

Exact Controllability for a Refined Stochastic Plate Equation*

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Abstract The classical stochastic plate equation suffers from a lack of exact controllability, even with controls that are effective in both the drift and diffusion terms and on the boundary. To address this issue, a one-dimensional refined stochastic plate equation was previously proposed and established as exactly controllable in [28]. In this paper, we establish the exact controllability of the multidimensional refined stochastic plate equation with two interior controls and two boundary controls by a new global Carleman estimate. Furthermore, we show that at least one boundary control and the action of two interior controls are necessary for exact controllability.

Keywords Stochastic plate equation, exact controllability, observability estimate, Carleman estimate.

2000 MR Subject Classification 93B05, 93B07.

1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Write $\mathbb{F}$ for the progressive σ -field with respect to $\mathbf{F}$. Let H be a Banach space. Denote by $L^2_{\mathbb{F}_t}(\Omega; H)$ the space of all $\mathcal{F}_t$ -measurable random variables ξ such that $\mathbb{E}|\xi|_H^2 < \infty$; by $L^2_{\mathbb{F}}(0, T; H)$ the space consisting of all H -valued $\mathbf{F}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty$; by $L^\infty_{\mathbb{F}}(0, T; H)$ the space consisting of all H -valued $\mathbf{F}$ -adapted bounded processes; and by $C_{\mathbb{F}}([0, T]; L^2(\Omega; H))$ the space consisting of all H -valued $\mathbf{F}$ -adapted processes $X(\cdot)$ such that $X(\cdot) : [0, T] \rightarrow L^2_{\mathbb{F}}(\Omega; H)$ is continuous. All these spaces are Banach spaces with the canonical norms (e.g., [24, Section 2.6]).

Let $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded domain with a C^4 boundary Γ . Set $Q = (0, T) \times G$ and $\Sigma = (0, T) \times \Gamma$. Denote by $\nu(x) = (\nu^1(x), \dots, \nu^n(x))$ the unit outward normal vector of Γ at point x .

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Consider the following refined stochastic plate equation:

$$\begin{cases} dy = \hat{y}dt + (a_3y + f)dW(t) & \text{in } Q, \\ d\hat{y} + \Delta^2 y dt = (a_1y + a_2 \cdot \nabla y + a_5g)dt + (a_4y + g)dW(t) & \text{in } Q, \\ y = h_1, \frac{\partial y}{\partial \nu} = h_2 & \text{on } \Sigma, \\ (y(0), \hat{y}(0)) = (y_0, \hat{y}_0) & \text{in } G. \end{cases} \quad (1.1)$$

Here, $(y_0, \hat{y}_0) \in H^{-1}(G) \times (H^3(G) \cap H_0^2(G))^*$ (where $(H^3(G) \cap H_0^2(G))^*$ is the dual space of $H^3(G) \cap H_0^2(G)$ with respect to the pivot space $L^2(G)$), the coefficients

$$a_1, a_3, a_4 \in L_{\mathbb{F}}^\infty(0, T; W^{1,\infty}(G)), \quad a_2 \in L_{\mathbb{F}}^\infty(0, T; W^{2,\infty}(G; \mathbb{R}^n)), \quad a_5 \in L_{\mathbb{F}}^\infty(0, T; W^{3,\infty}(G)).$$

and the controls

$$(f, g, h_1, h_2) \in L_{\mathbb{F}}^2(0, T; H^{-1}(G)) \times L_{\mathbb{F}}^2(0, T; (H^3(G) \cap H_0^2(G))^*) \times L_{\mathbb{F}}^2(0, T; L^2(\Gamma)) \\ \times L_{\mathbb{F}}^2(0, T; H^{-1}(\Gamma)).$$

Remark 1.1. The influence of the control g on the drift term is reflected by the term a_5g . Specifically, if a control g is introduced in the diffusion term, the term a_5g will appear as a side effect. This leads to technical difficulties in the study of the exact controllability of (1.1). It is worth noting that based on the same reasoning, an additional term of a_6f could be included in the drift term of the first equation of (1.1). However, we do not know how to handle this term now.

