

A Unified Kalman-Type Classification of Controllability for Linear Stochastic Systems

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Abstract

We provide a unified algebraic classification of approximate controllability, approximate null controllability, exact null controllability, and exact controllability for finite-dimensional linear stochastic control systems with constant coefficients and multidimensional Brownian noise. The controls may enter both the drift and the diffusion terms, and the joint diffusion-control matrix may have arbitrary rank. A finite decreasing sequence of subspaces defines a common geometric obstruction. Its vanishing characterizes the two approximate properties and exact null controllability. Exact controllability to arbitrary random targets additionally requires the joint diffusion-control matrix to be surjective. The codimension of the obstruction is given by a rank difference of explicit polynomial block matrices, yielding finite tests in the original coefficients. When the joint diffusion-control matrix is surjective, we obtain a finite Kalman-type full-rank condition for exact controllability. If the obstruction vanishes, a linear state feedback with a deterministic time-dependent gain drives every initial state to zero at any prescribed positive time with finite energy and polynomial small-time cost. An equivalent orthogonal elimination algorithm computes the rank differences, avoiding exponentially large block matrices.

Key words: Stochastic control; Controllability; State feedback; Rank conditions

1 Introduction

Linear stochastic systems with state- and control-dependent diffusion give a tractable setting for feedback design under multiplicative uncertainty, e.g., with actuation through uncertain input channels [13]. In multi-input systems, channel organization and allocation also shape control design, as in scheduling/control co-design over shared stochastic channels [5]. The model links mean-square stabilization to generalized Riccati equations [19]. This motivates a structural finite-horizon question: which input configurations admit a finite-energy adapted control meeting a prescribed terminal requirement? Reaching the origin at a prescribed time differs from reaching an arbitrary random terminal state in how control affects drift and diffusion terms.

1.1 Formulation of the controllability problems

Let $W = (W^1, \dots, W^m)$ be an m -dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}_t\}_{t \geq 0}$ be the usual augmentation of its natural filtration. Write $\mathbb{F}$ for the progressive σ -algebra and $\mathbb{E}$ for expectation.

For a Euclidean space H , let $L^2_{\mathcal{F}_t}(\Omega; H)$ denote the square-integrable $\mathcal{F}_t$ -measurable random variables. For $T > 0$, the space $L^2_{\mathbb{F}}(0, T; H)$ consists of progressively measurable processes with $\mathbb{E} \int_0^T |v(t)|_H^2 dt < \infty$. We use $L^2_{\mathbb{F}}(\Omega; C([0, T]; H))$ for continuous adapted processes with $\mathbb{E} \sup_{0 \leq t \leq T} |v(t)|_H^2 < \infty$.

For $t \in [0, T]$, consider the linear control system

$$\begin{cases} dy(t) = (Ay(t) + Bu(t))dt + \sum_{j=1}^m (C_j y(t) + D_j u(t))dW^j(t), \\ y(0) = y_0, \end{cases} \quad (1)$$

where $A, C_j \in \mathbb{R}^{n \times n}$, $B, D_j \in \mathbb{R}^{n \times r}$, $n, r, m \in \{1, 2, \dots\}$, $j = 1, \dots, m$. The control $u \in \mathcal{U}_T := L^2_{\mathbb{F}}(0, T; \mathbb{R}^r)$.

By [12, Theorem 3.2], for every $y_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}_T$, the control system (1) has a unique solution $y(\cdot; y_0, u) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n))$ satisfying

$$\mathbb{E} \sup_{0 \leq t \leq T} |y(t; y_0, u)|^2 \leq c(T) \left(|y_0|^2 + \mathbb{E} \int_0^T |u(t)|^2 dt \right). \quad (2)$$

Let us recall the following definition.

Definition 1 For $T > 0$, system (1) is said to be:

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- (EC) exactly controllable if, for every $y_0 \in \mathbb{R}^n$ and $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, there is $u \in \mathcal{U}_T$ such that $y(T; y_0, u) = \xi$, $\mathbb{P}$ -a.s.;
- (AC) approximately controllable if, for every $y_0 \in \mathbb{R}^n$, $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, and $\varepsilon > 0$, there is $u \in \mathcal{U}_T$ such that $\mathbb{E}|y(T; y_0, u) - \xi|^2 < \varepsilon$;
- (ENC) exactly null controllable if, for every $y_0 \in \mathbb{R}^n$, there is $u \in \mathcal{U}_T$ such that $y(T; y_0, u) = 0$, $\mathbb{P}$ -a.s.;
- (ANC) approximately null controllable if, for every $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there is $u \in \mathcal{U}_T$ such that $\mathbb{E}|y(T; y_0, u)|^2 < \varepsilon$.

Write $\mathcal{C} = [C_1^\top \cdots C_m^\top]^\top \in \mathbb{R}^{mn \times n}$ and define the joint diffusion-control matrix

$$\mathcal{D} = [D_1^\top \cdots D_m^\top]^\top \in \mathbb{R}^{mn \times r}. \quad (3)$$

For deterministic linear constant-coefficient systems, the Kalman rank condition characterizes controllability, and exact, null, and approximate controllability coincide for deterministic controls and targets.

The stochastic counterpart is less complete: for linear constant-coefficient SDEs with controls in drift and diffusion, the deterministic equivalences and rank condition do not carry over. For Eq. (1), two questions concern the relations among controllability notions and their algebraic characterization.

First, for scalar Brownian noise, Goreac [8] characterized AC and ANC by a common conditional-invariance obstruction. AC does not imply exact controllability to arbitrary random targets, as in Example 5. We ask whether ANC also guarantees finite-energy ENC at every prescribed $T > 0$ for Eq. (1). The gap is not merely technical: the terminal control map takes values in an infinite-dimensional space of random variables. ANC places the negative uncontrolled terminal state in the closure of the range, whereas ENC requires it in the range. Since approximately null controls need not have uniformly bounded energy, ENC is not automatic even in finite dimensions. We also ask what extra condition ensures exact attainment of arbitrary random targets.

Second, we seek a Kalman-type rank characterization in the original coefficients. For scalar noise with a full-row-rank diffusion-control matrix, Peng [14, pp. 281–282 and Theorem 3.2] gave a necessary and sufficient rank condition involving products of coefficient matrices after an equivalent change of controls. Lü and Zhang [12, Theorem 6.10] showed that products of length at most $n - 1$ suffice and raised the problem of a direct characterization for the original system [12, Remark 6.11]. We seek a unified finite-rank characterization in the original coefficients for multidimensional noise and arbitrary rank $\mathcal{D}$, within the same adapted L^2 control class.

We resolve these questions by identifying a common geometric obstruction and expressing its codimension as the rank difference of explicit polynomial block matrices in $A, B, (C_j, D_j)_{j=1}^m$.

From a design perspective, an original-coefficient criterion can test candidate input configurations before

controller construction. Our results distinguish exact return to the origin at a prescribed time from attaining arbitrary square-integrable random terminal states. They also reveal an input-structure restriction: every controllability property considered here requires $B \ker \mathcal{D} \neq \{0\}$, so some input combination must affect the drift without adding diffusion. Exact controllability further requires $\text{rank } \mathcal{D} = mn$, hence $r \geq mn + 1$; see Remark 16. The criteria in Corollary 7 can thus test feasibility before solving a minimum-energy steering problem.

1.2 Main results of this paper

We first state the geometric classification and feedback result, then express the classification as finite rank conditions in the original coefficient matrices.

In this paper, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean inner product and norm, while $\|\cdot\|$ and $\|\cdot\|_F$ denote matrix operator and Frobenius norms. Let $e_1, \dots, e_n$ be the coordinate vectors, $^\top$ the transpose, and $\otimes$ the Kronecker product. For a subspace V , $V^\perp$ is its Euclidean orthogonal complement and Π_V its orthogonal projector. Intermediate matrices with zero rows or columns are allowed under the usual rank, range, and kernel conventions; the system dimensions n, r, m remain positive. Throughout, c is a positive constant depending only on $A, B, (C_j, D_j)_{j=1}^m$, and $c(T)$ may also depend on T ; such generic constants may change between formulas. Constants used later are named separately.

To describe the obstruction to controllability, for $\eta \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, consider the adjoint BSDE

$$\begin{cases} dz(t) = -\left(A^\top z(t) + \sum_{j=1}^m C_j^\top Z_j(t)\right)dt + \sum_{j=1}^m Z_j(t)dW^j(t), \\ z(T) = \eta. \end{cases} \quad (4)$$

Here $Z = (Z_1, \dots, Z_m)$. By [12, Theorem 4.2], Eq. (4) has a unique solution with $(z, Z) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0, T; (\mathbb{R}^n)^m)$. Applying Itô's formula to $\langle y, z \rangle$ and taking expectations gives the duality identity

$$\mathbb{E}\langle y(T), \eta \rangle - \langle y_0, z(0) \rangle = \mathbb{E} \int_0^T \left\langle u, B^\top z + \sum_{j=1}^m D_j^\top Z_j \right\rangle dt. \quad (5)$$

Thus $B^\top z + \sum_{j=1}^m D_j^\top Z_j$ is the observation associated with

the control. An adjoint solution with zero observation is undetected by every admissible control in Eq. (5).

