

Null Controllability of a Stochastic Parabolic Equation with a Space-Dependent Analytic Noise Coefficient

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Abstract

We consider a stochastic parabolic equation on the n -dimensional flat torus $\mathbb{T}^n = (\mathbb{R}/(2\pi\mathbb{Z}))^n$, $n \in \mathbb{N}$, whose only control acts in the drift. The multiplicative-noise coefficient is adapted and may depend jointly on the sample point, time, and space. Its spatial profiles are assumed to be real analytic with a uniform positive radius of analyticity, while the corresponding analytic norm is only required to be square integrable in time, uniformly along sample paths. We prove a state-only observability inequality with an L^1 -in-time mean-square observation norm and deduce null controllability from every measurable spatial set of positive measure by one adapted drift control. The same one-time interpolation estimate also yields approximate controllability to arbitrary square-integrable random terminal targets. The proof is based on a direct analytic energy estimate for the adjoint state, followed by propagation of smallness and a telescoping argument.

Keywords. Stochastic parabolic equation; null controllability; approximate controllability; observability; analytic energy estimate.

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1 Introduction

Let $T > 0$, and set $\mathbb{T}^n = (\mathbb{R}/(2\pi\mathbb{Z}))^n$ ($n \in \mathbb{N}$). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ be a complete filtered probability space carrying a one-dimensional Brownian motion $W(\cdot)$. We assume that $\{\mathcal{F}_t\}_{t \in [0, T]}$ is the usual augmentation of the natural filtration of $W(\cdot)$, and write $\mathbb{F}$ for the progressive σ -algebra. For a separable Hilbert space H , let $L^2_{\mathcal{F}_t}(\Omega; H)$ denote the space of $\mathcal{F}_t$ -measurable H -valued random variables ξ satisfying $\mathbb{E}\|\xi\|_H^2 < \infty$. The space $L^2_{\mathbb{F}}(0, T; H)$ consists of progressively measurable H -valued processes X satisfying

$$\mathbb{E} \int_0^T \|X(t)\|_H^2 dt < \infty,$$

and $L^2_{\mathbb{F}}(\Omega; C([0, T]; H))$ denotes the space of adapted H -valued processes with continuous paths satisfying

$$\mathbb{E} \sup_{0 \leq t \leq T} \|X(t)\|_H^2 < \infty.$$

We use $L^\infty_{\mathbb{F}}(\Omega; L^2(0, T; H))$ for progressively measurable H -valued processes satisfying

$$\text{ess sup}_{\omega \in \Omega} \int_0^T \|X(\omega, t)\|_H^2 dt < \infty.$$

For $r \in \{1, \infty\}$, we denote by $L^r_{\mathbb{F}}(0, T; L^2(\Omega; H))$ the subspace of $L^r(0, T; L^2(\Omega; H))$ consisting of equivalence classes with progressively measurable H -valued representatives, equipped with the inherited Bochner norm.

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Thus the time norm is taken after the $L^2(\Omega; H)$ norm. For every complex Hilbert space used below, we denote its inner product by $\langle \cdot, \cdot \rangle_H$ and take it to be linear in the first variable. In particular,

$$\langle f, g \rangle_{L^2(\mathbb{T}^n)} = \int_{\mathbb{T}^n} f(x) \overline{g(x)} dx.$$

We use the same angle-bracket notation for real Hilbert spaces.

Let $G \subset \mathbb{T}^n$ be measurable with positive Lebesgue measure $|G|$. We consider the following stochastic parabolic equation:

$$\begin{cases} dy = (\Delta y + \mathbf{1}_G u) dt + by dW(t) & \text{in } (0, T) \times \mathbb{T}^n, \\ y(0) = y_0 & \text{in } \mathbb{T}^n. \end{cases} \quad (1.1)$$

Here the initial datum $y_0 \in L^2(\mathbb{T}^n)$, $b \in L^\infty_{\mathbb{F}}(\Omega; L^2(0, T; L^\infty(\mathbb{T}^n)))$ and the control $u \in L^\infty_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G)))$.

Classical well-posedness theory (e.g., [9, Theorems 3.18 and 3.20]) ensures that (1.1) admits a unique mild solution $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(\mathbb{T}^n))) \cap L^2_{\mathbb{F}}(0, T; H^1(\mathbb{T}^n))$. Moreover, this solution satisfies the estimate

$$\mathbb{E} \sup_{0 \leq t \leq T} \|y(t)\|_{L^2(\mathbb{T}^n)}^2 + \mathbb{E} \int_0^T \|\nabla y(t)\|_{L^2(\mathbb{T}^n)}^2 dt \leq C \left(\|y_0\|_{L^2(\mathbb{T}^n)}^2 + \mathbb{E} \int_0^T \|u(t)\|_{L^2(G)}^2 dt \right). \quad (1.2)$$

Throughout the paper, we use $C > 0$ for a constant whose value may change from line to line, and subscripts for constants that remain fixed within an argument.

We now recall the following definitions.

Definition 1.1 (Null controllability). Equation (1.1) is null controllable at time T if, for every $y_0 \in L^2(\mathbb{T}^n)$, there exists $u \in L^\infty_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G)))$ such that the corresponding solution to (1.1) satisfies $y(T) = 0$ in $L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))$.

Definition 1.2 (Approximate controllability). Equation (1.1) is approximately controllable at time T if, for every $y_0 \in L^2(\mathbb{T}^n)$, $y_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))$, and $\varepsilon > 0$, there exists $u \in L^\infty_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G)))$ such that the corresponding solution to (1.1) satisfies $\|y(T) - y_T\|_{L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))} < \varepsilon$.

The null controllability of deterministic parabolic equations has by now been studied extensively. A comprehensive list of references on this topic is beyond the scope of this paper; we refer interested readers to [3, 5, 13] and the extensive literature cited therein. By contrast, far fewer works address the controllability of stochastic parabolic equations (see [2, 4, 6–8, 10, 12]). The main difficulty lies in establishing the observability estimate for the dual system of a controlled stochastic parabolic equation. As pointed out in [2], a backward stochastic parabolic equation contains both a state and a martingale integrand. The martingale integrand acts as a “bad” (nonhomogeneous) term when one attempts to prove the null/approximate controllability of stochastic parabolic equations via the global Carleman estimate. Consequently, two controls are typically required, one acting on the drift term and the other on the diffusion term, in order to establish the null and approximate controllability of stochastic parabolic equations (see [4, 6, 10, 12]). When b is spatially independent, however, one can combine a spectral inequality with time iteration to establish the null and approximate controllability of (1.1) using only a drift control (see [7, 8, 11]).

In [7, 8, 11], spatial independence plays a crucial role in these spectral arguments. Multiplication by such a coefficient commutes with the spectral projectors of the Laplacian, so each projected adjoint equation involves only the corresponding frequencies of the martingale integrand. A space-dependent multiplier, however, generally mixes frequencies: the unobserved high-frequency part of the martingale integrand can enter the low-frequency equation for the state. Consequently, the projected equations no longer close in the form required by the scalar-coefficient argument.

In this paper, we establish null and approximate controllability for a class of spatially varying coefficients b . Our results allow the spatial control set to be an arbitrary measurable set of positive measure, while retaining the L^1 -in-time observation norm used in the duality argument.

Compared with the spatially independent single-control works [7, 8, 11], we remove the restriction that the noise coefficient be spatially independent. Compared with the spatially dependent two-control works [4, 6, 10, 12], we reduce the number of controls from two to one and obtain an L^1 -in-time observability estimate from an arbitrary measurable set, at the cost of restricting the coefficient class to real-analytic

functions with a uniform positive radius of analyticity. Section 5 shows that this analytic class is still rich: it contains random profiles, profiles with infinitely many nonzero Fourier modes, and time-unbounded profiles whose multiplication operators generally mix Fourier modes. Thus, the present paper establishes a new trade-off between the number of controls and the regularity of the noise coefficient: within the analytic coefficient class, a single drift control suffices to overcome the frequency-mixing effect caused by spatially dependent multiplicative noise. We study the control problem on the torus $\mathbb{T}^n$. The periodic setting plays a specific role in the proof: it makes complex translations compatible with Fourier projections and eliminates boundary contributions in the associated energy estimates. On domains with boundary, the compatibility of complex translations with the boundary conditions requires additional analysis, and the present argument does not apply directly.

To handle the spatial dependence of the coefficients, we impose a uniform holomorphic extension condition on their spatial profiles. For each $R > 0$, we define the following complex strip and the corresponding space of holomorphic functions.

For $R > 0$, set $S_R^n = \{\zeta \in \mathbb{C}^n : |\operatorname{Im} \zeta_j| < R, j = 1, \dots, n\}$, and write $\mathcal{O}(S_R^n)$ for the jointly holomorphic functions on S_R^n .

Definition 1.3 (Periodic polystrip-analytic space). Let $R > 0$, and define

$$H_{\text{per}}^\infty(S_R^n) = \left\{ F \in \mathcal{O}(S_R^n) : F(\zeta + 2\pi \mathbf{e}_j) = F(\zeta) \text{ for all } j = 1, \dots, n, \|F\|_{H^\infty(S_R^n)} := \sup_{\zeta \in S_R^n} |F(\zeta)| < \infty \right\},$$

where $\mathbf{e}_j$ denotes the j -th standard basis vector of $\mathbb{R}^n$.

Let $\mathcal{A}_R(\mathbb{T}^n)$ denote the space of all real-valued functions on $\mathbb{T}^n$ that arise as restrictions to $\mathbb{T}^n$ of functions $F \in H_{\text{per}}^\infty(S_R^n)$.