The control system (1.1) is a nonhomogeneous boundary value problem. Its solution is understood in the sense of transposition. For the readers' convenience, we recall it briefly below. A systematic introduction to that can be found in [24, Section 7.2].

First, we introduce the following reference equation:

$$\begin{cases} dz = \hat{z}dt + (Z - a_5z)dW(t) & \text{in } Q_\tau, \\ d\hat{z} + \Delta^2 z dt = [(a_1 - \operatorname{div} a_2 - a_4a_5)z - a_2 \cdot \nabla z - a_3\hat{Z} + a_4Z]dt + \hat{Z}dW(t) & \text{in } Q_\tau, \\ z = \frac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma_\tau, \\ (z(\tau), \hat{z}(\tau)) = (z^\tau, \hat{z}^\tau) & \text{in } G, \end{cases} \quad (1.2)$$

where $\tau \in (0, T]$, $Q_\tau \triangleq (0, \tau) \times G$, $\Sigma_\tau \triangleq (0, \tau) \times \Gamma$, and $(z^\tau, \hat{z}^\tau) \in L_{\mathcal{F}_\tau}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G))$. By the classical well-posedness result for backward stochastic evolution equations (e.g., [24, Section 4.2]), we know that (1.2) admits a unique weak solution

$$(z, Z, \hat{z}, \hat{Z}) \in L_{\mathbb{F}}^2(\Omega; C([0, \tau]; (H^3(G) \cap H_0^2(G)))) \times L_{\mathbb{F}}^2(0, \tau; (H^3(G) \cap H_0^2(G))) \\ \times L_{\mathbb{F}}^2(\Omega; C([0, \tau]; H_0^1(G))) \times L_{\mathbb{F}}^2(0, \tau; H_0^1(G)).$$

Furthermore, for $0 \leq s, t \leq \tau$, it holds that

$$\begin{aligned} & |(z(t), \hat{z}(t))|_{L_{\mathcal{F}_t}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_t}^2(\Omega; H_0^1(G))} \\ & \leq C(|(z(s), \hat{z}(s))|_{L_{\mathcal{F}_s}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_s}^2(\Omega; H_0^1(G))} \\ & \quad + |(Z, \hat{Z})|_{L_{\mathbb{F}}^2(0, \tau; H^3(G) \cap H_0^2(G)) \times L_{\mathbb{F}}^2(0, \tau; H_0^1(G))}). \end{aligned} \quad (1.3)$$

Here and in what follows, we denote by C a generic positive constant depending on G , T , τ and a_i , $i = 1, \dots, 5$, whose value may vary from line to line.

Next, we give the following hidden regularity for solutions to (1.2).

Proposition 1.1 Let $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; H^1_0(G))$. Then the solution $(z, Z, \hat{z}, \hat{Z})$ of (1.2) satisfies $|\nabla \Delta z|_{\Gamma} \in L^2_{\mathbb{F}}(0, \tau; L^2(\Gamma))$. Furthermore,

$$|\nabla \Delta z|_{L^2_{\mathbb{F}}(0, \tau; L^2(\Gamma))} \leq C|(z^\tau, \hat{z}^\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; H^1_0(G))}.$$

Proof of Proposition 1.1 is put in Section 2.

Now we are in a position to give the definition of the transposition solution to (1.1).