For such an adjoint motion to remain in a subspace $V \subset \mathbb{R}^n$, each martingale coefficient w_j and the drift must belong to V . We therefore define $\mathcal{R}(V)$ as the set of $v \in V$ for which some $w_1, \dots, w_m \in V$ satisfy

$$B^\top v + \sum_{j=1}^m D_j^\top w_j = 0, \quad A^\top v + \sum_{j=1}^m C_j^\top w_j \in V. \quad (6)$$

Starting from $\mathcal{V}_0 = \mathbb{R}^n$, set

$$\mathcal{V}_{k+1} = \mathcal{R}(\mathcal{V}_k) \quad (k \geq 0). \quad (7)$$

We denote $\mathcal{V}_n$ by $\mathcal{V}_*$.

The next lemma gives an intrinsic characterization of the limiting space.

Lemma 2 *The sequence $(\mathcal{V}_k)_{k \geq 0}$ decreases and is stationary from step n onward. Its limit $\mathcal{V}_*$ is the largest subspace V for which some $\Gamma_1, \dots, \Gamma_m \in \mathbb{R}^{n \times n}$ satisfy*

$$\Gamma_j V \subset V \quad (j = 1, \dots, m),$$

$$\left(A^\top + \sum_{j=1}^m C_j^\top \Gamma_j \right) V \subset V, \quad \left(B^\top + \sum_{j=1}^m D_j^\top \Gamma_j \right) V = \{0\}. \quad (8)$$

PROOF. For every subspace V , the linear conditions in Eq. (6) make $\mathcal{R}(V)$ a subspace of V . Thus $\mathcal{V}_{k+1} \subset \mathcal{V}_k$ for every $k \geq 0$. If equality holds at some step, then the sequence is stationary from that step onward. Otherwise, the dimension strictly decreases at each step; since the ambient space has dimension n , stationarity must occur no later than step n .

Next we identify the limit $\mathcal{V}_*$. At this fixed point, for each basis vector $v \in \mathcal{V}_*$, choose $w_1, \dots, w_m$ satisfying Eq. (6). Since the defining condition is linear, extending these assignments linearly on $\mathcal{V}_*$ and then to $\mathbb{R}^n$ defines matrices $\Gamma_1, \dots, \Gamma_m$ satisfying Eq. (8) with $V = \mathcal{V}_*$.

Conversely, suppose that a subspace V satisfies Eq. (8) for some $\Gamma_1, \dots, \Gamma_m$. We show by induction on k that $V \subset \mathcal{V}_k$. For $k = 0$, this holds since $\mathcal{V}_0$ is the ambient space. Assume $V \subset \mathcal{V}_k$. For $v \in V$, set $w_j = \Gamma_j v$. Then the invariance conditions ensure that $v \in \mathcal{V}_{k+1}$, because the defining condition for $\mathcal{V}_{k+1}$ is satisfied with these w_j . Hence $V \subset \mathcal{V}_{k+1}$. Therefore $V \subset \mathcal{V}_k$ for all k , and thus $V \subset \mathcal{V}_*$. This proves that $\mathcal{V}_*$ is the largest subspace with the stated property.

The following theorem classifies the four controllability properties through the obstruction space and the joint diffusion rank.

Theorem 3 *For system (1), the following statements hold.*

- (i) *For every $T > 0$, approximate controllability, approximate null controllability, and exact null controllability are equivalent, and each holds if and only if $\mathcal{V}_* = \{0\}$.*
- (ii) *For every $T > 0$, exact controllability to arbitrary targets in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ holds if and only if*

$$\mathcal{V}_* = \{0\} \quad \text{and} \quad \text{rank } \mathcal{D} = mn. \quad (9)$$

In particular, each of the four controllability properties is independent of the positive time horizon.

The next theorem achieves exact null controllability via deterministic linear feedback at each prescribed terminal time, with a polynomial small-time energy bound.

Theorem 4 *Assume that $\mathcal{V}_* = \{0\}$. Then for every $T > 0$ there exists a deterministic, piecewise-constant matrix function $K_T : [0, T] \rightarrow \mathbb{R}^{r \times n}$, locally bounded on $[0, T]$, such that, for every $y_0 \in \mathbb{R}^n$, the feedback control $u(\cdot) = K_T(\cdot)y(\cdot) \in \mathcal{U}_T$ and its state satisfies $y(T; y_0, u) = 0$, $\mathbb{P}$ -a.s. Moreover, there exist $c_0 > 0$ and an integer $N \geq 1$, depending only on $A, B, (C_j, D_j)_{j=1}^m$, such that*

$$\mathbb{E} \int_0^T |u(t)|^2 dt \leq c_0 T^{-N} |y_0|^2, \quad 0 < T \leq 1. \quad (10)$$

Table 1

The only possible joint regimes under the standing assumptions.

$\mathcal{V}_*$	$\text{rank } \mathcal{D}$	Properties
$\neq \{0\}$	arbitrary	none
$\{0\}$	$< mn$	AC, ANC, ENC
$\{0\}$	mn	AC, ANC, ENC, EC

The gain K_T may become unbounded as $t \uparrow T$, with switching times accumulating at T . The energy bound concerns the control $u = K_T y$.

The following example illustrates the role of the joint diffusion rank in Table 1.

Example 5 *Take $n = 1, r = 3, m = 2, A = C_1 = C_2 = 0$, and $B = (1, 0, 0), D_1 = (0, 1, 0), D_2 = (0, 1, \varepsilon)$, where $\varepsilon \in \mathbb{R}$. Then Eq. (1) becomes*

$$dy = u_1 dt + u_2 dW^1 + (u_2 + \varepsilon u_3) dW^2.$$

The zero-observation constraint is

$$B^\top v + D_1^\top w_1 + D_2^\top w_2 = \begin{pmatrix} v \\ w_1 + w_2 \\ \varepsilon w_2 \end{pmatrix} = 0.$$

Thus $\mathcal{V}_1 = \{0\}$ for every ε . By Theorem 3, AC, ANC and ENC hold at every $T > 0$, whereas EC holds if and only if $\varepsilon \neq 0$: the rank of $\mathcal{D}$ is one at $\varepsilon = 0$ and two otherwise. Both D_1 and D_2 , as well as their horizontal concatenation, have full row rank even at $\varepsilon = 0$.

Recall the definition of $\mathcal{D}$ in Eq. (3). Set

$$K_1 = B^\top, \quad N_1 = \mathcal{D}^\top. \quad (11)$$

For $k \geq 1$, write $K_k \in \mathbb{R}^{a_k \times n}$ and $N_k \in \mathbb{R}^{a_k \times b_k}$, and form

$$K_{k+1} = \begin{pmatrix} B^\top \\ 0_{ma_k \times n} \\ K_k A^\top \end{pmatrix}, \quad N_{k+1} = \begin{pmatrix} \mathcal{D}^\top & 0 & 0 \\ I_m \otimes K_k & I_m \otimes N_k & 0 \\ K_k C^\top & 0 & N_k \end{pmatrix}, \quad (12)$$

where $a_k = r \sum_{\ell=0}^{k-1} (m+1)^\ell$ and $b_k = mn \sum_{\ell=0}^{k-1} (m+1)^\ell$. The three column blocks of N_{k+1} have widths mn, mb_k and b_k , and its three row blocks have heights r, ma_k and a_k . Put $\delta_0 = 0$ and, for $k \geq 1$,

$$\delta_k = \text{rank}[N_k \ K_k] - \text{rank } N_k. \quad (13)$$

The block pair represents each obstruction space and determines its codimension.

Theorem 6 *For every $k \geq 1$,*

$$\mathcal{V}_k = \{v \in \mathbb{R}^n : \exists \alpha \in \mathbb{R}^{b_k}, \quad K_k v + N_k \alpha = 0\}, \quad (14)$$

and

$$\delta_k = n - \dim \mathcal{V}_k. \quad (15)$$

At step n , this identity yields the controllability criterion.

Corollary 7 *For every $T > 0$, approximate controllability, approximate null controllability, and exact null controllability are each equivalent to*

$$\text{rank}[N_n \ K_n] - \text{rank } N_n = n. \quad (16)$$

Exact controllability is equivalent to Eq. (16) together with $\text{rank } \mathcal{D} = mn$.