For $f \in \mathcal{A}_R(\mathbb{T}^n)$, define

$$\|f\|_{\mathcal{A}_R(\mathbb{T}^n)} = \|F\|_{H^\infty(S_R^n)}, \quad (1.3)$$

where $F \in H_{\text{per}}^\infty(S_R^n)$ is an extension of f to S_R^n . With this norm, $\mathcal{A}_R(\mathbb{T}^n)$ is a real Banach space.

Let $C^\omega(\mathbb{T}^n; \mathbb{R})$ denote the space of real-valued real-analytic functions on $\mathbb{T}^n$. Every such function belongs to $\mathcal{A}_R(\mathbb{T}^n)$ for some $R > 0$, possibly depending on the function. Indeed, by local real analyticity and compactness of the torus, it admits a periodic holomorphic extension to a polystrip of positive width; after possibly reducing the width, this extension is bounded. Conversely, every function in $\mathcal{A}_R(\mathbb{T}^n)$ is real analytic. Hence

$$C^\omega(\mathbb{T}^n; \mathbb{R}) = \bigcup_{R>0} \mathcal{A}_R(\mathbb{T}^n).$$

For the family $b(\omega, t, \cdot)$, the following assumption imposes a common extension width and an L^2 -in-time bound on the extension supremum norms that is uniform in ω .

Assumption 1.1. Let $b : \Omega \times (0, T) \times \mathbb{T}^n \rightarrow \mathbb{R}$ be progressively measurable. We assume that there exists $R > 0$ such that, outside a fixed $(\mathbb{P} \otimes dt)$ -null set, $b(\omega, t, \cdot) \in \mathcal{A}_R(\mathbb{T}^n)$. For each such (ω, t) , let $\tilde{b}(\omega, t, \cdot)$ denote the unique analytic extension of $b(\omega, t, \cdot)$ to S_R^n . We assume that these extensions admit an $\mathbb{F} \otimes \mathcal{B}(S_R^n)$ -measurable representative

$$\tilde{b} : \Omega \times (0, T) \times S_R^n \rightarrow \mathbb{C}.$$

For a.e. (ω, t) , set

$$\beta(\omega, t) := \|b(\omega, t, \cdot)\|_{\mathcal{A}_R(\mathbb{T}^n)}, \quad (1.4)$$

and define β arbitrarily on the exceptional null set. We further assume that, for some $C_1 > 0$,

$$\|\beta\|_{L^\infty_{\mathbb{F}}(\Omega; L^2(0, T))} \leq C_1. \quad (1.5)$$

Remark 1.1. Condition (1.5) imposes neither time continuity nor time differentiability on b , and allows β to be unbounded in time.

The process β is progressively measurable. Indeed, by (1.3), together with continuity and periodicity, it agrees with the supremum of $|\tilde{b}|$ over a fixed countable dense subset of $[0, 2\pi]^n \times (-R, R)^n$. Note also that

$$|b(\omega, t, x)| \leq \beta(\omega, t) \quad \text{for almost every } (\omega, t, x). \quad (1.6)$$

Throughout the paper, constants in estimates depend only on n, T, R , and C_1 , unless otherwise stated. Our main results are the following null and approximate controllability theorems.

Theorem 1.1. *Assume that Assumption 1.1 holds. Then (1.1) is null controllable at time T . Moreover, the control can be chosen to satisfy*

$$\|u\|_{L^\infty_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G)))} \leq C_{\text{obs}} \|y_0\|_{L^2(\mathbb{T}^n)}. \quad (1.7)$$

Here $C_{\text{obs}} > 0$ is a constant that may depend in addition on $|G|$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, equation (1.1) is approximately controllable at time T .*

By the classical duality argument, the null controllability in Theorem 1.1 is equivalent to an observability estimate for the following adjoint equation:

$$\begin{cases} dz = -(\Delta z + bZ) dt + Z dW(t) & \text{in } (0, T) \times \mathbb{T}^n, \\ z(T) = \eta & \text{in } \mathbb{T}^n. \end{cases} \quad (1.8)$$

Under Assumption 1.1, a standard adaptation of the well-posedness argument in [9, Section 4.4] yields, for every $\eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))$, a unique mild solution $(z, Z) \in [L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(\mathbb{T}^n))) \cap L^2_{\mathbb{F}}(0, T; H^1(\mathbb{T}^n))] \times L^2_{\mathbb{F}}(0, T; L^2(\mathbb{T}^n))$. Moreover, this solution satisfies the estimate

$$\mathbb{E} \sup_{0 \leq t \leq T} \|z(t)\|_{L^2(\mathbb{T}^n)}^2 + \mathbb{E} \int_0^T (\|\nabla z(t)\|_{L^2(\mathbb{T}^n)}^2 + \|Z(t)\|_{L^2(\mathbb{T}^n)}^2) dt \leq C \mathbb{E} \|\eta\|_{L^2(\mathbb{T}^n)}^2. \quad (1.9)$$

Our main observability result is the following.

Theorem 1.3. *Assume that Assumption 1.1 holds. Then, every solution (z, Z) of (1.8) satisfies*

$$\|z(0)\|_{L^2(\mathbb{T}^n)} \leq C_{\text{obs}} \|z\|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G)))}, \quad (1.10)$$

with C_{obs} as in (1.7).

The proof of Theorem 1.3 rests on a direct analytic energy estimate for the adjoint state. On each interval $[t, s]$, we apply a time-dependent complex translation to the Fourier–Galerkin equations. Since this translation vanishes at the terminal time s , no analytic regularity is required of the terminal datum. The positive Itô correction involving the translated martingale integrand absorbs the translated coupling term bZ , while the Laplacian controls the term generated by the time derivative of the translation. The analytic bound on b , together with (1.5), makes the resulting estimates uniform in the Galerkin dimension. Passing to the limit yields quantitative spatial analyticity of $z(t)$ without imposing spatial regularity on Z .

We then apply propagation of smallness to the $L^2(\Omega)$ -valued analytic state and obtain a one-time interpolation inequality from every measurable set G of positive measure. Combining this inequality with one-sided energy propagation and geometric telescoping yields the L^1 -in-time estimate in Theorem 1.3. Its L^1 – L^∞ duality yields Theorem 1.1, and the same interpolation inequality supplies the unique continuation needed for Theorem 1.2.

The remainder of the paper is organized as follows. Section 2 proves the analytic energy estimate. Section 3 derives the one-time interpolation and observability. Section 4 proves null and approximate controllability by duality. Section 5 illustrates the richness of the analytic coefficient class. Section 6 discusses the scope and limitations of the method and outlines directions for future research. Appendix A contains the proof of Lemma 2.2.

2 Analytic energy estimates for the adjoint state

Throughout Sections 2–4, we assume that the coefficient b satisfies Assumption 1.1 with fixed R and C_1 , and we let (z, Z) denote a solution of (1.8). We work in the complexifications of the periodic Hilbert spaces.

In this section, we prove the following analytic smoothing estimate for z , without assuming spatial regularity of Z . This estimate will be used in Section 3 to derive a one-time interpolation inequality and an observability estimate.

Theorem 2.1 (Quantitative analytic smoothing). *For every $0 \leq t < T$, the state $z(t)$ has an $L^2_{\mathcal{F}_t}(\Omega; \mathbb{C})$ -valued holomorphic extension $\mathcal{Z}_t$ to $S^n_{3R/4}$, periodic in each coordinate. There exists $C > 0$, depending only on n and R , such that, for every $0 \leq t < s \leq T$,*

$$\sup_{\zeta \in S^n_{R/2}} \|\mathcal{Z}_t(\zeta)\|_{L^2(\Omega)} \leq C e^{C_1^2/2} \exp\left(\frac{9nR^2}{64(s-t)}\right) \|z(s)\|_{L^2_{\mathcal{F}_s}(\Omega; L^2(\mathbb{T}^n))}. \quad (2.1)$$

2.1 Galerkin approximation and analytic energy estimates

To prove Theorem 2.1, we estimate the Galerkin solutions by a complex translation and then pass to the limit.

Let $e_{\mathbf{k}}(x) = (2\pi)^{-n/2} e^{i\mathbf{k} \cdot x}$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, be the orthonormal Fourier basis of $L^2(\mathbb{T}^n)$. Set $D_j = -i\partial_{x_j}$ and $D = (D_1, \dots, D_n)$. Let P_N be the orthogonal projection onto $H_N := \text{span}\{e_{\mathbf{k}} : |\mathbf{k}|_{\infty} \leq N\}$ with $|\mathbf{k}|_{\infty} = \max_{1 \leq j \leq n} |k_j|$. Fix $0 \leq t < s \leq T$ and set $\xi = z(s)$. On $[0, s]$, approximate (z, Z) by the H_N -valued solution (z_N, Z_N) of

$$\begin{cases} dz_N = -(\Delta z_N + P_N(bZ_N)) dr + Z_N dW(r) & \text{in } (0, s), \\ z_N(s) = P_N \xi. \end{cases} \quad (2.2)$$

Since P_N is an orthogonal projection, (1.6) gives

$$\|P_N(bv)\|_{L^2(\mathbb{T}^n)} \leq \|bv\|_{L^2(\mathbb{T}^n)} \leq \beta \|v\|_{L^2(\mathbb{T}^n)}, \quad v \in H_N.$$

In view of (1.5), a standard adaptation of the well-posedness argument in [9, Section 4.4] yields a unique solution of (2.2). The following convergence result is proved in Appendix A.