Definition 1.1. A pair of stochastic processes $(y, \hat{y}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; (H^3(G) \cap H^2_0(G))^*))$ is a transposition solution to (1.1) if for any $\tau \in (0, T]$ and $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; H^1_0(G))$, we have

$$\begin{aligned} & \mathbb{E} \langle \hat{y}(\tau), z^\tau \rangle_{(H^3(G) \cap H^2_0(G))^*, H^3(G) \cap H^2_0(G)} - \langle \hat{y}_0, z(0) \rangle_{(H^3(G) \cap H^2_0(G))^*, H^3(G) \cap H^2_0(G)} \\ & - \mathbb{E} \langle y(\tau), \hat{z}^\tau \rangle_{H^{-1}(G), H^1_0(G)} + \langle y_0, \hat{z}(0) \rangle_{H^{-1}(G), H^1_0(G)} \\ & = -\mathbb{E} \int_0^\tau \langle f, \hat{Z} \rangle_{H^{-1}(G), H^1_0(G)} dt + \mathbb{E} \int_0^\tau \langle g, Z \rangle_{(H^3(G) \cap H^2_0(G))^*, H^3(G) \cap H^2_0(G)} dt \\ & + \mathbb{E} \int_0^\tau \int_\Gamma \frac{\partial \Delta z}{\partial \nu} h_1 d\Gamma dt - \mathbb{E} \int_0^\tau \langle h_2, \Delta z \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} dt. \end{aligned}$$

Here, $(z, Z, \hat{z}, \hat{Z})$ solves (1.2).

Combining Theorem 1.1 and the well-posedness for stochastic evolution equation with an unbounded control operator in the sense of transposition solution (e.g., [24, Theorem 7.12]), we immediately get the following well-posedness result for (1.1).

Proposition 1.2 For each $(y_0, \hat{y}_0) \in H^{-1}(G) \times (H^3(G) \cap H^2_0(G))^*$, the system (1.1) admits a unique transposition solution $(y, \hat{y})$. Moreover,

$$\begin{aligned} & |(y, \hat{y})|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; (H^3(G) \cap H^2_0(G))^*))} \\ & \leq C(|y_0|_{H^{-1}(G)} + |\hat{y}_0|_{(H^3(G) \cap H^2_0(G))^*} + |f|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))} + |g|_{L^2_{\mathbb{F}}(0, T; (H^3(G) \cap H^2_0(G))^*)} \\ & + |h_1|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))} + |h_2|_{L^2_{\mathbb{F}}(0, T; H^{-1}(\Gamma))}). \end{aligned}$$

Now we give the definition of the exact controllability for (1.1).

Definition 1.2. The system (1.1) is called exactly controllable at time T if for any $(y_0, \hat{y}_0) \in H^{-1}(G) \times (H^3(G) \cap H^2_0(G))^*$ and $(y_1, \hat{y}_1) \in L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G)) \times L^2_{\mathcal{F}_T}(\Omega; (H^3(G) \cap H^2_0(G))^*)$, there exist controls

$$\begin{aligned} & (f, g, h_1, h_2) \in L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; (H^3(G) \cap H^2_0(G))^*) \times L^2_{\mathbb{F}}(0, T; L^2(\Gamma)) \\ & \times L^2_{\mathbb{F}}(0, T; H^{-1}(\Gamma)) \end{aligned}$$

such that the solution $(y, \hat{y})$ to (1.1) satisfies that $(y(T, \cdot), \hat{y}(T, \cdot)) = (y_1, \hat{y}_1)$, $\mathbb{P}$ -a.s.

Remark 1.2. In the definition of exact controllability for (1.1), we set the state space to be $L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G)) \times L^2_{\mathcal{F}_T}(\Omega; (H^3(G) \cap H_0^2(G))^*)$. However, it would be more natural to choose the state space as $L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(G))$. Furthermore, the controls $g \in L^2_{\mathbb{P}}(0, T; (H^3(G) \cap H_0^2(G))^*)$ and $h_2 \in L^2_{\mathbb{P}}(0, T; H^{-1}(\Gamma))$ are highly irregular. It is expected that more regular controls will be needed to achieve the desired goal, although the method to do so is currently unknown. Notably, even for the deterministic plate equation, the existing results (such as [15]) only prove the exact controllability in the space $H^{-1}(G) \times (H^3(G) \cap H_0^2(G))^*$ with controls in the Dirichlet and Neumann boundary conditions.