PROOF. Apply Theorems 3 and 6 with $\mathcal{V}_* = \mathcal{V}_n$.

When $\text{rank } \mathcal{D} = mn$, let $J = \mathcal{D}^\top (\mathcal{D}\mathcal{D}^\top)^{-1}$ and partition $J = [J_1 \cdots J_m]$, where $J_j \in \mathbb{R}^{r \times n}$. Define $E_j = BJ_j$, $F = A - BJC$, and $H = B(I_r - JD)$. Set $H_0 = H$ and, for $k \geq 0$, define

$$H_{k+1} = [E_1 H_k \cdots E_m H_k F H_k]. \quad (17)$$

The matrix H_k has n rows and $r(m+1)^k$ columns, corresponding to all products of exactly k factors from $\{E_1, \dots, E_m, F\}$ followed by H . Thus the concatenation of $H_0, \dots, H_{n-1}$ has $r \sum_{k=0}^{n-1} (m+1)^k$ columns.

These matrices give a Kalman-type rank condition for exact controllability.

Theorem 8 Assume that $\text{rank } \mathcal{D} = mn$. At every $T > 0$, system (1) is exactly controllable if and only if

$$\text{rank}[H_0 \ H_1 \ \cdots \ H_{n-1}] = n. \quad (18)$$

1.3 Relation to previous work

The spaces in Eqs. (6) and (7) extend Goreac's finite subspace iteration for AC and ANC [8, Theorem 3 and Remark 4] from scalar to multidimensional Brownian noise and arbitrary rank of $\mathcal{D}$. Theorem 3 further shows that the vanishing of the same obstruction characterizes ENC for every prescribed $T > 0$.

Guo, Yu, and Zhang [9] characterize stabilizability through approximate null controllability with cost, with horizon $T = T(\delta)$ for each contraction factor δ . In contrast, Theorem 4 achieves exact null control at every prescribed $T > 0$ by deterministic linear state feedback, with finite energy and a polynomial small-time bound.

For scalar noise and a surjective diffusion-control matrix, Liu and Peng [11] prove that, for the equivalent system with freely chosen diffusion, exact controllability is equivalent to arbitrary exponential convergence-index assignment by linear state feedback. Our classification also covers arbitrary joint diffusion rank, where ENC need not imply EC.

For scalar noise and $\text{rank } D_1 = n$, the matrices E_1, F, H before Theorem 8 express Peng's reduced criterion [14, Theorem 3.2], together with Lü and Zhang's finite truncation [12, Theorem 6.10], directly in A, B, C_1, D_1 without choosing a basis of $\ker D_1$. For the original system, Lü and Zhang prove sufficiency but leave necessity open, and ask in [12, Remark 6.11] for a rank characterization directly in the original coefficients. Theorem 8 gives the requested formulation and also covers multidimensional noise.

For broader background, see [4] for invariant-subspace criteria with drift-only control. Gramian and duality characterizations appear in [10, 17]; other terminal requirements include finite-chaos approximation [6], exact output control [18], and partial controllability with terminal constraints [15].

The remainder is organized as follows. Section 2 proves the classification and feedback theorems. Section 3 proves the rank criteria and discusses their deterministic reduction and nonrobustness. Section 4 presents the orthogonal elimination algorithm.

2 Proofs of Theorems 3 and 4

We first prove in Section 2.1 that AC, ANC, and $\mathcal{V}_* = \{0\}$ are equivalent. We then prove Theorem 4, yielding the ENC implication needed for Theorem 3(i).

The feedback construction has three steps. In Section 2.2, separation of the positive definite cone gives stabilizing feedbacks with any prescribed mean-square decay rate when the obstruction subspace vanishes. In Section 2.3, a semialgebraic argument bounds their gains and Lyapunov condition numbers polynomially in this rate. In Section 2.4, these feedbacks are applied on intervals accumulating at T ; exponential state decay dominates polynomial growth, making the control energy summable and proving Theorem 4. Section 2.4 then completes the proof of Theorem 3.

2.1 Approximate controllability and the obstruction space

We prove the geometric characterization of AC and ANC using the conditional-invariance approach of Goreac [8, Theorem 3].

Lemma 9 At every $T > 0$, system (1) is approximately controllable if and only if it is approximately null controllable, and these properties hold if and only if $\mathcal{V}_* = \{0\}$.

PROOF. Fix $T > 0$ and define $M_T : \mathcal{U}_T \rightarrow L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ by $M_T u = y(T; 0, u)$. This linear map is bounded by Eq. (2), and Eq. (5) gives $M_T^* \eta = B^\top z + \sum_{j=1}^m D_j^\top Z_j$, where (z, Z) solves Eq. (4). By linearity of the control system (1), AC is equivalent to density of the range of M_T in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, or equivalently to $\ker M_T^* = \{0\}$.

Suppose that $\mathcal{V}_* = \{0\}$ and $M_T^* \eta = 0$. We show by induction that $z(t) \in \mathcal{V}_k$ for all $t \in [0, T]$, $\mathbb{P}$ -a.s. For $k = 0$ this is trivial. Assume it holds for some k . Since $z(t) \in \mathcal{V}_k$, its projection onto $\mathcal{V}_k^\perp$ vanishes. Hence the projection of Eq. (4) onto $\mathcal{V}_k^\perp$ has zero drift and zero martingale part. Consequently, the projected diffusion coefficients satisfy

$$\sum_{j=1}^m \int_0^T |\Pi_{\mathcal{V}_k^\perp} Z_j(t)|^2 dt = 0, \quad \mathbb{P}\text{-a.s.}$$

Thus $Z_j(t) \in \mathcal{V}_k$ for every j , while the drift condition is $A^\top z(t) + \sum_{j=1}^m C_j^\top Z_j(t) \in \mathcal{V}_k$ almost everywhere with

respect to the product measure $dt \otimes d\mathbb{P}$. Together with $B^\top z + \sum_{j=1}^m D_j^\top Z_j = 0$, Eqs. (6) and (7) give $z(t) \in \mathcal{V}_{k+1}$ almost everywhere. Since z is continuous and $\mathcal{V}_{k+1}$ is closed, this inclusion holds for all $t \in [0, T]$, $\mathbb{P}$ -a.s. At $k = n$, we obtain $z = 0$ and hence $\eta = z(T) = 0$. Thus

AC holds, and ANC follows by taking the target to be zero.

Conversely, suppose that $\mathcal{V}_* \neq \{0\}$. Choose $0 \neq v \in \mathcal{V}_*$ and matrices $\Gamma_1, \dots, \Gamma_m$ as in Lemma 2, and let z solve

$$\begin{cases} dz(t) = -\left(A^\top + \sum_{j=1}^m C_j^\top \Gamma_j\right) z(t) dt + \sum_{j=1}^m \Gamma_j z(t) dW^j(t), \\ z(0) = v. \end{cases}$$

The solution z belongs to $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n))$ and remains in $\mathcal{V}_*$ by Eq. (8). Thus $Z = (\Gamma_1 z, \dots, \Gamma_m z) \in L^2_{\mathbb{F}}(0, T; (\mathbb{R}^n)^m)$ and $\eta = z(T) \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$. The pair (z, Z) solves Eq. (4) and satisfies $B^\top z + \sum_{j=1}^m D_j^\top Z_j = 0$.

Taking $y_0 = v$ in Eq. (5) gives

$$\mathbb{E}\langle y(T; v, u), \eta \rangle = |v|^2 \quad \text{for every } u \in \mathcal{U}_T.$$

If ANC held, there would be controls $u_\ell \in \mathcal{U}_T$ with $y(T; v, u_\ell) \rightarrow 0$ in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$. The Cauchy–Schwarz inequality would then force the left-hand side to tend to zero, contradicting $v \neq 0$. Hence ANC fails, which proves the converse implication.

2.2 Arbitrary-rate stabilizability

We first obtain stabilizing feedbacks with any prescribed mean-square decay rate. We write $H > 0$ (respectively, $H \geq 0$) when a symmetric matrix H is positive definite (respectively, positive semidefinite), and use $<, \leq$ analogously.

For the $m \times m$ identity matrix I_m and any matrix M , the Kronecker product is

$$I_m \otimes M = \begin{pmatrix} M & 0 & \cdots & 0 \\ 0 & M & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M \end{pmatrix},$$

with m copies of M on the diagonal.