Lemma 2.2 (Galerkin convergence). *For every $r \in [0, s]$, it holds that*

$$\lim_{N \rightarrow \infty} \|z_N(r) - z(r)\|_{L^2_{\mathcal{F}_r}(\Omega; L^2(\mathbb{T}^n))} = 0 \quad (2.3)$$

and

$$\begin{cases} \lim_{N \rightarrow \infty} \|z_N - z\|_{L^2_{\mathbb{F}}(\Omega; C([0, s]; L^2(\mathbb{T}^n))) \cap L^2_{\mathbb{F}}(0, s; H^1(\mathbb{T}^n))} = 0, \\ \lim_{N \rightarrow \infty} \|Z_N - Z\|_{L^2_{\mathbb{F}}(0, s; L^2(\mathbb{T}^n))} = 0. \end{cases} \quad (2.4)$$

For $v \in L^2(\mathbb{T}^n)$, we define its Fourier coefficients by

$$\widehat{v}(\mathbf{k}) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} v(x) e^{-i\mathbf{k} \cdot x} dx, \quad \mathbf{k} \in \mathbb{Z}^n.$$

For $a \in \mathbb{R}$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{+1, -1\}^n$, the Fourier multiplier $e^{a\varepsilon \cdot D}$ is defined on H_N by $e^{a\varepsilon \cdot D} e_{\mathbf{k}} = e^{a\varepsilon \cdot \mathbf{k}} e_{\mathbf{k}}$ and satisfies

$$(e^{a\varepsilon \cdot D} v)(x) = v(x - ia\varepsilon), \quad v \in H_N. \quad (2.5)$$

We work outside the fixed null set appearing in Assumption 1.1. Here $\widetilde{b}$ denotes the analytic extension from that assumption. For every $v \in H_N$, $|a| < R$, and $\varepsilon \in \{+1, -1\}^n$, we also have

$$e^{a\varepsilon \cdot D} P_N(bv) = P_N[\widetilde{b}(\cdot - ia\varepsilon) e^{a\varepsilon \cdot D} v]. \quad (2.6)$$

Indeed, Cauchy's theorem applied in each coordinate gives

$$\widetilde{b}(\cdot - ia\varepsilon)v(\cdot - ia\varepsilon)(\mathbf{k}) = e^{a\varepsilon \cdot \mathbf{k}} \widehat{b}v(\mathbf{k}).$$

The contours lie in a smaller polystrip, and the integrals over the vertical sides cancel by periodicity. By (2.5), retaining the Fourier modes with $|\mathbf{k}|_\infty \leq N$ yields (2.6).

To combine the estimates over all sign vectors, we use the Fourier multiplier $e^{\rho|D|_1}$, defined for $\rho > 0$ by

$$(e^{\rho|D|_1}v)(x) = \sum_{\mathbf{k} \in \mathbb{Z}^n} e^{\rho|\mathbf{k}|_1} \widehat{v}(\mathbf{k}) e^{i\mathbf{k} \cdot x},$$

where $|\mathbf{k}|_1 = \sum_{j=1}^n |k_j|$. Its domain consists of those $v \in L^2(\mathbb{T}^n)$ for which this series converges in $L^2(\mathbb{T}^n)$, equivalently,

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} e^{2\rho|\mathbf{k}|_1} |\widehat{v}(\mathbf{k})|^2 < \infty.$$

Proposition 2.3 (Finite-dimensional analytic energy estimate). *Let $0 < \rho < R$. For every $N \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}^n$,*

$$\mathbb{E} \|e^{\rho\varepsilon \cdot D} z_N(t)\|_{L^2(\mathbb{T}^n)}^2 \leq \exp\left(\frac{n\rho^2}{2(s-t)} + C_1^2\right) \mathbb{E} \|P_N z(s)\|_{L^2(\mathbb{T}^n)}^2. \quad (2.7)$$

Consequently,

$$\mathbb{E} \|e^{\rho|D|_1} z_N(t)\|_{L^2(\mathbb{T}^n)}^2 \leq 2^n \exp\left(\frac{n\rho^2}{2(s-t)} + C_1^2\right) \mathbb{E} \|z(s)\|_{L^2(\mathbb{T}^n)}^2. \quad (2.8)$$

Proof. We divide the proof into two steps.

Step 1. We first prove the estimate for a fixed sign vector. Fix $N \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}^n$. Set $\delta = s - t$ and $\sigma(r) = \rho(s - r)/\delta$, $r \in [t, s]$. Then $\sigma(t) = \rho$, $\sigma(s) = 0$, and $\sigma' = -\rho/\delta$. Define

$$X_N^\varepsilon(r) = e^{\sigma(r)\varepsilon \cdot D} z_N(r), \quad Y_N^\varepsilon(r) = e^{\sigma(r)\varepsilon \cdot D} Z_N(r), \quad b_\varepsilon(r, x) = \widetilde{b}(r, x - i\sigma(r)\varepsilon). \quad (2.9)$$

Since $0 \leq \sigma(r) \leq \rho < R$, (1.3) gives

$$|b_\varepsilon(\omega, r, x)| \leq \beta(\omega, r). \quad (2.10)$$

By (2.9) and (2.6), we have

$$e^{\sigma\varepsilon \cdot D} P_N(bZ_N) = P_N(b_\varepsilon Y_N^\varepsilon).$$

For fixed N , the multiplier $e^{\sigma(r)\varepsilon \cdot D}$ is continuously differentiable in r on H_N . Since D_j , P_N , and $e^{\sigma\varepsilon \cdot D}$ commute, Itô's formula gives

$$dX_N^\varepsilon = [-\Delta X_N^\varepsilon + \sigma'\varepsilon \cdot DX_N^\varepsilon - P_N(b_\varepsilon Y_N^\varepsilon)] dr + Y_N^\varepsilon dW(r). \quad (2.11)$$

Since $X_N^\varepsilon \in H_N$ and P_N is an orthogonal projection,

$$\langle X_N^\varepsilon, P_N(b_\varepsilon Y_N^\varepsilon) \rangle_{L^2(\mathbb{T}^n)} = \langle P_N X_N^\varepsilon, b_\varepsilon Y_N^\varepsilon \rangle_{L^2(\mathbb{T}^n)} = \langle X_N^\varepsilon, b_\varepsilon Y_N^\varepsilon \rangle_{L^2(\mathbb{T}^n)}.$$

Applying Itô's formula to $\|X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2$ in (2.11), we obtain

$$\begin{aligned} d\|X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 &= \left[2 \sum_{j=1}^n \|D_j X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 + 2\sigma' \sum_{j=1}^n \varepsilon_j \operatorname{Re} \langle D_j X_N^\varepsilon, X_N^\varepsilon \rangle_{L^2(\mathbb{T}^n)} \right. \\ &\quad \left. + \|Y_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 - 2 \operatorname{Re} \langle X_N^\varepsilon, b_\varepsilon Y_N^\varepsilon \rangle_{L^2(\mathbb{T}^n)} \right] dr + 2 \operatorname{Re} \langle X_N^\varepsilon, Y_N^\varepsilon \rangle_{L^2(\mathbb{T}^n)} dW(r). \end{aligned} \quad (2.12)$$

The first two terms on the right-hand side of (2.12) can be rewritten as

$$\begin{aligned} & 2 \sum_{j=1}^n \|D_j X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 + 2\sigma' \sum_{j=1}^n \varepsilon_j \operatorname{Re} \langle D_j X_N^\varepsilon, X_N^\varepsilon \rangle_{L^2(\mathbb{T}^n)} \\ &= 2 \sum_{j=1}^n \left\| \left(D_j + \frac{\sigma' \varepsilon_j}{2} \right) X_N^\varepsilon \right\|_{L^2(\mathbb{T}^n)}^2 - \frac{n\rho^2}{2\delta^2} \|X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2, \end{aligned} \quad (2.13)$$

and the third and fourth terms can be rewritten as

$$\|Y_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 - 2 \operatorname{Re} \langle X_N^\varepsilon, b_\varepsilon Y_N^\varepsilon \rangle_{L^2(\mathbb{T}^n)} = \|Y_N^\varepsilon - \bar{b}_\varepsilon X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 - \|b_\varepsilon X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2. \quad (2.14)$$

To compensate for the two negative terms in (2.13) and (2.14), set

$$\Gamma(r) = \exp \left(\int_t^r \left[\frac{n\rho^2}{2\delta^2} + \beta(\omega, \tau)^2 \right] d\tau \right).$$

Since $d\Gamma = (n\rho^2/(2\delta^2) + \beta^2)\Gamma dr$, the product rule together with (2.13)–(2.14) yields

$$\begin{aligned} & d(\Gamma \|X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2) \\ &= \Gamma d\|X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 + \|X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 d\Gamma \\ &= \Gamma \left[2 \sum_{j=1}^n \left\| \left(D_j + \frac{\sigma' \varepsilon_j}{2} \right) X_N^\varepsilon \right\|_{L^2(\mathbb{T}^n)}^2 + \|Y_N^\varepsilon - \bar{b}_\varepsilon X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 + \beta^2 \|X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 - \|b_\varepsilon X_N^\varepsilon\|_{L^2(\mathbb{T}^n)}^2 \right] dr + dM_r, \end{aligned} \quad (2.15)$$

where the stochastic integral is given by

$$dM_r = 2\Gamma(r) \operatorname{Re} \langle X_N^\varepsilon(r), Y_N^\varepsilon(r) \rangle_{L^2(\mathbb{T}^n)} dW(r), \quad M_t = 0.$$

By (1.5), $1 \leq \Gamma(r) \leq \exp(n\rho^2/(2\delta) + C_1^2)$. Since the Galerkin solution satisfies $z_N \in L_{\mathbb{F}}^2(\Omega; C([0, s]; H_N))$ and $Z_N \in L_{\mathbb{F}}^2(0, s; H_N)$, the bound on Γ and the boundedness of $e^{\sigma(r)\varepsilon \cdot D}$ on H_N yield

$$\mathbb{E} \left(\int_t^s |2\Gamma(r) \operatorname{Re} \langle X_N^\varepsilon(r), Y_N^\varepsilon(r) \rangle_{L^2(\mathbb{T}^n)}|^2 dr \right)^{1/2} < \infty.$$

Hence the stochastic integral has zero expectation by [9, Theorem 2.130, item 3)] with $p = 1$. The drift term in (2.15) is nonnegative by (2.10). Integrating from t to s and taking expectations, using $\Gamma(t) = 1$, $\sigma(t) = \rho$, and $\sigma(s) = 0$, we obtain

$$\mathbb{E} \|e^{\rho\varepsilon \cdot D} z_N(t)\|_{L^2(\mathbb{T}^n)}^2 \leq \mathbb{E} [\Gamma(s) \|P_N z(s)\|_{L^2(\mathbb{T}^n)}^2] \leq \exp \left(\frac{n\rho^2}{2\delta} + C_1^2 \right) \mathbb{E} \|P_N z(s)\|_{L^2(\mathbb{T}^n)}^2,$$

which proves (2.7).

