi, \quad \nabla \ell_t = 0. \quad (4.6.3)$$

The coefficients in [\(4.3.11\)](#)–[\(4.3.13\)](#) are unchanged. With

$$\mathcal{A}_0 = \lambda^2 D, \quad D = q^2 - k, \quad r = 4c_1 + c_0 - e, \quad (4.6.4)$$

the decomposition [\(4.3.10\)](#) holds with $u = z$. Its zeroth-order coefficient is

$$\begin{aligned} \mathcal{B}_0 &= \lambda^3 [(4c_1 + c_0)k + V \cdot \nabla k - (8c_1 + c_0)q^2 + De] \\ &\quad - \frac{\lambda}{2} \nabla \cdot (b \nabla d_\varepsilon). \end{aligned} \quad (4.6.5)$$

By [Condition 4.3.1](#),

$$V \cdot \nabla k \geq \mu_0 k. \quad (4.6.6)$$

The strict inequalities in [Condition 4.6.1](#) give a positive lower bound for the cubic coefficient when $e = 0$. We first fix ε small enough to preserve this lower bound and those in [\(4.3.11\)](#) and [\(4.3.13\)](#), and then fix d_ε . For sufficiently large λ_1 ,

$$\mathcal{B}_0 \geq C\lambda^3 \quad (\lambda \geq \lambda_1). \quad (4.6.7)$$

As in [\(4.3.17\)](#), the term $\lambda r I w$ is absorbed into a quarter of I^2 and the positive zeroth-order term by increasing λ_1 .

Step 2. We estimate the time-boundary terms. Since $\Psi_t = 0$, the time-boundary expression from the identity for [\(4.3.10\)](#) is

$$M(t) = \ell_t(\nabla w \cdot b \nabla w + w_t^2) - 2b \nabla \ell \cdot \nabla w w_t - \Psi w_t w + \mathcal{A}_0 \ell_t w^2. \quad (4.6.8)$$

At $t = 0$, [Condition 4.6.1](#) gives

$$2c_1 T - \sqrt{k} > 0, \quad D(0) = 4c_1^2 T^2 - k > 0. \quad (4.6.9)$$

The Cauchy–Schwarz inequality therefore yields

$$M(0) \geq \lambda(2c_1 T - \sqrt{k})(\nabla w \cdot b \nabla w + w_t^2) - \lambda d_\varepsilon w_t w + 2c_1 T \lambda^3 D(0) w^2.$$

Young’s inequality absorbs the mixed term for sufficiently large λ_2 , and hence

$$M(0) \geq C[\lambda(w_t^2 + |\nabla w|^2) + \lambda^3 w^2] \quad (\lambda \geq \lambda_2). \quad (4.6.10)$$

At $t = T$, $\ell_t(T) = 0$ and $w(T) = \nabla w(T) = 0$, so

$$M(T) = 0. \quad (4.6.11)$$

Step 3. We integrate the multiplier identity for [\(4.1.5\)](#).

Integrating the identity for [\(4.3.10\)](#) with $u = z$, taking mathematical expectation, and using [\(4.3.11\)–\(4.3.13\)](#), [\(4.3.20\)](#), [\(4.6.7\)](#), [\(4.6.10\)](#) and [\(4.6.11\)](#), we obtain

$$\begin{aligned} & \mathbb{E} \int_Q \theta \left\{ \left(-2\ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right) \left[dz_t - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] \right\} dx \\ & + \lambda \mathbb{E} \int_{\Sigma_0} \left(\sum_{j,k=1}^n b^{jk} \nu_j \nu_k \right) \left(\sum_{j',k'=1}^n b^{j'k'} \psi_{x_{j'}} \nu_{k'} \right) \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma dt \\ & \geq C \mathbb{E} \int_Q \theta^2 \left(\lambda z_t^2 + \lambda |\nabla z|^2 + \lambda^3 z^2 \right) dx dt \\ & + \frac{3}{4} \mathbb{E} \int_Q \left(-2\ell_t w_t + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} w_{x_k} + \Psi w \right)^2 dx dt \\ & + C \mathbb{E} \int_G \theta^2(0) \left[\lambda (|\nabla z_0|^2 + |z_1|^2) + \lambda^3 |z_0|^2 \right] dx \\ & + C \lambda \mathbb{E} \int_Q (T-t) \theta^2 |b_4 z + g|^2 dx dt, \end{aligned}$$

for $\lambda \geq \max\{\lambda_1, \lambda_2\}$.