The main result of this paper is the following.

Theorem 1.3 The system (1.1) is exactly controllable at any time $T > 0$.

Remark 1.3. Similar to the derivation process in [23, 28], the refined stochastic plate equation (1.1) can be obtained from the classical stochastic plate equation:

$$\begin{cases} dy_t + \Delta^2 y dt = (a_1 y + a_2 \cdot \nabla y + f) dt + (a_4 y + g) dW(t) & \text{in } Q, \\ y = h_1, \frac{\partial y}{\partial \nu} = h_2 & \text{on } \Sigma, \\ (y(0), y_t(0)) = (y_0, y_1) & \text{in } G. \end{cases} \quad (1.4)$$

Here, (y_0, y_1) are the initial data, and f, g, h_1, h_2 are controls.

The system (1.4) is widely used in structural engineering for modeling beams, bridges, and other structures, as demonstrated in various studies such as [2–4, 12]. For instance, (1.4) can be utilized to describe fluttering or large-amplitude vibration of an elastic panel induced by aerodynamic forces that are perturbed by random fluctuations, as explored in [3]. However, as proven in Theorem 4.1 of [28] and Theorem 2.1 of [23], the system (1.4) cannot be exactly controllable for any $T > 0$. Therefore, motivated by the negative controllability result and following a similar derivation process as in [23], we investigate a refined stochastic plate equation, namely (1.1).

We put four controls in the system (1.1). Similarly to Theorem 2.3 in [23], one can find that boundary controls h_1 and h_2 in (1.1) cannot be dropped simultaneously, and internal controls f and g must be acted on the whole domain G . More precisely, we have the following result.

Theorem 1.4 The system (1.1) is not exactly controllable at any time $T > 0$ provided that one of the following three conditions is satisfied:

- (1). $a_3 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$, $G \setminus \overline{G_0} \neq \emptyset$ and $\text{supp } f \subset G_0$;
- (2). $a_4 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$, $G \setminus \overline{G_0} \neq \emptyset$ and $\text{supp } g \subset G_0$;
- (3). $h_1 = h_2 = 0$.

Remark 1.4. It is worth investigating whether one of the boundary controls can be removed for the system (1.1), as has been done for the deterministic plate equation in [15, 18]. However, we currently do not have a method for achieving this for the stochastic case.

Remark 1.5. By letting $G = (0, 1)$, we can deduce from Theorem 3.2 that the system (1.1) is exactly controllable with controls in any nonempty subset of the boundary Γ . In fact, thanks to Remark 1.7, for any $(y_0, \hat{y}_0)$ and $(y_1, \hat{y}_1)$ satisfying Definition 1.2, one can find $(f, g, h_1, h_2) \in$

$L^2_{\mathbb{P}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{P}}(0, T; (H^3(G) \cap H^2_0(G))^*) \times (L^2_{\mathbb{P}}(0, T))^2$ such that the solution $(y, \hat{y})$ to (1.1), where the boundary conditions are

$$y(\cdot, 0) = h_1, \quad y_x(\cdot, 0) = h_2, \quad y(\cdot, 1) = 0, \quad y_x(\cdot, 1) = 0, \quad \text{on } (0, T)$$

satisfies that $(y(T, \cdot), \hat{y}(T, \cdot)) = (y_1, \hat{y}_1)$, $\mathbb{P}$ -a.s. In the multidimensional case, it is worth to studying whether (1.1) is still exactly controllable under the assumption that $(h_1, h_2) \in L^2_{\mathbb{P}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathbb{P}}(0, T; L^2(\Gamma_0))$, where Γ_0 is a nonempty subset of Γ .

By a standard duality argument, Theorem 1.3 is equivalent to the following observability estimate (e.g., [24, Theorem 7.17]).