For $\rho > 0$, a symmetric $P \in \mathbb{R}^{n \times n}$, and $U \in \mathbb{R}^{r \times n}$, put

$$\mathcal{M}_\rho(P, U) = \begin{pmatrix} AP + PA^\top + BU & (CP + DU)^\top \\ +U^\top B^\top + \rho P & \\ CP + DU & -I_m \otimes P \end{pmatrix}. \quad (19)$$

The matrix $\mathcal{M}_\rho(P, U)$ has size $(m+1)n$. A pair (P, U) is called feasible at rate ρ if $\mathcal{M}_\rho(P, U) < 0$; its lower-right block then gives $P > 0$. We prove feasibility at every positive rate directly from the vanishing of the obstruction subspace.

Lemma 10 *If $\mathcal{V}_* = \{0\}$, then for every $\rho > 0$ there exist a symmetric matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $U \in \mathbb{R}^{r \times n}$ such that $\mathcal{M}_\rho(P, U) < 0$.*

PROOF. Fix $\rho > 0$. Suppose that the linear space $\{-\mathcal{M}_\rho(P, U) : P = P^\top, U \in \mathbb{R}^{r \times n}\}$ is disjoint from the open cone of positive definite matrices. Finite-dimensional separation gives a nonzero symmetric matrix $Z \geq 0$ such that $\text{tr}(Z\mathcal{M}_\rho(P, U)) = 0$ for every

$P = P^\top$ and every U . Indeed, the separating functional annihilates the linear space, and its trace representative is positive semidefinite by testing on $vv^\top + \varepsilon I$, $\varepsilon > 0$, and letting $\varepsilon \downarrow 0$.

Write $Q = Z_{00}$, $S_j = Z_{j0}$, and $R_{ij} = Z_{ij}$, with the head block indexed by 0 and the noise blocks by $1, \dots, m$. The free symmetric variable P and the free variable U give

$$A^\top Q + QA + \rho Q + \sum_{j=1}^m (C_j^\top S_j + S_j^\top C_j) = \sum_{j=1}^m R_{jj}, \quad (20)$$

$$B^\top Q + \sum_{j=1}^m D_j^\top S_j = 0. \quad (21)$$

Here the trace pairing with P has the symmetric coefficient displayed in Eq. (20), while that with U is twice the pairing with the transpose of the coefficient in Eq. (21).

We use the elementary implication $H \geq 0$, $x^\top Hx = 0 \Rightarrow Hx = 0$. For $x \in \ker Q$, positivity of Z applied to $(x, 0, \dots, 0)$ gives $S_j x = 0$ for every j . Thus $Qx \mapsto S_j x$ is a well-defined linear map on $\text{im } Q$. Extend it linearly to $\Gamma_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$, so that $S_j = \Gamma_j Q$. Testing the principal block $\begin{pmatrix} Q & S_j^\top \\ S_j & R_{jj} \end{pmatrix}$ on $(-\Gamma_j^\top y, y)$ shows that

$$R_{jj} - \Gamma_j Q \Gamma_j^\top \geq 0. \quad (22)$$

This construction also applies when Q is singular.

If $Q = 0$, then every $S_j = 0$, and Eq. (20) gives $\sum_{j=1}^m R_{jj} = 0$. Since each $R_{jj} \geq 0$, all these diagonal blocks vanish. Applying the same zero-quadratic-form implication to vectors supported in a single block then gives $Z = 0$, a contradiction. Consequently $Q \neq 0$.

Put $F = A^\top + \sum_{j=1}^m C_j^\top \Gamma_j$ and $R = \sum_{j=1}^m R_{jj} \geq 0$. By

Eq. (20),

$$FQ + QF^\top + \rho Q = R. \quad (23)$$

For $x \in \ker Q$, taking its quadratic form in Eq. (23) gives $x^\top R x = 0$, hence $Rx = 0$ and $x^\top R_{jj} x = 0$ for every j . By Eq. (22), $(\Gamma_j^\top x)^\top Q (\Gamma_j^\top x) = 0$, so $Q \Gamma_j^\top x = 0$. Applying Eq. (23) to x also gives $QF^\top x = 0$. Thus $\ker Q$ is invariant under $F^\top$ and all $\Gamma_j^\top$. Since Q is symmetric, $V = \text{im } Q = (\ker Q)^\perp$ is invariant under F and all Γ_j . By Eq. (21), $(B^\top + \sum_{j=1}^m D_j^\top \Gamma_j) V = \{0\}$. The maximality assertion in Lemma 2 therefore yields $\{0\} \neq V \subset \mathcal{V}_*$, contrary to the hypothesis. The two sets cannot be disjoint, proving strict feasibility.

2.3 Polynomially bounded stabilizing feedbacks

We now seek feasible pairs for which the feedback gain and the condition number $\kappa(P) = \lambda_{\max}(P)/\lambda_{\min}(P)$ grow at most polynomially with the decay rate.

We use semialgebraic sets, i.e., finite Boolean combinations of polynomial equalities and inequalities with real coefficients; a function is semialgebraic if its graph is. The Tarski–Seidenberg theorem allows projection and elimination of real quantifiers [3, Theorem 2.2.1 and Proposition 2.2.4].

Lemma 11 *The image of a semialgebraic subset of $\mathbb{R}^{a+b}$ under coordinate projection onto $\mathbb{R}^a$ is semialgebraic. More generally, a set defined by polynomial equalities and inequalities, finite Boolean operations and finitely many real existential or universal quantifiers is semialgebraic.*

We also use the polynomial growth bound for continuous semialgebraic functions on closed semialgebraic sets [3, Proposition 2.6.2, p. 43], in the following form.

Lemma 12 *If $f : [1, \infty) \rightarrow \mathbb{R}$ is continuous and semialgebraic, then there exist $M \geq 1$ and an integer $p \geq 1$ such that*

$$|f(\rho)| \leq M\rho^p, \quad \rho \geq 1.$$

Related real-algebraic methods have been used in output-feedback and stabilization problems [1, 7]. We apply these lemmas to the minimum of $\|P\|_F^2 + \|U\|_F^2$ over a normalized feasible set for Eq. (19). The normalization bounds P^{-1} , while the minimum bounds both $\kappa(P)$ and $\|UP^{-1}\|^2$.

Lemma 13 *Suppose that a feasible pair exists for every $\rho \geq 1$. There exist a constant $M \geq 1$ and an integer $p \geq 1$, depending only on $A, B, (C_j, D_j)_{j=1}^m$, such that for every $\rho \geq 1$ one can choose a feasible pair $(P(\rho), U(\rho))$ satisfying*

$$\kappa(P(\rho)) \leq M\rho^p \quad (24)$$

and

$$\|U(\rho)P(\rho)^{-1}\|^2 \leq M\rho^p. \quad (25)$$

PROOF. *Step 1: a semialgebraic minimum.* For $\rho \geq 1$, define the closed set

$$\mathcal{F}_\rho = \{(P, U) : P = P^\top, \mathcal{M}_\rho(P, U) \leq -I_{(m+1)n}\}$$

and put $g(P, U) = \|P\|_F^2 + \|U\|_F^2$. By the feasibility assumption and the homogeneity

$$\mathcal{M}_\rho(cP, cU) = c\mathcal{M}_\rho(P, U), \quad c > 0,$$

$\mathcal{F}_\rho$ is nonempty. Its lower-right block gives $I_m \otimes P \geq I_{mn}$, hence $P \geq I_n$. Since g is coercive, it attains the finite minimum

$$\mu(\rho) = \min_{(P, U) \in \mathcal{F}_\rho} g(P, U).$$

We first show that μ is semialgebraic. The set of triples (ρ, P, U) with $\rho \geq 1$ and $(P, U) \in \mathcal{F}_\rho$ is semialgebraic: the matrix inequality is equivalent to nonnegativity of all principal minors of $-\mathcal{M}_\rho(P, U) - I_{(m+1)n}$. For $\rho \geq 1$, a point (ρ, s) belongs to the graph of μ precisely when some $(P, U) \in \mathcal{F}_\rho$ satisfies $g(P, U) = s$ and no $(P', U') \in \mathcal{F}_\rho$ satisfies $g(P', U') < s$. By Lemma 11, this graph is semialgebraic.

Step 2: continuity of the minimum. Let $\rho_j \rightarrow \rho$ in $[1, \infty)$. To prove lower semicontinuity, it suffices to consider $\liminf_j \mu(\rho_j) < \infty$. Pass to a subsequence realizing this lower limit and choose minimizing pairs $(P_j, U_j) \in \mathcal{F}_{\rho_j}$. Coercivity gives a further subsequence converging

to a pair (P, U) . Passing to the limit in the matrix inequality gives $(P, U) \in \mathcal{F}_\rho$, and hence

$$\mu(\rho) \leq g(P, U) = \liminf_{j \rightarrow \infty} \mu(\rho_j).$$

For upper semicontinuity, choose a minimizing pair (P, U) at ρ and fix $c > 1$. For ρ' near ρ ,

$$\begin{aligned} \mathcal{M}_{\rho'}(cP, cU) &= c\mathcal{M}_\rho(P, U) + c(\rho' - \rho) \begin{pmatrix} P & 0 \\ 0 & 0_{mn} \end{pmatrix} \\ &\leq -cI_{(m+1)n} + c(\rho' - \rho) \begin{pmatrix} P & 0 \\ 0 & 0_{mn} \end{pmatrix}. \end{aligned}$$

Thus $(cP, cU) \in \mathcal{F}_{\rho'}$ whenever $\rho' \geq 1$ is sufficiently close to ρ , so

$$\limsup_{\rho' \rightarrow \rho} \mu(\rho') \leq c^2 \mu(\rho).$$

Letting $c \downarrow 1$ proves upper semicontinuity.