Since $|b_4 z + g|^2 \geq \frac{1}{2}|g|^2 - 2|b_4 z|^2$, we deduce that

$$\lambda \mathbb{E} \int_Q (T-t) \theta^2 |b_4 z + g|^2 dx dt \geq \frac{1}{2} \lambda \mathbb{E} \int_Q (T-t) \theta^2 g^2 dx dt - 2\lambda T \mathbb{E} \int_Q \theta^2 b_4^2 z^2 dx dt. \quad (4.6.12)$$

Similar to the proof of (4.3.29), taking $\tilde{\lambda} = \max\{\lambda_1, \lambda_2, \mathcal{C}(r_1^2 + r_2^{\frac{1}{3/2-n/p}} + 1)\}$ in (4.6.12), for any $\lambda \geq \tilde{\lambda}$, we deduce (4.6.1). $\square$

Remark 4.6.1. In contrast to the inverse source problem (Problem 3.10) for the stochastic parabolic equation in Chapter 3, here we can determine not only the source term g but also simultaneously determine the initial datum of the system. Further, we do not need assumptions on g as the one in Theorem 3.11.1.

Remark 4.6.2. An interesting question arises: Can we determine f that is in the drift term and (z_0, z_1) through the measurement $\left(\frac{\partial z}{\partial \nu}\Big|_{\Sigma_0}, z(T)\right)$? To be more specific, we consider the following stochastic hyperbolic equation:

$$\begin{cases} dz_t - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt = (b_1 z_t + b_2 \cdot \nabla z + b_3 z + f) dt + b_4 z dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } G, \end{cases}$$

in which z_0, z_1 and f are unknowns. From Example 1.1.2, we see that the answer is no even for deterministic wave equations.

4.7 Solution to inverse source problem with the boundary measurement and the final time measurement. II

This section focuses on the following problem:

- **Problem 4.5:** For a given R , determine the source function h and g by means of the measurement $\left(\frac{\partial z}{\partial \nu}\Big|_{\Sigma}, \frac{\partial z_{x_1}}{\partial \nu}\Big|_{\Sigma}, z(T), z_t(T)\right)$.

This problem involves simultaneously determining the two unknown source terms for (4.1.10). Boundary measurements in this section are understood as weak normal traces defined by Green's formula. The zero Cauchy and terminal data give the following spatial regularity.

Lemma 4.7.1. *Under the coefficient and source assumptions in (4.1.10), let $z \in \mathbb{H}_T$ satisfy that equation. If $\partial_\nu z = 0$ on Σ and $z(T) = z_t(T) = 0$ in G , then $z_{x_1} \in \mathbb{H}_T$.*

Proof. Let $A = -\Delta_D$ be the positive Dirichlet Laplacian, and set

$$E_+ = H_0^1(G) \times L^2(G), \quad E_- = L^2(G) \times H^{-1}(G).$$

The Dirichlet wave group $S(t)$ acts continuously on both spaces. Initially, $q = z_{x_1}$ and $q_t = \partial_{x_1} z_t$ give $X = (q, q_t) \in L^2(\Omega; C([0, T]; E_-))$, with $X(T) = 0$.

We first verify the Dirichlet equation for q in the transposition sense. Take a finite linear combination v of tensor products of Dirichlet eigenfunctions on $(0, l)$ and G' . Such functions form a core for A , belong to $H^2(G) \cap H_0^1(G)$, and have $\partial_{x_1} \Delta v \in L^2(G)$. Apply the Green formula for z with the spatial test function $-v_{x_1}$. The zero normal trace permits this test, although v_{x_1} need not vanish on the boundary. Since $z|_\Sigma = 0$,

$$-\int_G \nabla z \cdot \nabla v_{x_1} dx = \int_G z \partial_{x_1} \Delta v dx = -\int_G q \Delta v dx.$$