Step 2. We now combine the estimates over all sign vectors. For every $\mathbf{k} \in \mathbb{Z}^n$, choose a sign vector ε such that $\varepsilon \cdot \mathbf{k} = |\mathbf{k}|_1$. Hence

$$e^{2\rho|\mathbf{k}|_1} \leq \sum_{\varepsilon \in \{+1, -1\}^n} e^{2\rho\varepsilon \cdot \mathbf{k}}.$$

Parseval's identity then gives

$$\|e^{\rho|\mathbf{k}|_1} z_N(t)\|_{L^2(\mathbb{T}^n)}^2 \leq \sum_{\varepsilon \in \{+1, -1\}^n} \|e^{\rho\varepsilon \cdot D} z_N(t)\|_{L^2(\mathbb{T}^n)}^2.$$

Applying (2.7) to each term and using $\|P_N z(s)\|_{L^2(\mathbb{T}^n)} \leq \|z(s)\|_{L^2(\mathbb{T}^n)}$ proves (2.8). $\square$

2.2 Proof of the analytic smoothing estimate

Proof of Theorem 2.1. We begin by passing to the limit in the estimate (2.8). Fix $0 \leq t < s \leq T$, $0 < \rho < R$, and $K \in \mathbb{N}$. By Lemma 2.2, $P_K z_N(t) \rightarrow P_K z(t)$ strongly in $L^2_{\mathcal{F}_t}(\Omega; L^2(\mathbb{T}^n))$. Since $e^{\rho|D|_1}$ is bounded on H_K , (2.8) gives

$$\begin{aligned} \mathbb{E} \|e^{\rho|D|_1} P_K z(t)\|_{L^2(\mathbb{T}^n)}^2 &= \lim_{N \rightarrow \infty} \mathbb{E} \|e^{\rho|D|_1} P_K z_N(t)\|_{L^2(\mathbb{T}^n)}^2 \\ &\leq 2^n \exp\left(\frac{n\rho^2}{2(s-t)} + C_1^2\right) \mathbb{E} \|z(s)\|_{L^2(\mathbb{T}^n)}^2. \end{aligned}$$

Letting $K \rightarrow \infty$ and using monotone convergence of the Fourier series yields

$$\mathbb{E} \|e^{\rho|D|_1} z(t)\|_{L^2(\mathbb{T}^n)}^2 \leq 2^n \exp\left(\frac{n\rho^2}{2(s-t)} + C_1^2\right) \mathbb{E} \|z(s)\|_{L^2(\mathbb{T}^n)}^2. \quad (2.16)$$

To construct the extension, set $z_{\mathbf{k}}(t) = \widehat{z(t)}(\mathbf{k})$ and write

$$z(t, x) = \sum_{\mathbf{k} \in \mathbb{Z}^n} z_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot x}, \quad z_{\mathbf{k}}(t) \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{C}).$$

For $\zeta \in \overline{S_r^n}$ with $0 < r < \rho$, the Cauchy–Schwarz inequality gives

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \|z_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \zeta}\|_{L^2(\Omega)} \leq \left(\sum_{\mathbf{k} \in \mathbb{Z}^n} e^{-2(\rho-r)|\mathbf{k}|_1} \right)^{1/2} \left(\sum_{\mathbf{k} \in \mathbb{Z}^n} e^{2\rho|\mathbf{k}|_1} \|z_{\mathbf{k}}(t)\|_{L^2(\Omega)}^2 \right)^{1/2}. \quad (2.17)$$

The first factor on the right-hand side of (2.17) is finite, while the second is bounded by (2.16). Hence the Fourier series converges absolutely and uniformly in $L^2_{\mathcal{F}_t}(\Omega; \mathbb{C})$ on every smaller closed polystrip. Its sum $\mathcal{Z}_t$ is holomorphic on S_ρ^n , periodic, and agrees with $z(t)$ on the real torus $\mathbb{T}^n$. The same estimate yields

$$\sup_{\zeta \in \overline{S_r^n}} \|\mathcal{Z}_t(\zeta)\|_{L^2(\Omega)} \leq C \exp\left(\frac{n\rho^2}{4(s-t)} + \frac{C_1^2}{2}\right) \|z(s)\|_{L^2_{\mathcal{F}_s}(\Omega; L^2(\mathbb{T}^n))}. \quad (2.18)$$

Here C depends only on n and $\rho - r$. Taking $\rho = 3R/4$ and $r = R/2$ proves (2.1). $\square$

3 Propagation of smallness and observability

We combine analytic propagation of smallness with (2.1) and then apply a telescoping argument.

3.1 Analytic propagation and one-time interpolation

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0, 1, \dots\}$, we use

$$|\alpha| = \sum_{j=1}^n \alpha_j, \quad \alpha! = \prod_{j=1}^n \alpha_j!, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

We first recall [1, Theorem 4].

Lemma 3.1 (Multidimensional analytic propagation). *Let $\ell > 0$, let $E \subset B_\ell(x_0) \subset \mathbb{R}^n$ be measurable with positive Lebesgue measure $|E|$, and let $f : B_{2\ell}(x_0) \rightarrow \mathbb{R}$ be real-analytic. Assume that, for some $\mathcal{M} > 0$ and $0 < \varrho \leq 1$,*

$$|\partial^\alpha f(x)| \leq \mathcal{M} \frac{|\alpha|!}{(\varrho\ell)^{|\alpha|}}, \quad x \in B_{2\ell}(x_0), \quad \alpha \in \mathbb{N}_0^n.$$

Then there exist constants $N \geq 1$ and $\vartheta \in (0, 1)$, depending only on n , ϱ , and $|E|/|B_\ell(x_0)|$, such that

$$\|f\|_{L^\infty(B_\ell(x_0))} \leq N \left(\frac{1}{|E|} \int_E |f(x)| dx \right)^\vartheta \mathcal{M}^{1-\vartheta}.$$

We next extend this estimate to periodic Hilbert space-valued functions.

Proposition 3.2 (Periodic Hilbert space-valued propagation). *Let H be a separable complex Hilbert space, let $r > 0$, and let $E \subset \mathbb{T}^n$ be a measurable set of positive measure. Then there exist constants $C > 0$ and $\theta \in (0, 1)$, depending only on n , r , and $|E|$, such that the following holds. If $f : S_r^n \rightarrow H$ is holomorphic, 2π -periodic in each coordinate, and satisfies*

$$\sup_{\zeta \in S_r^n} \|f(\zeta)\|_H \leq \mathcal{M}$$

for some $\mathcal{M} \geq 0$, then

$$\|f\|_{L^2(\mathbb{T}^n; H)} \leq C \|f\|_{L^2(E; H)}^\theta \mathcal{M}^{1-\theta}. \quad (3.1)$$

In particular, the constants C and θ do not depend on H .

Proof. If $\mathcal{M} = 0$ or $H = \{0\}$, the conclusion is immediate. Hence we may assume that $\mathcal{M} > 0$ and $H \neq \{0\}$. The proof is divided into two steps.

Step 1. We first apply the scalar propagation estimate. Regard E as a subset of $Q = [-\pi, \pi]^n$, and set $\varrho = \min\{1, r/(4\pi\sqrt{n})\}$. Then $E \subset Q \subset B_{2\pi\sqrt{n}}(0)$ and $2\pi\sqrt{n}\varrho \leq r/2$. For each unit vector $h \in H$ and each $\gamma \in \mathbb{R}$, define

$$g_{h,\gamma}(x) = \operatorname{Re}(e^{-i\gamma} \langle f(x), h \rangle_H), \quad x \in \mathbb{R}^n.$$

Since the inner product is linear in its first variable, the function $\zeta \mapsto \langle f(\zeta), h \rangle_H$ is holomorphic on S_r^n and bounded by $\mathcal{M}$. Every polydisc of radius $r/2$ centered at a real point is contained in S_r^n . Thus Cauchy's inequality yields, for $x \in B_{4\pi\sqrt{n}}(0)$,

$$|\partial^\alpha g_{h,\gamma}(x)| \leq |\langle \partial^\alpha f(x), h \rangle_H| \leq \mathcal{M} \alpha! \left(\frac{2}{r}\right)^{|\alpha|} \leq \mathcal{M} \frac{|\alpha|!}{(2\pi\sqrt{n}\varrho)^{|\alpha|}}.$$

Here we used $\alpha! \leq |\alpha|!$ and $2\pi\sqrt{n}\varrho \leq r/2$. Therefore Lemma 3.1 applies and gives

$$\sup_{x \in B_{2\pi\sqrt{n}}(0)} |g_{h,\gamma}(x)| \leq C \left(\frac{1}{|E|} \int_E |g_{h,\gamma}(x)| dx \right)^\theta \mathcal{M}^{1-\theta}.$$

For the observation term, we have

$$\frac{1}{|E|} \int_E |g_{h,\gamma}(x)| dx \leq \frac{1}{|E|} \int_E \|f(x)\|_H dx \leq |E|^{-1/2} \|f\|_{L^2(E; H)}.$$

Consequently,

$$\sup_{x \in B_{2\pi\sqrt{n}}(0)} |g_{h,\gamma}(x)| \leq C |E|^{-\theta/2} \|f\|_{L^2(E; H)}^\theta \mathcal{M}^{1-\theta},$$

where C and θ depend only on n , r , and $|E|$, and are independent of h , γ , and H .