Theorem 1.5 There exists a constant $C > 0$ such that for every $(z^T, \hat{z}^T) \in L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))$, it holds that

$$\begin{aligned} & |(z^T, \hat{z}^T)|^2_{L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))} \\ & \leq C \mathbb{E} \int_{\Sigma} (|\nabla \Delta z|^2 + |\Delta z|^2) d\Gamma dt + C |(Z, \hat{Z})|^2_{L^2_{\mathbb{F}}(0, T; H^3(G) \cap H^2_0(G)) \times L^2_{\mathbb{F}}(0, T; H^1_0(G))}, \end{aligned}$$

where $(z, Z, \hat{z}, \hat{Z})$ is the solution to the equation (1.2) with $\tau = T$, $z(T) = z^T$, and $\hat{z}(T) = \hat{z}^T$.

Remark 1.6. In order to estimate $|\nabla \Delta z|$ on the boundary of the dual system (1.2), we may need better regularity than $H^2_0(G) \times L^2(G)$ for the initial data, as suggested by the sharp trace estimate established for the deterministic plate equation in [17]. Therefore, we choose the final data of (1.2) in $L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))$.

Remark 1.7. Let $G = (0, 1)$. By choosing $x_0 > 1$ in (3.4), from the proof of Theorem 3.2, we can deduce that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 (s\lambda\xi|\hat{v}_x|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\hat{v}|^2 + \lambda|v_{xxx}|^2 + s^3\lambda^4\xi^3|v_{xx}|^2 + s^4\lambda^6\xi^4|v_x|^2 \\ & \quad + s^6\lambda^8\xi^6|v|^2) dx dt \\ & \leq C \mathbb{E} \int_Q \theta^2 (s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|f_{1x}|^2 + s^2\lambda^2\xi^2|f_{1xx}|^2 + f_2^2 + s^6\lambda^6\xi^6 g_1^2 \\ & \quad + s^4\lambda^4\xi^4|g_{1x}|^2 + s^2\lambda^2\xi^2|g_{1xx}|^2 + |g_{1xxx}|^2 + s^2\lambda^2\xi^2|g_2|^2) dx dt \\ & \quad + C \mathbb{E} \int_0^T \theta^2 (s\lambda\xi|v_{xxx}(0)|^2 + s^3\lambda^3\xi^3|v_{xx}(0)|^2) dt. \end{aligned}$$

In fact, on Σ , we have

$$\begin{aligned} V_1 \cdot \nu &= -2s\lambda\xi\eta_x\nu w_{xxx}^2 - 10s^3\lambda^3\xi^3\eta_x^3\nu w_{xx}^2 \\ &\geq Cs\lambda\xi(w_{xxx}^2(1) - w_{xxx}^2(0)) + Cs^3\lambda^3\xi^3(w_{xx}^2(1) - w_{xx}^2(0)), \end{aligned}$$

and

$$V_2 \cdot \nu \geq O(e^{C\lambda})(w_{xxx}^2(1) + w_{xxx}^2(0)) + s^2 O(e^{C\lambda})(w_{xx}^2(1) + w_{xx}^2(0)).$$

The remainder of the proof follows a similar approach to that of Theorem 1.5. Hence, we arrive the following observability inequality.

$$|(z^T, \hat{z}^T)|^2_{L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))}$$

$$\leq C\mathbb{E} \int_0^T (|z_{xxx}(0)|^2 + |z_{xx}(0)|^2)dt + C|(Z, \hat{Z})|_{L^2_{\mathbb{F}}(0,T;H^3(G)\cap H^2_0(G))\times L^2_{\mathbb{F}}(0,T;H^1_0(G))}.$$

Remark 1.8. One can also consider the exact controllability of a refined stochastic plate equation when the boundary controls are as follows:

$$y = h_1 \quad \text{and} \quad \Delta y = h_2 \quad \text{on } \Sigma.$$

By the standard duality argument and the technique of transforming the controllability of the forward stochastic equation into the controllability of a backward equation (e.g., [24, Section 7.5]), we need to prove the observability estimate for the following equation:

$$\begin{cases} dz = \hat{z}dt - a_5 z dW(t) & \text{in } Q, \\ d\hat{z} + \Delta^2 z dt = [(a_1 - \operatorname{div} a_2)z - a_2 \cdot \nabla z - a_4 a_5 z]dt & \text{in } Q, \\ z = \Delta z = 0 & \text{on } \Sigma, \\ (z(0), \hat{z}(0)) = (z_0, \hat{z}_0) & \text{in } G. \end{cases} \quad (1.5)$$