The zero trace of v also removes the boundary terms when differentiating the lower-order terms and the noise. Since h is independent of x_1 , this yields the weak Dirichlet evolution equation on E_- ,

$$dX = [\mathcal{A}_D X + \mathcal{B}(t)X + (0, F_0)]dt + (0, G_0)dW(t), \quad (4.7.1)$$

where $\mathcal{A}_D(q, p) = (p, -Aq)$ and $\mathcal{B}(t)(q, p) = (0, b_1 p + b_2 \cdot \nabla q + b_3 q)$. The forcing terms are

$$F_0 = (b_1)_{x_1} z_t + (b_2)_{x_1} \cdot \nabla z + (b_3)_{x_1} z + h R_{x_1}$$

and $G_0 = (b_4 z + g)_{x_1}$, both in $L_{\mathbb{F}}^2(0, T; L^2(G))$. Multiplication by a $W^{1, \infty}$ function is bounded on $H^{-1}(G)$, by duality with $H_0^1(G)$, and $b_2 \cdot \nabla : L^2(G) \rightarrow H^{-1}(G)$ is bounded. Thus $\mathcal{B}(t)$ is uniformly bounded on both E_- and E_+ . Applying variation of constants to finite spectral projections of this weak equation and passing to the limit gives its mild formulation in E_- .

Define the forward stochastic convolution

$$Y(t) = \int_0^t S(t-s)(0, G_0(s)) dW(s).$$

Writing $Y(t) = S(t) \int_0^t S(-s)(0, G_0(s)) dW(s)$ and applying the martingale maximal inequality gives

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y(t)|_{E_+}^2 \leq C \mathbb{E} \int_0^T |G_0(s)|_{L^2(G)}^2 ds.$$

The process $V = X - Y$ satisfies, pathwise,

$$V' = \mathcal{A}_D V + \mathcal{B}(t)V + \mathcal{B}(t)Y + (0, F_0), \quad V(T) = -Y(T).$$

The variation-of-constants formula, solved backwards from T , and Gronwall's inequality give a unique solution in $C([0, T]; E_+)$ with

$$\mathbb{E} \sup_{0 \leq t \leq T} |V(t)|_{E_+}^2 \leq C \left(\mathbb{E} \sup_{0 \leq t \leq T} |Y(t)|_{E_+}^2 + \mathbb{E} \int_0^T |F_0(t)|_{L^2(G)}^2 dt \right).$$

Uniqueness of the same deterministic integral equation in E_- identifies this solution with $X - Y$. Hence $X \in L^2(\Omega; C([0, T]; E_+))$. Its adaptedness is inherited from $q = z_{x_1}$ and $q_t = \partial_{x_1} z_t$, so $q \in \mathbb{H}_T$. $\square$

Theorem 4.7.2. *Let $h \in L_{\mathbb{F}}^2(0, T; H^1(G'))$, $g \in L_{\mathbb{F}}^2(0, T; H^1(G))$ and $R \in C^3(\overline{Q})$ satisfy*

$$|R(t, x)| \neq 0 \quad \text{for all } (t, x) \in [0, T] \times \overline{G}. \quad (4.7.2)$$

If $z \in \mathbb{H}_T$ satisfies (4.1.10) and

$$\begin{aligned} \frac{\partial z}{\partial \nu} &= \frac{\partial z_{x_1}}{\partial \nu} = 0 \quad \text{on } (0, T) \times \partial G, \quad \mathbb{P}\text{-a.s.}, \\ z(T) &= z_t(T) = 0 \quad \text{in } G, \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

then

$$\begin{aligned} h(t, x') &= 0 \quad \text{for a.e. } (t, x') \in (0, T) \times G', \quad \mathbb{P}\text{-a.s.} \\ g(t, x) &= 0 \quad \text{for a.e. } (t, x) \in (0, T) \times G, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

To prove [Theorem 4.7.2](#), we need a new global Carleman estimate for the special case of [\(4.1.5\)](#), where $(b^{jk})_{1 \leq j, k \leq n}$ is the identity matrix and $G = (0, l) \times G'$.