Step 2. We now recover the Hilbert-space norm. For every $x \in Q$, the dual characterization of the norm yields

$$\|f(x)\|_H = \sup_{\|h\|_H=1} |\langle f(x), h \rangle_H| = \sup_{\|h\|_H=1, \gamma \in \mathbb{R}} |\operatorname{Re}(e^{-i\gamma} \langle f(x), h \rangle_H)|.$$

Taking the supremum over h and γ in the estimate from Step 1, we obtain

$$\sup_{x \in Q} \|f(x)\|_H \leq C |E|^{-\theta/2} \|f\|_{L^2(E; H)}^\theta \mathcal{M}^{1-\theta}.$$

Since f is periodic, integrating over Q gives

$$\|f\|_{L^2(\mathbb{T}^n; H)} = \left(\int_Q \|f(x)\|_H^2 dx \right)^{1/2} \leq |Q|^{1/2} \sup_{x \in Q} \|f(x)\|_H \leq C \|f\|_{L^2(E; H)}^\theta \mathcal{M}^{1-\theta}.$$

This proves (3.1), with constants independent of H . □

Combining (3.1) with (2.1) gives the following interpolation estimate.

Corollary 3.3 (One-time interpolation). *Let (z, Z) solve (1.8). Then there exist constants $C > 0$ and $\theta \in (0, 1)$, where C depends only on $n, R, C_1, |G|$ and θ depends only on $n, R, |G|$, such that for all $0 \leq t < s \leq T$,*

$$\|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(\mathbb{T}^n))} \leq C \exp\left(\frac{C}{s-t}\right) \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))}^\theta \|z(s)\|_{L^2_{\mathcal{F}_s}(\Omega; L^2(\mathbb{T}^n))}^{1-\theta}. \quad (3.2)$$

Equivalently, for every $\varepsilon > 0$,

$$\|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(\mathbb{T}^n))} \leq \varepsilon \|z(s)\|_{L^2_{\mathcal{F}_s}(\Omega; L^2(\mathbb{T}^n))} + C\varepsilon^{-\frac{1-\theta}{\theta}} \exp\left(\frac{C}{s-t}\right) \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))}. \quad (3.3)$$

Proof. Fix $0 \leq t < s \leq T$ and set $H = L^2_{\mathcal{F}_t}(\Omega; \mathbb{C})$. By Theorem 2.1, $\mathcal{Z}_t$ is an H -valued holomorphic extension of $z(t)$ to $S^1_{3R/4}$. Since it agrees with $z(t)$ on the real torus $\mathbb{T}^n$, Fubini's theorem yields

$$\begin{aligned} \|\mathcal{Z}_t\|_{L^2(\mathbb{T}^n; H)}^2 &= \mathbb{E} \int_{\mathbb{T}^n} |z(t, x)|^2 dx = \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(\mathbb{T}^n))}^2, \\ \|\mathcal{Z}_t\|_{L^2(G; H)}^2 &= \mathbb{E} \int_G |z(t, x)|^2 dx = \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))}^2. \end{aligned}$$

Applying Proposition 3.2 with polystrip width $R/2$ and observation set G , and using the preceding identities together with (2.1), we obtain

$$\begin{aligned} &\|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(\mathbb{T}^n))} \\ &\leq C \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))}^\theta \left(\sup_{\zeta \in S^1_{R/2}} \|\mathcal{Z}_t(\zeta)\|_{L^2(\Omega)} \right)^{1-\theta} \\ &\leq C e^{(1-\theta)C_1^2/2} \exp\left(\frac{9nR^2(1-\theta)}{64(s-t)}\right) \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))}^\theta \|z(s)\|_{L^2_{\mathcal{F}_s}(\Omega; L^2(\mathbb{T}^n))}^{1-\theta}. \end{aligned}$$

Absorbing the fixed factors into the constant C proves (3.2). The exponent θ comes from Proposition 3.2 and therefore depends only on n, R , and $|G|$.

We now derive the additive estimate. Young's inequality with conjugate exponents $1/(1-\theta)$ and $1/\theta$ gives, for every $\varepsilon > 0$,

$$\begin{aligned} &C \exp\left(\frac{C}{s-t}\right) \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))}^\theta \|z(s)\|_{L^2_{\mathcal{F}_s}(\Omega; L^2(\mathbb{T}^n))}^{1-\theta} \\ &\leq \varepsilon \|z(s)\|_{L^2_{\mathcal{F}_s}(\Omega; L^2(\mathbb{T}^n))} + C\varepsilon^{-\frac{1-\theta}{\theta}} \exp\left(\frac{C}{\theta(s-t)}\right) \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))}. \end{aligned}$$

Combining this with (3.2) and absorbing $1/\theta$ into the constant in the exponential proves (3.3). $\square$

3.2 Energy propagation and observability

To pass from one-time interpolation to observability, we use the following energy bound.

Lemma 3.4 (One-sided energy propagation). *Let (z, Z) solve (1.8). Then, for $0 \leq t_1 < t_2 \leq T$,*

$$\mathbb{E}\|z(t_1)\|_{L^2(\mathbb{T}^n)}^2 \leq e^{C_1^2} \mathbb{E}\|z(t_2)\|_{L^2(\mathbb{T}^n)}^2. \quad (3.4)$$

Proof. The following Itô identities are first obtained for (z_N, Z_N) with $s = t_2$ and then passed to the limit using Lemma 2.2 and (1.5). In particular, Itô's formula and periodic integration by parts yield

$$d\mathbb{E}\|z(r)\|_{L^2(\mathbb{T}^n)}^2 = \left[2\mathbb{E}\|\nabla z(r)\|_{L^2(\mathbb{T}^n)}^2 + \mathbb{E}\|Z(r)\|_{L^2(\mathbb{T}^n)}^2 - 2\mathbb{E}\operatorname{Re}\langle b(r)Z(r), z(r) \rangle_{L^2(\mathbb{T}^n)} \right] dr.$$

The two terms involving Z can be rewritten as

$$\begin{aligned} & \mathbb{E}\|Z(r)\|_{L^2(\mathbb{T}^n)}^2 - 2\mathbb{E}\operatorname{Re}\langle b(r)Z(r), z(r) \rangle_{L^2(\mathbb{T}^n)} \\ &= \mathbb{E}\|Z(r)\|_{L^2(\mathbb{T}^n)}^2 - 2\mathbb{E}\operatorname{Re}\langle Z(r), \overline{b(r)}z(r) \rangle_{L^2(\mathbb{T}^n)} + \mathbb{E}\|\overline{b(r)}z(r)\|_{L^2(\mathbb{T}^n)}^2 - \mathbb{E}\|b(r)z(r)\|_{L^2(\mathbb{T}^n)}^2 \\ &= \mathbb{E}\|Z(r) - \overline{b(r)}z(r)\|_{L^2(\mathbb{T}^n)}^2 - \mathbb{E}\|b(r)z(r)\|_{L^2(\mathbb{T}^n)}^2. \end{aligned}$$

To control the negative term, set

$$\Gamma(r) = \exp\left(\int_{t_1}^r \beta(\omega, \tau)^2 d\tau\right), \quad t_1 \leq r \leq t_2.$$

Then $\Gamma(t_1) = 1$, $\Gamma' = \beta^2\Gamma$ almost everywhere, and (1.5) implies

$$1 \leq \Gamma(r) \leq e^{C_1^2}, \quad t_1 \leq r \leq t_2, \quad \text{almost surely.}$$

Applying the product formula to $\Gamma(r)\|z(r)\|_{L^2(\mathbb{T}^n)}^2$ and integrating from t_1 to t_2 , we obtain, after taking expectations,

$$\begin{aligned} & \mathbb{E}[\Gamma(t_2)\|z(t_2)\|_{L^2(\mathbb{T}^n)}^2] - \mathbb{E}\|z(t_1)\|_{L^2(\mathbb{T}^n)}^2 \\ &= \mathbb{E} \int_{t_1}^{t_2} \Gamma(r) \left[2\|\nabla z(r)\|_{L^2(\mathbb{T}^n)}^2 + \|Z(r) - \overline{b(r)}z(r)\|_{L^2(\mathbb{T}^n)}^2 + \beta(r)^2\|z(r)\|_{L^2(\mathbb{T}^n)}^2 - \|b(r)z(r)\|_{L^2(\mathbb{T}^n)}^2 \right] dr. \end{aligned}$$

Here the stochastic integral has zero expectation by the same integrability argument as in the proof of Proposition 2.3. By (1.6),

$$\beta(r)^2\|z(r)\|_{L^2(\mathbb{T}^n)}^2 - \|b(r)z(r)\|_{L^2(\mathbb{T}^n)}^2 \geq 0 \quad \text{almost surely,}$$

so the entire drift integrand is nonnegative. Using the bound on Γ , we obtain

$$\mathbb{E}\|z(t_1)\|_{L^2(\mathbb{T}^n)}^2 \leq \mathbb{E}[\Gamma(t_2)\|z(t_2)\|_{L^2(\mathbb{T}^n)}^2] \leq e^{C_1^2} \mathbb{E}\|z(t_2)\|_{L^2(\mathbb{T}^n)}^2,$$

which proves (3.4). $\square$

Proof of Theorem 1.3. We follow the telescoping argument in [1, proof of Theorem 1].