Step 3: polynomial bounds. Applying Lemma 12 to the continuous semialgebraic function $\mu : [1, \infty) \rightarrow \mathbb{R}$, we obtain a constant $M \geq 1$ and an integer $p \geq 1$ such that

$$\mu(\rho) \leq M\rho^p, \quad \rho \geq 1.$$

For any minimizing pair $(\bar{P}, \bar{U})$, we have $\bar{P} \geq I_n$ and $\mu(\rho) \geq n \geq 1$. In particular, $\|\bar{P}^{-1}\| \leq 1$, so

$$\kappa(\bar{P}) \leq \|\bar{P}\|_F \leq \sqrt{\mu(\rho)} \leq \mu(\rho).$$

The gain satisfies $\|\bar{U}\bar{P}^{-1}\|^2 \leq \|\bar{U}\|_F^2 \leq \mu(\rho)$. Taking $P(\rho) = \bar{P}$ and $U(\rho) = \bar{U}$ proves Eqs. (24) and (25).

We next record the decay estimate that will be used on each switching interval. For a feasible pair (P, U) at rate ρ , set $K = UP^{-1}$. The Schur complement of Eq. (19), followed by multiplication on both sides by P^{-1} , gives

$$\begin{aligned} (A + BK)^\top P^{-1} + P^{-1}(A + BK) + \rho P^{-1} \\ + \sum_{j=1}^m (C_j + D_j K)^\top P^{-1}(C_j + D_j K) < 0. \end{aligned} \quad (26)$$

Define $V(x) = \langle P^{-1}x, x \rangle$ and apply the feedback $u = Ky$. The initial state $y(t)$ may be any square-integrable $\mathcal{F}_t$ -measurable vector. On the switching interval, write $Y_s = y(t + s)$, put $f(s) = \mathbb{E}V(Y_s)$, and

$$\begin{aligned} g(s) &= \mathbb{E} \left[2 \langle P^{-1}Y_s, (A + BK)Y_s \rangle \right. \\ &\quad \left. + \sum_{j=1}^m \langle P^{-1}(C_j + D_j K)Y_s, (C_j + D_j K)Y_s \rangle \right]. \end{aligned}$$

The expected-product identity for the integral equation gives $f(s) = f(0) + \int_0^s g(a) da$. The continuous S^2 solution makes these quadratic terms integrable and, by dominated convergence, makes g continuous. By Eq. (26), $g(s) \leq -\rho f(s)$. Consequently, $\frac{d}{ds}(e^{\rho s} f(s)) = e^{\rho s}(g(s) + \rho f(s)) \leq 0$ in the interior of the interval, and continuity gives

$$\mathbb{E}V(y(t + s)) \leq e^{-\rho s} \mathbb{E}V(y(t)).$$

Finally, the eigenvalue bounds $\frac{|x|^2}{\lambda_{\max}(P)} \leq V(x) \leq \frac{|x|^2}{\lambda_{\min}(P)}$ give

$$\mathbb{E}|y(t+s)|^2 \leq \kappa(P)e^{-\rho s} \mathbb{E}|y(t)|^2. \quad (27)$$

Both K and P are deterministic, so Eq. (27) applies without changing the feedback when $y(t)$ is random.

2.4 Feedback construction and exact controllability

Proof of Theorem 4 Assume $\mathcal{V}_* = \{0\}$. We first consider $0 < T \leq 1$ and treat $T > 1$ at the end of the construction. By Lemmas 10 and 13, we may choose feedbacks satisfying Eqs. (24) and (25) at every rate $\rho \geq 1$. We will shorten the switching intervals while increasing the decay rates fast enough that the factor $e^{-\rho s}$ in Eq. (27) compensates for both polynomial factors in Eqs. (24) and (25).

Step 1: choosing the intervals and decay rates. Choose $L \geq \max(1, 4p)$ so large that $4^{2p}ML^pe^{-L/2} \leq \frac{1}{2}$, which is possible because $e^{-L/2}$ decays faster than L^p . Set

$$t_j = T(1 - 2^{-j}), \quad j \geq 0. \quad (28)$$

The interval lengths are $\ell_j = t_{j+1} - t_j = T2^{-j-1}$. Choose $\lambda_j = LT^{-2}4^j$ and $K_j = U(\lambda_j)P(\lambda_j)^{-1}$, and define

$$K_T(t) = K_j, \quad t \in [t_j, t_{j+1}), \quad j \geq 0.$$

In particular, $K_T(t_j) = K_j$ at every switching time. On each interval, we solve the linear SDE with drift matrix $A + BK_j$ and diffusion matrices $C_i + D_iK_j$, $i = 1, \dots, m$, using the terminal state from the preceding interval as its initial state, starting with $y(0) = y_0$. By [12, Theorem 3.2, p. 132], these equations have unique square-integrable continuous adapted solutions. They therefore join continuously at the switching times. Since K_T is bounded on every $[0, s]$ with $s < T$, this gives a unique continuous adapted state on $[0, T)$, with $u(t) = K_T(t)y(t)$ locally square-integrable. Although $\ell_j \rightarrow 0$, the products $\lambda_j \ell_j = L2^{j-1}/T$ tend to infinity. Writing $a_j = \mathbb{E}|y(t_j)|^2$, Eqs. (24) and (27) give

$$a_{j+1} \leq M\lambda_j^p e^{-\lambda_j \ell_j} a_j. \quad (29)$$

Step 2: summability of the control energy. The energy on $[t_j, t_{j+1}]$ contains the product of $\|K_j\|^2$ and $\kappa(P(\lambda_j))$. Indeed, Eqs. (24), (25) and (27) imply

$$\begin{aligned} \mathbb{E} \int_{t_j}^{t_{j+1}} |u(t)|^2 dt &\leq \|K_j\|^2 \kappa(P(\lambda_j)) a_j \int_0^{\ell_j} e^{-\lambda_j s} ds \\ &\leq M^2 \lambda_j^{2p-1} a_j. \end{aligned} \quad (30)$$

This motivates the definition $b_j = \lambda_j^{2p} a_j$, for which Eq. (30) is bounded by $M^2 b_j / \lambda_j$. Since $\lambda_{j+1} = 4\lambda_j$, Eq. (29) gives

$$b_{j+1} = \lambda_{j+1}^{2p} a_{j+1} \leq 4^{2p} M \lambda_j^p e^{-\lambda_j \ell_j} b_j \leq \frac{1}{2} b_j. \quad (31)$$

To verify the last inequality, put $x_j = 2^j/T \geq 1$. Then

$$4^{2p} M \lambda_j^p e^{-\lambda_j \ell_j} = 4^{2p} M L^p x_j^{2p} e^{-Lx_j/2} \leq 4^{2p} M L^p e^{-L/2} \leq \frac{1}{2}.$$

Here $x \mapsto x^{2p} e^{-Lx/2}$ is nonincreasing on $[1, \infty)$ because $L \geq 4p$. It follows from Eq. (31) that $b_j \leq 2^{-j} b_0$. This, together with $\lambda_j = 4^j \lambda_0$, gives $\frac{b_j}{\lambda_j} \leq 8^{-j} \frac{b_0}{\lambda_0}$. Summing Eq. (30), we therefore obtain

$$\begin{aligned} \mathbb{E} \int_0^T |u(t)|^2 dt &\leq M^2 \sum_{j=0}^{\infty} \frac{b_j}{\lambda_j} \leq \frac{M^2 b_0}{\lambda_0} \sum_{j=0}^{\infty} 8^{-j} \\ &= \frac{8M^2 b_0}{7\lambda_0} = \frac{8}{7} M^2 L^{2p-1} T^{-(4p-2)} |y_0|^2, \end{aligned} \quad (32)$$

where we used $b_0 = \lambda_0^{2p} |y_0|^2$ and $\lambda_0 = LT^{-2}$.