For any $x'_0 \in \mathbb{R}^{n-1} \setminus \overline{G'}$, let

$$\theta = e^\ell, \quad \ell = \lambda \varphi, \quad \varphi(t, x) = |x' - x'_0|^2 - \beta |x_1|^2 + |t + m|^2, \quad (4.7.3)$$

where constants β and m satisfy

$$\beta \in \left(0, \frac{4}{5}\right) \text{ and } m > \max_{x' \in \overline{G'}} |x' - x'_0| + \beta l. \quad (4.7.4)$$

We have the following global Carleman estimate:

Theorem 4.7.3. *Let $(b^{jk})_{1 \leq j, k \leq n}$ be the identity matrix and φ, θ be given in (4.7.3). Then there exist constants $C > 0$ and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, for any solution $z \in \mathbb{H}_T$ to the equation (4.1.5), it holds that*

$$\begin{aligned} & \lambda \mathbb{E} \int_Q \theta^2 (g^2 + z_t^2 + |\nabla z|^2 + \lambda^2 z^2) dx dt \\ & + \lambda \mathbb{E} \int_G \theta^2 (0) (z_t^2(0) + |\nabla z(0)|^2 + \lambda^2 z^2(0)) dx \\ & \leq C \mathbb{E} \left(\lambda \int_\Sigma \theta^2 \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt + \int_Q \theta^2 f^2 dx dt \right) \\ & + C \lambda \mathbb{E} \int_G \theta^2 (T) (z_t^2(T) + |\nabla z(T)|^2 + \lambda^2 z^2(T)) dx. \end{aligned} \quad (4.7.5)$$

Proof. The proof is divided into two steps.

Step 1. In this step, we compute the order of the terms in the equality (4.2.1) with respect to λ .

Choosing

$$\phi \equiv 1, \quad u = z, \quad b^{jk} = \delta_{jk}, \quad \Psi = \ell_{tt} + \Delta \ell - c_0 \lambda, \quad (4.7.6)$$

in Theorem 4.2.1, we will estimate the terms in (4.2.1) one by one. For $j = 1, \dots, n$, it is straightforward to verify that

$$\ell_t = 2\lambda(t + m), \quad \ell_{tt} = 2\lambda, \quad \ell_{x_1} = -2\beta\lambda x_1, \quad \ell_{tx_j} = 0, \quad (4.7.7)$$

and for $k = 2, \dots, n$,

$$\ell_{x_k} = 2\lambda(x_k - x'_{0,k}). \quad (4.7.8)$$

Combining (4.7.6)–(4.7.8), we obtain

$$(\ell_{tt} + \Delta \ell - \Psi) w_t^2 = c_0 \lambda w_t^2, \quad (4.7.9)$$

and

$$2 \sum_{j=1}^n \ell_{tx_j} w_{x_j} w_t = 0. \quad (4.7.10)$$

Choose $0 < c_0 < \min\{4(1 - \beta), 1\}$ and $0 < c_1 < \min\{8 - c_0, 4(1 - \beta) - c_0\}$. By (4.2.2) and (4.7.6)–(4.7.8), it holds that

$$\begin{aligned} \sum_{j,k=1}^n c^{jk} w_{x_j} w_{x_k} &= \sum_{j=1}^n (\ell_{tt} + 2\ell_{x_j x_j} - \Delta \ell + \Psi) w_{x_j}^2 \\ &= \sum_{j=2}^n (8 - c_0) \lambda w_{x_j}^2 + [4(1 - \beta) - c_0] \lambda w_{x_1}^2 \\ &\geq c_1 \lambda |\nabla w|^2. \end{aligned} \quad (4.7.11)$$

Now we compute the coefficient of w^2 . Thanks to (4.2.2) and (4.7.6)–(4.7.8), we have

$$\begin{aligned} \mathcal{A} &= \ell_t^2 - \ell_{tt} - |\nabla \ell|^2 + \Delta \ell - \Psi \\ &= 4\lambda^2[(t + m)^2 - |x' - x'_0|^2 - \beta^2 x_1^2] + (c_0 - 4)\lambda. \end{aligned} \quad (4.7.12)$$

From (4.7.7)–(4.7.9) and (4.7.12), we obtain

$$(\mathcal{A} \ell_t)_t = 8\lambda^3[3(t + m)^2 - |x' - x'_0|^2 - \beta^2 x_1^2] + O(\lambda^2), \quad (4.7.13)$$

and

$$\begin{aligned}
-\sum_{j=1}^n (\mathcal{A}\ell_{x_j})_{x_j} &= -8(n-1-\beta)\lambda^3[(t+m)^2 - |x' - x'_0|^2 - \beta^2 x_1^2] \\
&\quad + 16\lambda^3|x' - x'_0|^2 - 16\beta^3\lambda^3 x_1^2 + O(\lambda^2).
\end{aligned} \tag{4.7.14}$$