Step 1. We derive an estimate on each time interval. With θ from Corollary 3.3, set

$$\begin{aligned} \alpha &= \frac{1-\theta}{\theta}, \quad \rho_0 = 1 - \frac{\theta}{2}, \quad t_k = T(1 - \rho_0^k), \\ \delta_k &= t_{k+1} - t_k = T(1 - \rho_0)\rho_0^k, \quad I_k = [t_k, t_k + \delta_k/2], \quad k \geq 0. \end{aligned}$$

Then $t_0 = 0$, $t_k \uparrow T$, and $\delta_{k+1} = \rho_0\delta_k$. For $t \in I_k$, we have $t_{k+1} - t \geq \delta_k/2$, while Lemma 3.4 gives

$$\|z(t_k)\|_{L_{\mathcal{F}_{t_k}}^2(\Omega; L^2(\mathbb{T}^n))} \leq e^{C_1^2/2} \|z(t)\|_{L_{\mathcal{F}_t}^2(\Omega; L^2(\mathbb{T}^n))}.$$

Apply (3.3) with $s = t_{k+1}$ and with $\varepsilon e^{-C_1^2/2}$ in place of ε . Combining the two estimates gives

$$\|z(t_k)\|_{L_{\mathcal{F}_{t_k}}^2(\Omega; L^2(\mathbb{T}^n))} \leq \varepsilon \|z(t_{k+1})\|_{L_{\mathcal{F}_{t_{k+1}}}^2(\Omega; L^2(\mathbb{T}^n))} + C\varepsilon^{-\alpha} e^{2C/\delta_k} \|z(t)\|_{L_{\mathcal{F}_t}^2(\Omega; L^2(G))}, \quad t \in I_k.$$

The first two norms are independent of t . Integrating over I_k and dividing by $|I_k| = \delta_k/2$, we obtain

$$\|z(t_k)\|_{L_{\mathcal{F}_{t_k}}^2(\Omega; L^2(\mathbb{T}^n))} \leq \varepsilon \|z(t_{k+1})\|_{L_{\mathcal{F}_{t_{k+1}}}^2(\Omega; L^2(\mathbb{T}^n))} + \frac{2C}{\delta_k} \varepsilon^{-\alpha} e^{2C/\delta_k} \int_{I_k} \|z(t)\|_{L_{\mathcal{F}_t}^2(\Omega; L^2(G))} dt.$$

Since $\delta_k^{-1} \leq e^{1/\delta_k}$, the factor from averaging can be absorbed into the exponential. Hence there are constants $C, C_0 > 0$, depending only on $n, T, R, C_1, |G|$, such that

$$\|z(t_k)\|_{L_{\mathcal{F}_{t_k}}^2(\Omega; L^2(\mathbb{T}^n))} \leq \varepsilon \|z(t_{k+1})\|_{L_{\mathcal{F}_{t_{k+1}}}^2(\Omega; L^2(\mathbb{T}^n))} + C\varepsilon^{-\alpha} e^{C_0/\delta_k} \int_{I_k} \|z(t)\|_{L_{\mathcal{F}_t}^2(\Omega; L^2(G))} dt, \quad \varepsilon > 0. \quad (3.5)$$

We keep C_0 fixed in the remaining argument.

Step 2. We choose weights that make the estimate telescope. Set

$$C_2 = (2 - \theta)(C_0 + 1), \quad w_k = e^{-C_2/\delta_k}, \quad \varepsilon_k = \frac{w_{k+1}}{w_k}.$$

The relation $\delta_{k+1} = \rho_0 \delta_k$ gives

$$\varepsilon_k = \exp\left(-\frac{C_2(\rho_0^{-1} - 1)}{\delta_k}\right), \quad 1 - \alpha(\rho_0^{-1} - 1) = 1 - \frac{1 - \theta}{2 - \theta} = \frac{1}{2 - \theta}.$$

Consequently, the coefficient of the observation term satisfies

$$w_k \varepsilon_k^{-\alpha} e^{C_0/\delta_k} = \exp\left(\frac{C_0 - C_2 + \alpha C_2(\rho_0^{-1} - 1)}{\delta_k}\right) = \exp\left(\frac{C_0 - C_2/(2 - \theta)}{\delta_k}\right) = e^{-1/\delta_k} \leq 1.$$

Substitute $\varepsilon = \varepsilon_k$ into (3.5) and multiply by w_k . Since $w_k \varepsilon_k = w_{k+1}$, we obtain

$$\begin{aligned} & w_k \|z(t_k)\|_{L^2_{\mathcal{F}_{t_k}}(\Omega; L^2(\mathbb{T}^n))} - w_{k+1} \|z(t_{k+1})\|_{L^2_{\mathcal{F}_{t_{k+1}}}(\Omega; L^2(\mathbb{T}^n))} \\ & \leq C e^{-1/\delta_k} \int_{I_k} \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))} dt \leq C \int_{I_k} \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))} dt. \end{aligned}$$

Step 3. We sum the estimates and pass to the limit. Summing over $k = 0, \dots, m$ cancels all the intermediate weighted norms. Since the intervals I_k are disjoint subsets of $[0, T]$, this gives

$$\begin{aligned} & w_0 \|z(0)\|_{L^2(\mathbb{T}^n)} - w_{m+1} \|z(t_{m+1})\|_{L^2_{\mathcal{F}_{t_{m+1}}}(\Omega; L^2(\mathbb{T}^n))} \\ & \leq C \sum_{k=0}^m \int_{I_k} \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))} dt \leq C \int_0^T \|z(t)\|_{L^2_{\mathcal{F}_t}(\Omega; L^2(G))} dt = C \|z\|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G)))}. \end{aligned}$$

By (1.9),

$$0 \leq w_{m+1} \|z(t_{m+1})\|_{L^2_{\mathcal{F}_{t_{m+1}}}(\Omega; L^2(\mathbb{T}^n))} \leq C e^{-C_2/\delta_{m+1}} \|\eta\|_{L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))} \longrightarrow 0 \quad \text{as } m \rightarrow \infty,$$

because $\delta_{m+1} \rightarrow 0$. Moreover, $\mathcal{F}_0$ is trivial up to null sets, so the norm of $z(0)$ above equals its $L^2(\mathbb{T}^n)$ norm. Letting $m \rightarrow \infty$ and dividing by w_0 therefore yields

$$\|z(0)\|_{L^2(\mathbb{T}^n)} \leq C w_0^{-1} \|z\|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G)))}. \quad (3.6)$$

Finally, $\delta_0 = T\theta/2$ gives $w_0^{-1} = \exp(2C_2/(T\theta))$. Setting $C_{\text{obs}} = C w_0^{-1}$ proves (1.10). $\square$

4 Null and approximate controllability by duality

We work over the real scalar field. For each $\eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))$, let (z^η, Z^η) solve (1.8) with terminal value η . For every solution y of (1.1), we apply Itô's formula to $\langle P_N y, P_N z^\eta \rangle_{L^2(\mathbb{T}^n)}$. By (1.2), (1.9), and (1.5), the stochastic integrands belong to $L^1_{\mathbb{F}}(\Omega; L^2(0, T))$, and the drift and endpoint terms converge as $N \rightarrow \infty$. Taking expectations as above and passing to the limit gives the standard duality identity [9, Section 4.4]:

$$\mathbb{E} \langle y(T), \eta \rangle_{L^2(\mathbb{T}^n)} - \langle y_0, z^\eta(0) \rangle_{L^2(\mathbb{T}^n)} = \mathbb{E} \int_0^T \langle u(t), z^\eta(t) \rangle_{L^2(G)} dt. \quad (4.1)$$

Here the coefficient terms cancel because $\langle by, Z^\eta \rangle_{L^2(\mathbb{T}^n)} = \langle y, bZ^\eta \rangle_{L^2(\mathbb{T}^n)}$.