Step 3: extending the state to the terminal time. So far, (y, u) has been constructed on $[0, T)$ by solving the feedback equation on successive intervals. Set $u(T) = 0$; this single endpoint value does not affect the integrals. The resulting control is progressively measurable, and Eq. (32) gives $u \in \mathcal{U}_T$. Regarding u as a fixed admissible control, we apply [12, Theorem 3.2, p. 132] to Eq. (1) on $[0, T]$. The resulting solution belongs to $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n))$ and, by uniqueness, agrees with the successively constructed state on every $[0, t_j]$. We retain the notation y for this extension. Any other continuous adapted closed-loop solution agrees with y on every $[0, t_j]$ by local uniqueness, and hence also at T by continuity. Since $\lambda_j \geq 1$, Eq. (31) implies $a_j = \lambda_j^{-2p} b_j \leq 2^{-j} b_0 \rightarrow 0$. In particular, y is continuous in $L^2(\Omega; \mathbb{R}^n)$ at T . Consequently,

$$\mathbb{E}|y(T)|^2 = \lim_{j \rightarrow \infty} \mathbb{E}|y(t_j)|^2 = \lim_{j \rightarrow \infty} a_j = 0.$$

The deterministic gain schedule is independent of y_0 , so the resulting null-control map is linear and bounded. Together with Eq. (32), the identity $y(T) = 0$ proves the theorem for $0 < T \leq 1$, with $N = 4p - 2$ in Eq. (10).

For $T > 1$, let $\hat{K}$ be the feedback schedule constructed above for the unit horizon. Set $K_T(t) = 0$ on $[0, T-1)$ and $K_T(t) = \hat{K}(t-T+1)$ on $[T-1, T)$. The uncontrolled state $y(T-1)$ is square-integrable. Since Eq. (27) applies to square-integrable $\mathcal{F}_{T-1}$ -measurable initial states, the preceding estimates apply on the last unit interval with $|y_0|^2$ replaced by $\mathbb{E}|y(T-1)|^2$. They give finite control energy and $y(T) = 0$. This schedule is locally bounded on $[0, T)$ and independent of y_0 , completing the proof.

Proof of Theorem 3, part (i) By Lemma 9, AC and ANC are each equivalent to $\mathcal{V}_* = \{0\}$. By Theorem 4, this condition implies ENC. Conversely, ENC implies ANC, completing the equivalence.

Proof of Theorem 3, part (ii) Fix $T > 0$. Exact controllability implies approximate null controllability and hence $\mathcal{V}_* = \{0\}$. It also makes the map $M_T : \mathcal{U}_T \rightarrow L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, $M_T u = y(T; 0, u)$, surjective. Hilbert-space duality therefore gives a constant $c(T) > 0$ such that every adjoint solution satisfies

$$\mathbb{E}|\eta|^2 \leq c(T)\mathbb{E} \int_0^T \left| B^\top z(t) + \sum_{j=1}^m D_j^\top Z_j(t) \right|^2 dt. \quad (33)$$

We show that this requires $\text{rank } \mathcal{D} = mn$. Otherwise, choose $q = [q_1^\top \cdots q_m^\top]^\top \in \ker \mathcal{D}^\top$ with $|q|^2 = \sum_{j=1}^m |q_j|^2 = 1$. For $0 < \varepsilon < \min(T, 1)$, prescribe

$$Z_j^\varepsilon(t) = \varepsilon^{-1/2} q_j \mathbf{1}_{[T-\varepsilon, T]}(t), \quad j = 1, \dots, m,$$

solve the adjoint differential equation forward from $z^\varepsilon(0) = 0$, and put $\eta_\varepsilon = z^\varepsilon(T)$. By uniqueness, this is the adjoint solution for terminal value η_ε . Set $g = \sum_{j=1}^m C_j^\top q_j$.

The solution is zero before $T - \varepsilon$. On the last interval, write α, β, Δ_j for the coordinate matrices of the maps $-A^\top$, $x \mapsto -xg$, and $x \mapsto xq_j$, respectively, using the standard orthonormal bases and regarding the last two maps as maps from $\mathbb{R}$ to $\mathbb{R}^n$. For a real matrix M , set $s(M) = \sum_{k,\ell} |M_{k\ell}|$. The scalar input $\varepsilon^{-1/2}$ on this interval has L^2 norm one. The integral equation, the Cauchy–Schwarz inequality for the drift, and the Itô isometry give, with $H_\varepsilon = \max_k \|z_k^\varepsilon\|_{L^2((T-\varepsilon, T) \times \Omega)}$,

$$H_\varepsilon \leq \varepsilon s(\alpha) H_\varepsilon + \varepsilon s(\beta) + \sqrt{\varepsilon} \sum_{j=1}^m s(\Delta_j).$$

Consequently, for $\varepsilon s(\alpha) < 1$, zero extension of the actual local solution gives

$$\|z^\varepsilon\|_{L^2((0, T) \times \Omega)} \leq r(\varepsilon),$$

$$r(\varepsilon) := \frac{n(\varepsilon s(\beta) + \sqrt{\varepsilon} \sum_{j=1}^m s(\Delta_j))}{1 - \varepsilon s(\alpha)} \longrightarrow 0.$$

All coefficients in this bound are independent of ε . Let $M_\varepsilon = \sum_{j=1}^m \int_0^T Z_j^\varepsilon(t) dW^j(t)$. Independence of the Brownian channels and the Itô isometry give $\|M_\varepsilon\|_{L^2(\Omega)} = 1$. The actual terminal integral equation is

$$\eta_\varepsilon = M_\varepsilon - \int_0^T A^\top z^\varepsilon(t) dt - \int_0^T \sum_{j=1}^m C_j^\top Z_j^\varepsilon(t) dt.$$

Hence the triangle and Cauchy–Schwarz inequalities imply

$$1 \leq \|\eta_\varepsilon\|_{L^2(\Omega)} + \sqrt{T} \|A^\top\| r(\varepsilon) + \sqrt{\varepsilon} |g|.$$

On the other hand, $\sum_{j=1}^m D_j^\top Z_j^\varepsilon = 0$, so

$$\left\| B^\top z^\varepsilon + \sum_{j=1}^m D_j^\top Z_j^\varepsilon \right\|_{L^2((0, T) \times \Omega)} \leq \|B^\top\| r(\varepsilon).$$

Taking square roots in Eq. (33) and combining these bounds yields

$$1 \leq (\sqrt{c(T)} \|B^\top\| + \sqrt{T} \|A^\top\|) r(\varepsilon) + \sqrt{\varepsilon} |g| \longrightarrow 0,$$

a contradiction. Thus $\mathcal{D}$ is onto.

Conversely, suppose that $\mathcal{V}_* = \{0\}$ and $\mathcal{D}$ is onto. Set $J = \mathcal{D}^\top (\mathcal{D} \mathcal{D}^\top)^{-1}$. For $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, let $(x, \zeta_1, \dots, \zeta_m)$ solve

$$\begin{cases} dx(t) = [(A - BJC)x(t) + BJ\zeta(t)] dt + \sum_{j=1}^m \zeta_j(t) dW^j(t), \\ x(T) = \xi. \end{cases}$$

where $\zeta = [\zeta_1^\top \cdots \zeta_m^\top]^\top$. By [12, Theorem 4.2], this BSDE has a square-integrable solution. Define $\hat{u} = J(\zeta - Cx) \in \mathcal{U}_T$. Since $\mathcal{D}J = I_{mn}$, its diffusion coefficients satisfy $Cx + \mathcal{D}\hat{u} = \zeta$. The drift coefficient is $Ax + B\hat{u} = (A - BJC)x + BJ\zeta$. Thus $(x, \hat{u})$ solves Eq. (1) with terminal state ξ . Since $\mathcal{F}_0$ is trivial modulo null sets, $x(0)$ is deterministic. Theorem 4 supplies a control u^0 whose trajectory y^0 starts from $y_0 - x(0)$ and satisfies $y^0(T) = 0$. By linearity, $y = x + y^0$ is the state corresponding to $u = \hat{u} + u^0$, with initial state $y(0) = x(0) + (y_0 - x(0)) = y_0$. Its terminal state is $y(T) = x(T) + y^0(T) = \xi$. This proves Eq. (9), and hence part (ii).

Remark 14 Following [11], exact terminal controllability allows choosing a deterministic x_ξ with the control for each target $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, instead of fixing the initial state. When $\mathcal{D}$ is onto, the BSDE construction yields $x_\xi = x(0)$; exact null controllability then corrects $y_0 - x_\xi$ without changing the target. Thus, by linearity, exact controllability is equivalent to exact terminal plus exact null controllability. Here x_ξ is deterministic, not noise-dependent.

3 Proofs of Theorems 6 and 8

We now prove the criteria in the original coefficients stated in Theorems 6 and 8.