Combining (4.2.2), (4.7.4), (4.7.6) and (4.7.12)–(4.7.14), for $c_2 \leq m^2 - \beta^2 l^2$, we conclude that

$$\begin{aligned}
\mathcal{B} &= 4\lambda^3[(8-c_0)(t+m)^2 + c_0|x' - x'_0|^2 + \beta^2(-4-4\beta+c_0)x_1^2] + O(\lambda^2) \\
&\geq 28\lambda^3(m^2 - \beta^2 l^2) + O(\lambda^2) \\
&\geq c_2\lambda^3 + O(\lambda^2).
\end{aligned}$$

Hence, there exists $\lambda_1 > 0$ so that for any $\lambda > \lambda_1$,

$$\mathcal{B}w^2 \geq c_2\lambda^3 w^2. \tag{4.7.15}$$

Step 2. In this step, we apply Theorem 4.2.1 to the equation (4.1.5). Put

$$M(t, x) = \ell_t |\nabla w|^2 - 2\nabla \ell \cdot \nabla w w_t + \ell_t w_t^2 - \Psi w_t w + \mathcal{A}\ell_t w^2.$$

Noting that $z = 0$ on Σ , integrating (4.2.1) in Q , taking expectation and using (4.7.9)–(4.7.11) and (4.7.15), for $\lambda \geq \lambda_1$ we obtain

$$\begin{aligned}
&\mathbb{E} \int_Q \theta [(-2\ell_t w_t + 2\nabla \ell \cdot \nabla w + \Psi w)(dz_t - \Delta z dt)] dx + C\lambda \mathbb{E} \int_\Sigma \theta^2 \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \\
&\quad + \mathbb{E} \int_G [M(T, x) - M(0, x)] dx \\
&\geq C\lambda \mathbb{E} \int_Q \theta^2 (z_t^2 + |\nabla z|^2 + \lambda^2 z^2) dx dt + \mathbb{E} \int_Q \theta^2 \ell_t |dz_t|^2 dx \\
&\quad + \mathbb{E} \int_Q (-2\ell_t w_t + 2\nabla \ell \cdot \nabla w + \Psi w)^2 dx dt.
\end{aligned} \tag{4.7.16}$$

At $t = 0$, from (4.7.3), (4.7.4), (4.7.6) and (4.7.12), there exists $\lambda_2 > 0$ such that for all $\lambda \geq \lambda_2$, we have

$$\begin{aligned}
&\ell_t |\nabla w|^2 - 2\nabla \ell \cdot \nabla w w_t + \ell_t w_t^2 - \Psi w_t w + \mathcal{A}\ell_t w^2 \\
&\geq 2\lambda m(|\nabla w|^2 + w_t^2) - 4\lambda(|x' - x'_0| + \beta l)|\nabla w||w_t| - C\lambda|w_t||w| \\
&\quad + 8\lambda^3 m(m^2 - |x' - x'_0|^2 - \beta^2 l^2)w^2 + O(\lambda^2)w^2 \\
&\geq C\lambda(w_t^2 + |\nabla w|^2 + \lambda^2 w^2) \\
&\geq C\lambda\theta^2(z_t^2 + |\nabla z|^2 + \lambda^2 z^2).
\end{aligned} \tag{4.7.17}$$

At $t = T$, the same formulas give

$$|M(T, x)| \leq C\lambda\theta^2(T, x)(z_t^2(T, x) + |\nabla z(T, x)|^2 + \lambda^2 z^2(T, x)). \quad (4.7.18)$$

It follows from (4.7.3) that

$$\mathbb{E} \int_Q \theta^2 \ell_t |dz_t|^2 dx \geq c\lambda \mathbb{E} \int_Q \theta^2 g^2 dx dt - C\lambda \|b_4\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(G))}^2 \mathbb{E} \int_Q \theta^2 z^2 dx dt. \quad (4.7.19)$$