Proof of Theorem 1.1. Fix $y_0 \in L^2(\mathbb{T}^n)$. Define the linear functional on the subspace

$$\mathcal{X} = \{\mathbf{1}_G z^\eta : \eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))\} \subset L^1_{\mathbb{F}}(0, T; L^2(\Omega; L^2(G))),$$

as

$$\mathcal{L}(\mathbf{1}_G z^\eta) = -\langle y_0, z^\eta(0) \rangle_{L^2(\mathbb{T}^n)}, \quad \eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n)).$$

By Theorem 1.3, we have

$$|\mathcal{L}(\mathbf{1}_G z^\eta)| = |\langle y_0, z^\eta(0) \rangle_{L^2(\mathbb{T}^n)}| \leq C_{\text{obs}} \|y_0\|_{L^2(\mathbb{T}^n)} \|\mathbf{1}_G z^\eta\|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;L^2(G)))}, \quad \forall \eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n)).$$

Hence, $\mathcal{L}$ is a bounded linear functional with the norm $\|\mathcal{L}\| \leq C_{\text{obs}} \|y_0\|_{L^2(\mathbb{T}^n)}$. The Hahn–Banach theorem extends $\mathcal{L}$ to the space $L^1_{\mathbb{F}}(0,T;L^2(\Omega;L^2(G)))$ without increasing its norm. The representation theorem [9, Theorem 2.73], with $p = 1$, $q = 2$, and $H = L^2(G; \mathbb{R})$, then gives a real $u \in L^\infty_{\mathbb{F}}(0,T;L^2(\Omega;L^2(G)))$ satisfying (1.7) and

$$\mathbb{E} \int_0^T \langle u(t), z^\eta(t) \rangle_{L^2(G)} dt = -\langle y_0, z^\eta(0) \rangle_{L^2(\mathbb{T}^n)}, \quad \forall \eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n)).$$

This, together with (4.1), gives

$$\mathbb{E} \langle y(T), \eta \rangle_{L^2(\mathbb{T}^n)} = 0, \quad \forall \eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n)).$$

Taking $\eta = y(T)$ yields $y(T) = 0$ almost surely. $\square$

Proof of Theorem 1.2. By linearity, it suffices to consider $y_0 = 0$. Let $\eta \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))$ be orthogonal to every terminal state reached by controls in $L^\infty_{\mathbb{F}}(0,T;L^2(\Omega;L^2(G)))$. By (4.1),

$$\mathbb{E} \int_0^T \langle u(t), z^\eta(t) \rangle_{L^2(G)} dt = 0, \quad \forall u \in L^\infty_{\mathbb{F}}(0,T;L^2(\Omega;L^2(G))). \quad (4.2)$$

Taking $u = \mathbf{1}_G z^\eta$ in (4.2) gives

$$\|z^\eta(t)\|_{L^2_{\mathcal{F}_t}(\Omega;L^2(G))} = 0 \quad \text{for almost every } t < T.$$

The interpolation estimate (3.2), with $s = T$, implies that

$$\|z^\eta(t)\|_{L^2_{\mathcal{F}_t}(\Omega;L^2(\mathbb{T}^n))} = 0 \quad \text{for almost every } t < T. \quad (4.3)$$

Since $z^\eta(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0,T]; L^2(\mathbb{T}^n)))$, we get from (4.3) that $\eta = z^\eta(T) = 0$, $\mathbb{P}$ -a.s. Thus the reachable set has trivial orthogonal complement and is dense in $L^2_{\mathcal{F}_T}(\Omega; L^2(\mathbb{T}^n))$, which proves the claim. $\square$

5 The analytic coefficient class: criterion and examples

We give a Fourier criterion for Assumption 1.1 and use it to construct coefficients with arbitrarily high spatial frequencies and infinitely many nonzero Fourier modes.

Proposition 5.1 (Fourier realization of the analytic class). *Fix $R > 0$. Let $\{a_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^n}$ be a family of complex-valued progressively measurable processes satisfying*

$$a_{-\mathbf{k}}(\omega, t) = \overline{a_{\mathbf{k}}(\omega, t)}, \quad \mathbf{k} \in \mathbb{Z}^n, \quad (5.1)$$

and define

$$\mathcal{C}(\omega, t) := \sum_{\mathbf{k} \in \mathbb{Z}^n} |a_{\mathbf{k}}(\omega, t)| e^{R|\mathbf{k}|_1}. \quad (5.2)$$

Suppose that, for some $C > 0$,

$$\|\mathcal{C}\|_{L^\infty_{\mathbb{F}}(\Omega; L^2(0,T))} \leq C^2. \quad (5.3)$$

Then, after possibly modifying the $a_{\mathbf{k}}$ on a common progressive $(\mathbb{P} \otimes dt)$ -null set, the series

$$\tilde{b}(\omega, t, \zeta) := \sum_{\mathbf{k} \in \mathbb{Z}^n} a_{\mathbf{k}}(\omega, t) e^{i\mathbf{k} \cdot \zeta} \quad (5.4)$$

admits a real trace b that satisfies Assumption 1.1 and

$$\|b(\omega, t, \cdot)\|_{\mathcal{A}_R(\mathbb{T}^n)} \leq \mathcal{C}(\omega, t) \quad (5.5)$$

for every $(\omega, t) \in \Omega \times (0, T)$.

Proof. The finite sums in (5.2) increase to $\mathcal{C}(\cdot, \cdot)$, so $\mathcal{C}(\cdot, \cdot)$ is progressively measurable. By (5.3), the set $\{\mathcal{C} = \infty\}$ is a progressive $(\mathbb{P} \otimes dt)$ -null set. We set each $a_{\mathbf{k}}$ equal to zero on this common set. Conjugate symmetry is preserved, and the modified weighted sum, still denoted by $\mathcal{C}(\cdot, \cdot)$, is finite everywhere and still satisfies (5.3).

Step 1. We first establish uniform convergence on the polystrip. Fix (ω, t) and define

$$\tilde{b}_N(\omega, t, \zeta) = \sum_{|\mathbf{k}|_\infty \leq N} a_{\mathbf{k}}(\omega, t) e^{i\mathbf{k} \cdot \zeta}, \quad N \in \mathbb{N}.$$

For $\zeta = x + iy \in S_R^n$,

$$|e^{i\mathbf{k} \cdot \zeta}| = e^{-\mathbf{k} \cdot y} \leq e^{\sum_{j=1}^n |k_j| |y_j|} \leq e^{R|\mathbf{k}|_1}.$$

Consequently, for $M > N$,

$$\begin{aligned} \sup_{\zeta \in S_R^n} |\tilde{b}_M(\omega, t, \zeta) - \tilde{b}_N(\omega, t, \zeta)| &\leq \sum_{N < |\mathbf{k}|_\infty \leq M} |a_{\mathbf{k}}(\omega, t)| e^{R|\mathbf{k}|_1} \\ &\leq \sum_{|\mathbf{k}|_\infty > N} |a_{\mathbf{k}}(\omega, t)| e^{R|\mathbf{k}|_1} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

The last limit follows from $\mathcal{C}(\omega, t) < \infty$ and is uniform in $M > N$. Thus (5.4) converges absolutely and uniformly on S_R^n for each fixed (ω, t) . Each partial sum is holomorphic and 2π -periodic in each coordinate, so these properties pass to the limit. Moreover,

$$\sup_{\zeta \in S_R^n} |\tilde{b}(\omega, t, \zeta)| \leq \sum_{\mathbf{k} \in \mathbb{Z}^n} |a_{\mathbf{k}}(\omega, t)| e^{R|\mathbf{k}|_1} = \mathcal{C}(\omega, t).$$

Step 2. We now verify the coefficient assumptions. For $x \in \mathbb{T}^n$, conjugate symmetry and the change of index $\mathbf{k} \mapsto -\mathbf{k}$ yield

$$\overline{\tilde{b}_N(\omega, t, x)} = \sum_{|\mathbf{k}|_\infty \leq N} \overline{a_{\mathbf{k}}(\omega, t) e^{-i\mathbf{k} \cdot x}} = \sum_{|\mathbf{k}|_\infty \leq N} a_{-\mathbf{k}}(\omega, t) e^{-i\mathbf{k} \cdot x} = \tilde{b}_N(\omega, t, x).$$

Passing to the limit shows that the real trace $b(\omega, t, x) = \tilde{b}(\omega, t, x)$ is real-valued. Each $\tilde{b}_N$ is $\mathbb{F} \otimes \mathcal{B}(S_R^n)$ -measurable, since its coefficients are progressively measurable and its spatial factors are continuous. The pointwise limit $\tilde{b}$ has the same measurability, and its restriction b to the real torus is progressively measurable.

We have therefore shown that $b(\omega, t, \cdot) \in \mathcal{A}_R(\mathbb{T}^n)$ for every (ω, t) , and that

$$\beta(\omega, t) = \|b(\omega, t, \cdot)\|_{\mathcal{A}_R(\mathbb{T}^n)} = \sup_{\zeta \in S_R^n} |\tilde{b}(\omega, t, \zeta)| \leq \mathcal{C}(\omega, t).$$

By continuity in ζ , this supremum can be taken over a fixed countable dense subset of S_R^n , so β is progressively measurable. Finally, (5.3) yields

$$\|\beta\|_{L_{\mathbb{F}}^\infty(\Omega; L^2(0, T))} \leq \|\mathcal{C}\|_{L_{\mathbb{F}}^\infty(\Omega; L^2(0, T))} \leq C.$$

This proves (5.5) and all the requirements of Assumption 1.1. $\square$

By choosing finitely many nonzero coefficients in Proposition 5.1, one obtains random trigonometric polynomials with arbitrary frequencies. A deterministic conjugate-symmetric sequence with infinitely many nonzero coefficients and finite weighted sum in (5.2) yields a profile that is not a trigonometric polynomial. Consequently, Assumption 1.1 encompasses an infinite-dimensional class of spatial profiles, whose multiplication operators generally mix Fourier modes.

Example 5.1 (A random, time- and space-dependent coefficient). Let $M > 0$ and define

$$\tilde{b}(\omega, t, \zeta) = \frac{M}{2} \tanh(W(t)) + \frac{M}{2n \cosh R} \tanh\left(\int_0^t W(r) dr\right) \sum_{j=1}^n \cos \zeta_j.$$

The random amplitudes are bounded and progressively measurable, which ensures the required joint measurability of $\tilde{b}$. Since $|\cos \zeta_j| \leq \cosh R$ on S_R^n , the real trace $b(\omega, t, \cdot)$ of $\tilde{b}(\omega, t, \cdot)$ belongs to $\mathcal{A}_R(\mathbb{T}^n)$ and satisfies $\|b(\omega, t, \cdot)\|_{\mathcal{A}_R(\mathbb{T}^n)} \leq M$. Hence Assumption 1.1 holds with $C_1 = M\sqrt{T}$.