Proof of Theorem 6 We prove Eq. (14) by induction. For $k = 1$, the drift condition in Eq. (6) is unrestricted, so Eq. (11) gives the representation. At step $k + 1$, the conditions $w_j \in \mathcal{V}_k$ are represented by

$$K_k w_j + N_k \alpha_j = 0, \quad j = 1, \dots, m. \quad (34)$$

The drift constraint is

$$K_k \left(A^\top v + \sum_{j=1}^m C_j^\top w_j \right) + N_k \beta = 0. \quad (35)$$

Combining Eqs. (34) and (35) with $B^\top v + \sum_{j=1}^m D_j^\top w_j = 0$, we obtain

$$K_{k+1} v + N_{k+1} [w_1^\top \cdots w_m^\top \alpha_1^\top \cdots \alpha_m^\top \beta^\top]^\top = 0,$$

with the block matrices from Eq. (12).

Conversely, the block equation imposes the zero-observation constraint and places every w_j and the drift $A^\top v + \sum_{j=1}^m C_j^\top w_j$ in $\mathcal{V}_k$. These conditions also imply $v \in \mathcal{V}_k$. Indeed, starting with $v \in \mathcal{V}_0$, if $v \in \mathcal{V}_\ell$ for $0 \leq \ell < k$, then $\mathcal{V}_k \subset \mathcal{V}_\ell$ and Eqs. (6) and (7)

give $v \in \mathcal{V}_{\ell+1}$. The same constraints therefore yield $v \in \mathcal{V}_{k+1}$, proving Eq. (14).

For matrices K, N with the same number of rows, the linear map $v \mapsto Kv + \text{im } N$ into the quotient by $\text{im } N$ has kernel $\{v : Kv \in \text{im } N\}$ and rank $(\text{rank}[N K] - \text{rank } N)$. This proves Eq. (15).

Every block is polynomial in $A, B, (C_j, D_j)_{j=1}^m$, so the criterion does not require computing the obstruction spaces or inverting a coefficient matrix.

Remark 15 When the diffusion coefficients vanish, Corollary 7 recovers the classical Kalman rank condition. The distinction between deterministic and random terminal targets explains why EC is not equivalent to the other three properties here.

If $C_j = D_j = 0$ for all j , Eq. (6) yields

$$\mathcal{V}_k = \bigcap_{\ell=0}^{k-1} \ker\{B^\top (A^\top)^\ell\}, \quad k \geq 1,$$

so $\delta_n = \text{rank}[B A B \cdots A^{n-1} B]$. Thus Eq. (16) becomes the Kalman rank condition for AC, ANC, and ENC. In the classical deterministic-control, deterministic-target problem, this condition is equivalent to all four controllability properties. Here, however, controls lie in $\mathcal{U}_T$ and targets range over $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$: adapted controls can generate random terminal states even without diffusion. Exact controllability to all such targets fails since $\mathcal{D} = 0$ violates the necessary condition in Corollary 7.

Adapted controls approximate arbitrary random targets even when diffusion vanishes as follows. Assume the Kalman condition. For $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ and $0 < h < T$, let $\xi_h = \mathbb{E}[\xi \mid \mathcal{F}_{T-h}]$, which is known at time $T - h$. With zero control on $[0, T-h]$, the deterministic Gramian formula gives an adapted control on $[T-h, T]$ reaching ξ_h exactly; its expected energy is finite for each h , since ξ_h is $\mathcal{F}_{T-h}$ -measurable and square-integrable. Under the Brownian natural filtration, $\xi_h \rightarrow \xi$ in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ as $h \downarrow 0$, yielding AC. The corresponding control energies need not stay bounded.

Remark 16 The first iterate is $\mathcal{V}_1 = (B \ker \mathcal{D})^\perp$. If $B \ker \mathcal{D} = \{0\}$, the iteration is stationary from the start. Hence all four controllability properties require $B \ker \mathcal{D} \neq \{0\}$. Exact controllability also requires $\text{rank } \mathcal{D} = mn$, so $r \geq mn + 1$. Thus, beyond enough input directions to prescribe all diffusion channels, at least one input combination vanishing in every diffusion channel must act nontrivially through the drift.

Adapted from [12, Section 6.3 and Remark 6.24], the following example shows that arbitrarily small diffusion perturbations can destroy AC, ANC, and ENC, ruling out any full-rank characterization by a fixed-size matrix depending continuously on the coefficients.

Example 17 For fixed $m \geq 1$, take $n = 2$, $r = 1$ and consider

$$dy_1 = y_2 dt + \varepsilon y_2 dW^1(t), \quad dy_2 = u dt.$$

Its drift coefficients are $B = (0, 1)^\top$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The diffusion coefficients are $C_1 = \varepsilon A$, $C_j = 0$ for $j \geq 2$, and $D_j = 0$ for every j . Since $B^\top v = v_2$, the first step of Eq. (7) gives $\mathcal{V}_1 = \text{span}\{e_1\}$ for every ε . At the second step, write $v = ae_1$ and $w_j = b_j e_1$. The remaining constraint is

$$A^\top v + \sum_{j=1}^m C_j^\top w_j = (a + \varepsilon b_1)e_2 \in \text{span}\{e_1\},$$

or equivalently $a + \varepsilon b_1 = 0$. If $\varepsilon = 0$, this forces $a = 0$; if $\varepsilon \neq 0$, every a is admissible by choosing $b_1 = -a/\varepsilon$ and $b_j = 0$ for $j \geq 2$. Since $n = 2$, it follows that

$$\mathcal{V}_* = \mathcal{V}_2 = \begin{cases} \{0\}, & \varepsilon = 0, \\ \text{span}\{e_1\}, & \varepsilon \neq 0. \end{cases}$$

By Theorem 3, AC, ANC and ENC hold at every $T > 0$ when $\varepsilon = 0$, and all three fail at every $T > 0$ when $\varepsilon \neq 0$. Exact controllability fails throughout this family because $\mathcal{D} = 0$.

For fixed n, r, m , maximal rank of a fixed-size matrix depending continuously on the coefficient matrices is an open condition. The family above therefore precludes such a full-rank characterization of AC, ANC or ENC in general. For the rank-difference criterion, Eq. (15) gives $\delta_2 = 2$ when $\varepsilon = 0$ and $\delta_2 = 1$ otherwise. Thus the condition $\delta_2 = n$ in Eq. (16) is not open along this family.

Proof of Theorem 8 Put $w = [w_1^\top \cdots w_m^\top]^\top$. Since $J\mathcal{D}$ is the orthogonal projector onto $\text{im } \mathcal{D}^\top$, we have $(I_r - J\mathcal{D})\mathcal{D}^\top = 0$ and $J^\top \mathcal{D}^\top = I_{mn}$. Multiplying $B^\top v + \mathcal{D}^\top w = 0$ by $I_r - J\mathcal{D}$ and $J^\top$ therefore gives $H^\top v = 0$ and $w = -J^\top B^\top v$. Conversely, these conditions imply

$$B^\top v + \mathcal{D}^\top w = (I_r - \mathcal{D}^\top J^\top) B^\top v = H^\top v = 0.$$

The second condition reads $w_j = -E_j^\top v$ for every j , and the drift becomes

$$A^\top v + \sum_{j=1}^m C_j^\top w_j = (A^\top - \mathcal{C}^\top J^\top B^\top) v = F^\top v.$$

By Lemma 2, $\mathcal{V}_*$ is therefore the largest common $(E_1^\top, \dots, E_m^\top, F^\top)$ -invariant subspace contained in $\ker H^\top$.

Put $S_k = \text{im}[H_0 H_1 \cdots H_k]$. By Eq. (17),

$$S_{k+1} = S_k + \sum_{j=1}^m E_j S_k + F S_k.$$

Thus the sequence is increasing, and an equality $S_{k+1} = S_k$ implies invariance under every E_j and F , so all subsequent spaces coincide. If $H = 0$, all these spaces are zero. Otherwise $\dim S_0 \geq 1$; if no equality occurs before S_{n-1} , then $\dim S_{n-1} \geq \dim S_0 + n - 1 \geq n$. In either case, $S := S_{n-1}$ is invariant under all these matrices. If U is any other common invariant subspace containing $\text{im } H$, the recursion gives $S_k \subset U$ by induction. Hence S is the smallest such subspace.

For $v \in S^\perp$, $x \in S$, and $M \in \{E_1, \dots, E_m, F\}$, we have $\langle M^\top v, x \rangle = \langle v, Mx \rangle = 0$. Consequently, $S^\perp$ is invariant under $E_1^\top, \dots, E_m^\top, F^\top$. Since $\text{im } H \subset S$, it is

also contained in $\ker H^\top$. The maximality of $\mathcal{V}_*$ therefore gives $S^\perp \subset \mathcal{V}_*$. Conversely, the same inner-product identity shows that $\mathcal{V}_*^\perp$ is invariant under every E_j and F , and $\mathcal{V}_* \subset \ker H^\top$ implies $\text{im } H \subset \mathcal{V}_*^\perp$. The minimality of S gives $S \subset \mathcal{V}_*^\perp$, and hence $\mathcal{V}_* \subset S^\perp$. We conclude that $S_{n-1}^\perp = \mathcal{V}_*$. Thus Eq. (18) is equivalent to $\mathcal{V}_* = \{0\}$, and the conclusion follows from Theorem 3.