By (4.1.5), we get that

$$\begin{aligned} & \mathbb{E} \int_Q \theta [(-2\ell_t w_t + 2\nabla \ell \nabla w + \Psi w)(dz_t - \Delta z dt)] dx \\ & \leq C \left[\mathbb{E} \int_Q \theta^2 (|f|^2 + |g|^2) dx dt + |b_1|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(G))}^2 \mathbb{E} \int_Q \theta^2 z_t^2 dx dt \right. \\ & \quad + |b_2|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(G, \mathbb{R}^n))}^2 \mathbb{E} \int_Q \theta^2 |\nabla z|^2 dx dt + \mathbb{E} \int_Q \theta^2 b_3^2 z^2 dx dt \\ & \quad + |b_4|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(G))}^2 \mathbb{E} \int_Q \theta^2 z^2 dx dt \Big] \\ & \quad + \mathbb{E} \int_Q (-2\ell_t w_t + 2\nabla \ell \cdot \nabla w + \Psi w)^2 dx dt. \end{aligned} \quad (4.7.20)$$

Combining (4.7.16)–(4.7.20) and absorbing the lower-order terms as in (4.3.29), we obtain (4.7.5) for all $\lambda \geq \lambda_0$, with $\lambda_0 \geq \max\{\lambda_1, \lambda_2\}$ sufficiently large. $\square$

Proof of Theorem 4.7.2. By Theorem 4.7.1, $z_{x_1} \in \mathbb{H}_T$. Set $z = Ru$. Then $u, u_{x_1} \in \mathbb{H}_T$, and their displacements and velocities vanish at T . Moreover, since $z = z_{x_1} = 0$ on Σ , the product rule for weak normal traces gives

$$\partial_\nu u = R^{-1} \partial_\nu z = 0, \quad \partial_\nu u_{x_1} = R^{-1} \partial_\nu z_{x_1} - R^{-2} R_{x_1} \partial_\nu z = 0.$$

From (4.1.10), we have

$$\begin{cases} du_t - \Delta u dt = [(b_1 - 2R^{-1}R_t)u_t + (2R^{-1}\nabla R + b_2) \cdot \nabla u \\ \quad + R^{-1}(-R_{tt} + \Delta R + b_1 R_t + b_2 \cdot \nabla R + b_3 R)u] dt \\ \quad + h(t, x') dt + (R^{-1}g + b_4 u) dW(t) & \text{in } Q, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma. \end{cases} \quad (4.7.21)$$

Differentiate both side of (4.7.21) with respect to x_1 , and set $w = u_{x_1}$. Noting

that $\nabla u = \frac{\partial u}{\partial \nu} \nu = 0$ on Σ , we obtain that

$$\left\{ \begin{array}{l} dw_t - \Delta w dt = [(b_1 - 2R^{-1}R_t)w_t + (2R^{-1}\nabla R + b_2) \cdot \nabla w \\ \quad + R^{-1}(-R_{tt} + \Delta R + b_1R_t + b_2 \cdot \nabla R + b_3R)w]dt \\ \quad + \{(b_1 - 2R^{-1}R_t)_{x_1}u_t + (2R^{-1}\nabla R + b_2)_{x_1} \cdot \nabla u \\ \quad + [R^{-1}(-R_{tt} + \Delta R + b_1R_t + b_2 \cdot \nabla R + b_3R)]_{x_1}u\}dt \\ \quad + ((R^{-1}g)_{x_1} + b_4w + b_{4,x_1}u)dW(t) \end{array} \right. \begin{array}{l} \text{in } Q, \\ \text{on } \Sigma. \end{array} \quad (4.7.22)$$

Apply [Theorem 4.7.3](#) to $w = u_{x_1}$. Its terminal term vanishes because $w(T) = w_t(T) = 0$. Thus, for some $\lambda_1 > \lambda_0$ and all $\lambda \geq \lambda_1$, it holds that

$$\begin{aligned} & \lambda \mathbb{E} \int_Q \theta^2 (|(R^{-1}g)_{x_1} + b_{4,x_1}u|^2 + w_t^2 + |\nabla w|^2 + \lambda^2 w^2) dx dt \\ & \leq C \mathbb{E} \left\{ \lambda \int_{\Sigma} \theta^2 \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma dt + \int_Q \theta^2 \tilde{f}^2 dx dt \right\}, \end{aligned} \quad (4.7.23)$$

where

$$\begin{aligned} \tilde{f} &= (b_1 - 2R^{-1}R_t)_{x_1}u_t + (2R^{-1}\nabla R + b_2)_{x_1} \cdot \nabla u \\ & \quad + [R^{-1}(-R_{tt} + \Delta R + b_1R_t + b_2 \cdot \nabla R + b_3R)]_{x_1}u. \end{aligned} \quad (4.7.24)$$