Example 5.2 (A random coefficient with infinitely many Fourier modes). Let $M > 0$. For $\zeta \in S_R^n$, set

$$F_R(\zeta) = \frac{1}{n} \sum_{j=1}^n \sum_{m=1}^{\infty} 2^{-m} e^{-mR} \cos(m\zeta_j), \quad A(t) = \frac{2 + \tanh\left(\int_0^t W(r) dr\right)}{3}.$$

We define the coefficient by

$$\tilde{b}(\omega, t, \zeta) = \frac{M}{2} \tanh(W(t)) + \frac{M}{2} A(t) F_R(\zeta).$$

The random amplitudes are bounded and progressively measurable, with $1/3 < A(t) < 1$. Since $2^{-m} e^{-mR} |\cos(m\zeta_j)| \leq 2^{-m}$ on S_R^n , the series defining F_R converges absolutely and uniformly to a periodic holomorphic function with real trace and supremum norm at most 1. The partial sums ensure the required joint measurability of $\tilde{b}$. Thus the real trace $b(\omega, t, \cdot)$ belongs to $\mathcal{A}_R(\mathbb{T}^n)$ and satisfies

$$\|b(\omega, t, \cdot)\|_{\mathcal{A}_R(\mathbb{T}^n)} \leq \frac{M}{2} |\tanh(W(t))| + \frac{M}{2} A(t) \leq M.$$

Hence Assumption 1.1 holds with $C_1 = M\sqrt{T}$. Since $A(t) > 1/3$, the Fourier coefficients at $\pm m\mathbf{e}_j$ are nonzero for every $m \geq 1$ and $1 \leq j \leq n$.

The analytic norm may also be unbounded in time for a profile with infinitely many nonzero Fourier modes. With F_R as above, set

$$b(t, x) = M(T - t)^{-1/4} F_R(x), \quad 0 < t < T.$$

Then $b(t, \cdot) \in \mathcal{A}_R(\mathbb{T}^n)$ and

$$\beta(t) \leq M(T - t)^{-1/4}, \quad \int_0^T \beta(t)^2 dt \leq 2M^2\sqrt{T}.$$

Hence Assumption 1.1 holds with $C_1^2 = 2M^2\sqrt{T}$. Since $F_R(0) > 0$, this coefficient is genuinely unbounded as $t \uparrow T$.

6 Concluding remarks

In this paper we established null and approximate controllability for a stochastic parabolic equation with a space-dependent analytic noise coefficient, together with an observability inequality with an L^1 -in-time mean-square observation norm. The examples in Section 5 show that Assumption 1.1 accommodates random trigonometric polynomials, profiles with infinitely many nonzero Fourier modes, and coefficients that are unbounded in time. These coefficients must nevertheless admit bounded holomorphic extensions to a complex

strip of fixed positive width, with the corresponding analytic norms satisfying a square-integrability bound in time that is uniform over sample paths.

The core of our proof follows the chain

$$\begin{aligned} &\text{analytic energy estimate (Theorem 2.1)} \longrightarrow \text{propagation of smallness (Proposition 3.2)} \\ &\longrightarrow \text{one-time interpolation (Corollary 3.3)} \longrightarrow \text{observability (Theorem 1.3)}. \end{aligned}$$

The key step that propagates information from the local observation set G to the whole torus $\mathbb{T}^n$ is the analytic propagation of smallness used in Proposition 3.2. Its proof rests on the multidimensional analytic propagation theorem [1, Theorem 4], which provides quantitative propagation of smallness from a set of positive measure under a uniform analytic a priori bound. In contrast with qualitative uniqueness, this quantitative form allows local observations to be combined with the global analytic bound, thereby yielding the interpolation estimate required for observability.

A natural question is whether spatial analyticity can be weakened to Gevrey regularity of order greater than one. Two obstacles prevent a direct extension of the present argument. First, general nonanalytic Gevrey functions do not admit holomorphic extensions to complex neighborhoods. Gevrey regularity alone therefore does not provide the holomorphic extensions and multiplier identities required by the complex-translation method. An approach based on almost-analytic extensions would require additional control of the errors caused by their failure to be holomorphic. Second, Gevrey classes of order greater than one contain nonzero compactly supported functions, which may vanish on a nonempty open set. Consequently, a propagation-of-smallness estimate of the type in Proposition 3.2, controlling the global norm by observations on an arbitrary set of positive measure, cannot hold throughout such a class. This second obstruction already occurs in the scalar setting and is unrelated to the passage to Hilbert-space-valued functions.

Another direction is to extend the results to domains with boundary. The periodic setting makes complex translations compatible with Fourier projections and eliminates boundary contributions in the energy estimates. On a domain with boundary, complex translations generally do not preserve the prescribed boundary conditions. An extension would therefore require regularity estimates compatible with those conditions and an appropriate treatment of the resulting boundary terms.

These obstacles delimit the scope of the present method; they do not establish the failure of single-control controllability under weaker spatial regularity or on domains with boundary. In particular, the failure of propagation of smallness for a general function class does not preclude quantitative unique continuation or observability for solutions of the adjoint equation. A central task for future work is to develop alternative estimates that exploit the equation's structure and handle spatially dependent noise coupling without observing the martingale integrand.

Provided that the corresponding well-posedness results and duality identity hold, an observability estimate involving only the adjoint state, with the same temporal observation norm, would imply null controllability via the duality argument in Section 4.

A Proof of Lemma 2.2

The proof of Lemma 2.2 follows a standard Galerkin stability argument. We include it here for completeness.

Proof of Lemma 2.2. Fix $s \in (0, T]$, and let (z_N, Z_N) be the solution of (2.2) with $\xi = z(s)$. Set $e_N = z_N - P_N z$ and $h_N = Z_N - P_N Z$. Subtracting the projection of (1.8) from (2.2) yields

$$\begin{cases} de_N = (-\Delta e_N - P_N(bh_N) + f_N) dr + h_N dW(r) & \text{in } (0, s), \\ e_N(s) = 0, \end{cases}$$

where $f_N = P_N(b(Z - P_N Z))$. By (1.5) and the strong convergence of P_N ,

$$\mathbb{E} \left(\int_0^s \|f_N(r)\|_{L^2(\mathbb{T}^n)} dr \right)^2 \leq C_1^2 \mathbb{E} \int_0^s \|Z(r) - P_N Z(r)\|_{L^2(\mathbb{T}^n)}^2 dr \longrightarrow 0. \quad (\text{A.1})$$

A standard adaptation of the stability argument in [9, Section 4.4], using (1.6) and (1.5), yields

$$\mathbb{E} \sup_{0 \leq r \leq s} \|e_N(r)\|_{L^2(\mathbb{T}^n)}^2 \leq C \mathbb{E} \left(\int_0^s \|f_N(r)\|_{L^2(\mathbb{T}^n)} dr \right)^2,$$

where, throughout this proof, C depends only on T and C_1 . To estimate the integral norms, define $\gamma(r) = \exp(2 \int_0^r \beta(\tau)^2 d\tau)$. Itô's formula gives

$$\begin{aligned} d(\gamma \|e_N\|_{L^2(\mathbb{T}^n)}^2) &= \gamma (2 \|\nabla e_N\|_{L^2(\mathbb{T}^n)}^2 + \|h_N\|_{L^2(\mathbb{T}^n)}^2 + 2\beta^2 \|e_N\|_{L^2(\mathbb{T}^n)}^2 - 2 \operatorname{Re} \langle b h_N, e_N \rangle_{L^2(\mathbb{T}^n)} \\ &\quad + 2 \operatorname{Re} \langle f_N, e_N \rangle_{L^2(\mathbb{T}^n)}) dr + 2\gamma \operatorname{Re} \langle e_N, h_N \rangle_{L^2(\mathbb{T}^n)} dW(r). \end{aligned}$$

Since $1 \leq \gamma \leq e^{2C_1^2}$, the same integrability argument as in the proof of Proposition 2.3 shows that the stochastic integral has zero expectation. Using

$$2|\langle b h_N, e_N \rangle_{L^2(\mathbb{T}^n)}| \leq \frac{1}{2} \|h_N\|_{L^2(\mathbb{T}^n)}^2 + 2\beta^2 \|e_N\|_{L^2(\mathbb{T}^n)}^2,$$

integrating from 0 to s , and taking expectations, noting that $e_N(s) = 0$, we obtain from the Cauchy–Schwarz inequality for the forcing term, together with the preceding stability estimate and $1 \leq \gamma \leq e^{2C_1^2}$,

$$\mathbb{E} \sup_{0 \leq r \leq s} \|e_N(r)\|_{L^2(\mathbb{T}^n)}^2 + \mathbb{E} \int_0^s (\|\nabla e_N(r)\|_{L^2(\mathbb{T}^n)}^2 + \|h_N(r)\|_{L^2(\mathbb{T}^n)}^2) dr \leq C \mathbb{E} \left(\int_0^s \|f_N(r)\|_{L^2(\mathbb{T}^n)} dr \right)^2. \quad (\text{A.2})$$

Combining (A.1) and (A.2) yields

$$\lim_{N \rightarrow \infty} e_N = 0 \quad \text{in } L^2_{\mathbb{F}}(\Omega; C([0, s]; L^2(\mathbb{T}^n))) \cap L^2_{\mathbb{F}}(0, s; H^1(\mathbb{T}^n))$$

and

$$\lim_{N \rightarrow \infty} h_N = 0 \quad \text{in } L^2_{\mathbb{F}}(0, s; L^2(\mathbb{T}^n)).$$

Consequently, (2.4) holds. $\square$

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