Remark 18 Since $I_r - JD$ is the orthogonal projector onto $\ker \mathcal{D}$, $\text{im } H = B \ker \mathcal{D}$. Thus H represents the drift-control directions left after the diffusion has been prescribed, without choosing a basis of $\ker \mathcal{D}$. For $m = 1$, the matrices E_1, F, H express Peng's reduced word criterion [14, Theorem 3.2] directly in the original coefficients, with the finite truncation in [12, Theorem 6.10]. Related finite-word characterizations for backward-structured systems are given in [16, Propositions 3.2–3.3].

The block dimensions in Eq. (12) grow exponentially. Section 4 gives an equivalent reduced computation.

4 Computational method

The rank differences in Eq. (13) can be computed by orthogonal elimination on the current obstruction space. We give the exact algebraic algorithm and prove its equivalence to subspace iteration, then describe a QR/SVD implementation, avoiding the exponentially large matrices in Eq. (12).

4.1 Orthogonal elimination

Suppose $[Q_k \ V_k]$ is orthogonal, with $\text{im } V_k = \mathcal{V}_k$ and $\text{im } Q_k = \mathcal{V}_k^\perp$. Write $q_k = \dim \mathcal{V}_k^\perp$ and $d_k = n - q_k$, so that $Q_k \in \mathbb{R}^{n \times q_k}$ and $V_k \in \mathbb{R}^{n \times d_k}$. Initially, Q_0 has no columns and $V_0 = I_n$. By Lemma 2, $\mathcal{V}_{k+1} \subset \mathcal{V}_k$, so a candidate for the next space has the form $v = V_k b$, with $b \in \mathbb{R}^{d_k}$. Write the auxiliary vectors as $w_j = V_k a_j$, with $a_j \in \mathbb{R}^{d_k}$, and put $a = [a_1^\top \cdots a_m^\top]^\top$. Since a vector belongs to $\mathcal{V}_k$ precisely when $Q_k^\top$ annihilates it, the input and drift constraints become

$$B^\top V_k b + \sum_{j=1}^m D_j^\top V_k a_j = 0,$$

$$Q_k^\top A^\top V_k b + \sum_{j=1}^m Q_k^\top C_j^\top V_k a_j = 0.$$

The reduced block multiplying b is

$$L_k = \begin{pmatrix} B^\top V_k \\ Q_k^\top A^\top V_k \end{pmatrix} \in \mathbb{R}^{(r+q_k) \times d_k}, \quad (36)$$

and the block multiplying a is

$$M_k = \begin{pmatrix} D_1^\top V_k & \cdots & D_m^\top V_k \\ Q_k^\top C_1^\top V_k & \cdots & Q_k^\top C_m^\top V_k \end{pmatrix} \in \mathbb{R}^{(r+q_k) \times (md_k)}. \quad (37)$$

The constraints read

$$L_k b + M_k a = 0. \quad (38)$$

The coordinates of a already impose $w_j \in \mathcal{V}_k$ for every j , so these membership conditions require no additional equations.

To eliminate a from Eq. (38), let $U_{0,k}$ have orthonormal columns spanning $\ker M_k^\top$ and set

$$R_k = U_{0,k}^\top L_k. \quad (39)$$

Because $(\ker M_k^\top)^\perp = \text{im } M_k$, eliminating a from Eq. (38) gives

$$V_k b \in \mathcal{V}_{k+1} \iff L_k b \in \text{im } M_k \iff U_{0,k}^\top L_k b = 0 \\ \iff b \in \ker R_k.$$

Thus $\mathcal{V}_{k+1} = V_k \ker R_k$. Let $\Delta q_k = \text{rank } R_k$, and choose an orthogonal matrix $O_k = [O_{1,k} \ O_{0,k}]$ whose first Δq_k columns span $(\ker R_k)^\perp$ and whose remaining columns span $\ker R_k$. The update is

$$Q_{k+1} = [Q_k \ V_k O_{1,k}], \quad V_{k+1} = V_k O_{0,k}. \quad (40)$$

The columns of $V_k O_{1,k}$ are the directions removed at this stage; they are appended to the orthogonal complement already computed.

This construction computes exactly the rank differences of the prescribed block pair.

Proposition 19 *In exact arithmetic, the updates in Eqs. (36), (37), (39) and (40) preserve orthogonality of $[Q_k \ V_k]$ and satisfy $\text{im } V_k = \mathcal{V}_k$, $q_k = \delta_k$, and $\Delta q_k = \delta_{k+1} - \delta_k$. The iteration reaches a stationary space or the zero space after at most n stages, and returns the classification in Table 1 after checking rank $\mathcal{D}$.*

PROOF. The elimination above gives $\text{im } V_{k+1} = V_k \ker R_k = \mathcal{V}_{k+1}$. Since V_k is an isometry on $\mathbb{R}^{d_k}$, the two blocks $V_k O_{1,k}$ and $V_k O_{0,k}$ are orthonormal and mutually orthogonal. Both are orthogonal to Q_k , so Eq. (40) preserves a complete orthogonal state basis and gives $q_{k+1} = q_k + \Delta q_k$. Starting from $q_0 = \delta_0 = 0$, Eq. (15) yields $q_k = \delta_k$ and the asserted increment identity.

If $\Delta q_k = 0$, then $\ker R_k = \mathbb{R}^{d_k}$ and $\mathcal{V}_{k+1} = \mathcal{V}_k$; by Lemma 2, all subsequent spaces coincide. Otherwise the dimension drops by at least one. Hence the algorithm terminates within n stages. The returned space is $\mathcal{V}_*$, and Theorem 3 gives the classification.

4.2 Numerical realization

The elimination uses Householder QR and SVD [2]. At a stage, set $p = r + q$, $d = n - q$, $h = md$, $s = \text{rank } M$, with $M \in \mathbb{R}^{p \times h}$, $L \in \mathbb{R}^{p \times d}$. If $p \geq h$, apply Householder QR to $[M \ L]$, then SVD to the leading $h \times h$ triangular block of the transformed M . Apply the same left transformations to L and keep the $p - s$ rows corresponding to zero rows of M ; these form R . Reflectors avoid storing a full $p \times p$ factor. If $p < h$, a full left singular-vector matrix of M gives these rows directly; its full right factor is unnecessary.

An SVD of R gives Eq. (40). Its full $d \times d$ right factor is required when R is wide, since a thin SVD omits part of the nullspace. If R has no rows, its kernel is $\mathbb{R}^d$ and the

Algorithm 1. Orthogonal elimination for the controllability criterion (exact arithmetic)

Require: $A, C_j \in \mathbb{R}^{n \times n}$, $B, D_j \in \mathbb{R}^{n \times r}$, $j = 1, \dots, m$.

- 1: $Q \leftarrow [], V \leftarrow I_n$; form $\mathcal{D} = [D_1^\top \dots D_m^\top]^\top$.
- 2: **for** $k = 0, \dots, n - 1$ **do**
- 3: Form L and M from Q, V by Eqs. (36) and (37).
- 4: Reduce M by left orthogonal transformations and apply the same transformations to L .
- 5: Let R consist of the rows of the transformed L corresponding to the zero rows of the transformed M .
- 6: Find $\Delta q = \text{rank } R$ and an orthogonal matrix $O = [O_1 \ O_0]$, with $\text{im } O_0 = \ker R$.
- 7: **if** $\Delta q = 0$ **then**
- 8: **return** nonzero subspace $\text{im } V$; all four controllability properties fail.
- 9: **end if**
- 10: $Q \leftarrow [Q \ VO_1]$, $V \leftarrow VO_0$ (using the old V on both right sides).
- 11: **if** V has no columns **then**
- 12: **return** AC, ANC and ENC hold; EC holds if and only if $\text{rank } \mathcal{D} = mn$.
- 13: **end if**
- 14: **end for**

iteration stops. After updating, Householder QR may reorthogonalize $[Q \ V]$ in its existing order.

The residual R_k is formed through the orthogonal transformations of L_k , and its singular values are computed directly. In floating-point arithmetic, the numerical rank of each matrix X is the number of singular values above $\tau_{\text{abs}} + \tau_{\text{rel}} \|X\|$, with tolerances chosen for the data scale and precision.

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