Since $w = u_{x_1}$, and $u(t, 0, x') = z(t, 0, x') = 0$ for $(t, x') \in (0, T) \times G'$, we have

$$u(t, x_1, x') = \int_0^{x_1} w(t, \eta, x') d\eta. \quad (4.7.25)$$

For $0 \leq \eta \leq x_1 \leq l$, the weight in [\(4.7.3\)](#) satisfies $\theta(t, x_1, x') \leq \theta(t, \eta, x')$. Thus the Cauchy-Schwarz argument in [\(3.11.5\)](#) and [\(3.11.6\)](#) applies also to this weight. Using [\(4.7.24\)](#) and [\(4.7.25\)](#), we obtain

$$\mathbb{E} \int_Q \theta^2 \tilde{f}^2 dx dt \leq C \mathbb{E} \int_Q \theta^2 (w_t^2 + |\nabla w|^2 + w^2) dx dt, \quad (4.7.26)$$

and

$$\begin{aligned} & \lambda \mathbb{E} \int_Q \theta^2 |(R^{-1}g)_{x_1} + b_{4,x_1}u|^2 dx dt \\ & \geq \frac{1}{2} \lambda \mathbb{E} \int_Q \theta^2 |(R^{-1}g)_{x_1}|^2 dx dt - C \lambda \mathbb{E} \int_Q \theta^2 w^2 dx dt. \end{aligned} \quad (4.7.27)$$

From [\(4.7.23\)](#), [\(4.7.26\)](#) and [\(4.7.27\)](#), we know there exists $\lambda_2 \geq \lambda_1$ such that for all $\lambda \geq \lambda_2$, it holds that

$$\lambda \mathbb{E} \int_Q \theta^2 (|(R^{-1}g)_{x_1}|^2 + w_t^2 + |\nabla w|^2 + \lambda^2 w^2) dx dt \leq C \mathbb{E} \lambda \int_{\Sigma} \theta^2 \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma dt.$$

Since $\frac{\partial w}{\partial \nu} = \frac{\partial u_{x_1}}{\partial \nu} = 0$ on Σ , for $\lambda \geq \lambda_2$, we have

$$\lambda \mathbb{E} \int_Q \theta^2 (|(R^{-1}g)_{x_1}|^2 + w_t^2 + |\nabla w|^2 + \lambda^2 w^2) dx dt \leq 0.$$

This yields that

$$w = 0 \text{ in } (0, T) \times G, \mathbb{P}\text{-a.s.},$$

which, together with (4.7.25), implies that

$$u = 0 \text{ in } (0, T) \times G, \mathbb{P}\text{-a.s.}$$

Consequently,

$$z = 0 \text{ in } (0, T) \times G, \mathbb{P}\text{-a.s.} \quad (4.7.28)$$

From (4.1.10) and (4.7.28), for any $t \in (0, T)$, we have

$$\int_0^t h(s, x') R(s, x) ds + \int_0^t g(s, x) dW(s) = 0. \quad (4.7.29)$$

Noting that $\int_0^t h(s, x') R(s, x) ds$ is absolutely continuous with respect to t , we get from (4.7.29) that $\int_0^t g(s, x) dW(s)$ is also absolutely continuous with respect to t . Consequently, its quadratic variation is zero, which concludes that

$$g = 0 \text{ in } (0, T) \times G, \mathbb{P}\text{-a.s.}$$

This, together with (4.7.29), implies that

$$\int_0^t h(s, x') R(s, x) ds = 0, \quad \forall t \in (0, T).$$

Therefore,

$$h(s, x') = 0 \text{ in } (0, T) \times G, \mathbb{P}\text{-a.s.}$$

□

4.8 Further comments

To the best of our knowledge, [79] is the pioneering work that utilized Carleman estimates in investigating inverse problems for stochastic hyperbolic equations. Subsequently, multiple studies have delved into the realm of inverse problems for stochastic hyperbolic equations using Carleman estimates, such as [54, 58, 59, 74, 77]. Several of these works are introduced in this chapter. Theorem 4.2.1 is first established in [58], while Theorem 4.2.3 make its debut in [79]. The energy

estimate in [Theorem 4.2.5](#) is established in [\[54\]](#). [Theorem 4.3.1](#) is proved in [\[54\]](#), and the primary content of [Section 4.6](#) is taken from [\[59\]](#). [Theorem 4.7.2](#) is taken from [\[77\]](#), and the core material of [Section 4.4](#) is referenced from [\[21\]](#).

In this chapter, we focus on the inverse state problems and inverse source problems for stochastic hyperbolic equations. However, there are still many open problems related to inverse problems for that equation.

- *Inverse problems for stochastic hyperbolic equations with less restrictions*

To prove inequality [\(4.3.2\)](#), we assume [Conditions 4.3.1](#) and [4.3.2](#) hold. However, for deterministic hyperbolic equations, the time T does not need to be that large. It is an interest question to weaken the condition on time T for [Theorem 4.3.2](#) holds.

- *Inverse problems for stochastic hyperbolic equations with general coefficients*

Replacing $\Delta \tilde{z}$ in [\(4.1.7\)](#) and [\(4.1.10\)](#) with $\sum_{j,k=1}^n (b^{jk} \tilde{z}_{x_j})_{x_k}$, it is also quite interesting to consider [Problems 4.5](#) and [4.3](#). In this case, the main challenge lies in constructing an appropriate weight function for Carleman estimates to ensure the validity of [Theorems 4.5.1](#) and [4.7.3](#).

- *Inverse state and source problems for semilinear stochastic hyperbolic equations*

For a general semilinear stochastic hyperbolic equation [\(4.1.2\)](#), can we investigate its inverse source problem with both boundary and final time measurements, for instance, [Problem 4.4](#)?

- *Reconstruct problem for the unknown state and source term*

For the equation [\(4.1.5\)](#), we obtain an estimate of the initial values (z_0, z_1) and the diffusion term g in [Theorem 4.6.1](#). Analogous to [Section 4.4](#), can we reconstruct the unknown state and source problem with a boundary measurement and a final time measurement?

- *Efficient algorithm for the construction of unknowns*

In [Section 4.4](#), we discuss reconstructing the solution from boundary measurement using the Tikhonov regularization approach. When applying gradient methods to solve these optimization problems, challenges arise in solving backward stochastic partial differential equations numerically, similar to stochastic parabolic equations (see [Section 3.12](#)). However, it is highly challenging to solve backward stochastic partial differential equations numerically. Therefore, novel numerical algorithms must be developed for stochastic problems.

An interesting direction for future research might be the adaptation of methods from the deterministic setting. For instance, in [\[3, 4\]](#), Carleman estimate methods are applied to reconstruct the potential or the wave speed in wave

equations. A key feature is that the Carleman approach is used only for the forward equation. It would be interesting to investigate whether these ideas could be extended to the stochastic case, potentially leading to a convergent algorithm that avoids the numerical difficulties associated with backward SPDEs.

- *The strategy for selecting the prior information F*

In the reconstruct problem (Section 4.4), our objective is to minimize the Tikhonov functional (4.4.3), which requires selecting a function F . How does the selection of different prior information F affect the efficacy of numerically solving the approximate solution? Can a suitable method be proposed to find a better F ?

- *The strategy for quantitatively selecting the regularization parameter*

In the Tikhonov functional (4.4.3), we introduce a regularization parameter γ . In general, the regularization parameter γ has a certain impact on the numerical error. In Theorem 4.4.3, we set a prior selection of the regularization parameter as δ^2 . Can a method be provided for selecting the regularization parameter based on the input data?

- *The optimal measurement method*

In practical problems, measurements are typically made through the deployment of costly sensors. Numerous instances in engineering applications emphasize the significant impact of sensor placement on the achievable accuracy of inverse problems. Therefore, determining where to locate the observation becomes an important problem. Some interesting findings already exist in PDEs [69]. However, as far as we know, there has been no published work on this problem for SPDEs.

- *What can we benefit from uncertainties?*

In Example 1.1.2, we mention that Problem 4.4 is essentially stochastic. It can be observed that the uncertainty in SPDEs benefits certain inverse problems. What benefits can we derive from uncertainties in solving inverse problems for SPDEs remains a largely unresolved puzzle. We believe that studying this problem can lead to new insights into uncertainty.

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