Following the idea in [29], sufficiently regular solutions of (1.5) can formally be rewritten as two coupled stochastic Schrödinger equations. In fact, letting $u = i\hat{z} + \Delta z$, we have

$$idz + \Delta z dt = udt - ia_5 z dW(t),$$

and

$$-idu + \Delta udt = [(a_1 - \operatorname{div} a_2)z - a_2 \cdot \nabla z - a_4 a_5 z]dt + i\Delta(a_5 z)dW(t).$$

Clearly, it holds that $z = u = 0$ on Σ . This formal reduction does not by itself prove observability under the coefficients assumed in this paper. Indeed, on the Navier boundary,

$$\Delta(a_5 z) = 2(\partial_\nu a_5)(\partial_\nu z),$$

which need not vanish. Thus the transformed noise need not belong to $H^1_0(G)$, and multiplication by a_5 need not preserve $\{\eta \in H^3(G) \cap H^1_0(G) : \Delta\eta \in H^1_0(G)\}$. For example, on $(0, 1)$, $a_5(x) = x$ and $z(x) = \sin(\pi x)$ give $(a_5 z)''(0) = 2\pi$. Additional boundary compatibility, such as $\partial_\nu a_5 = 0$, and a separate observability argument would therefore be needed for this Navier extension. No such extension is asserted here.

A plenty of works on exact controllability for deterministic plate equations exist in the literature, including works such as [1, 5, 9–11, 15, 18, 19, 26, 29] and the references therein. However, to the best of our knowledge, the work in [28] is the only published study that investigates the exact controllability of stochastic beam equations. In their study, the authors demonstrate that the equation given in (1.1) is exactly controllable when G is an interval. In this article, we employ a stochastic Carleman estimate to prove Theorem 1.5. This type of estimate is one of the most useful tools in studying the controllability of stochastic partial differential equations, as evident from previous works such as [6, 8, 20–22, 24, 27] and references therein. However, we note that [28] is the only published study to use this method for investigating the exact controllability of stochastic beam equations.

The rest of this paper is organized as follows. In Section 2, we provide some preliminaries. Section 3 is devoted to establishing a Carleman estimate for the adjoint equation (1.2). By means of that Carleman estimate, we prove Theorem 1.5 in Section 4. At last, Section 5 is addressed to the proof of Theorem 1.4.

2 Some preliminary results

This section provides some preliminary results. In the rest of this paper, the notation $y_{x_i} \equiv y_{x_i}(x) = \partial y(x)/\partial x_i$ will be used for simplicity, where x_i is the i -th coordinate of a generic point $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$. In a similar manner, we use notations z_{x_j} , v_{x_j} , etc. for the partial derivatives of z and v with respect to x_j .

We first prove the hidden regularity for solutions to (1.2).

Proof of Theorem 1.1. For any $\rho \triangleq (\rho^1, \dots, \rho^n) \in C^2(\mathbb{R}^{n+1}; \mathbb{R}^n)$, by Itô's formula and (1.2), we have

$$\begin{aligned}
& \sum_{j=1}^n (2\rho \cdot \nabla \Delta z \Delta z_{x_j} - \rho^j |\nabla \Delta z|^2)_{x_j} dt \\
&= -\operatorname{div} \rho |\nabla \Delta z|^2 dt + 2\rho \cdot \nabla \Delta z [(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2 \cdot \nabla z + a_4 Z - a_3 \hat{Z}] dt \\
&\quad + 2\rho \cdot \nabla \Delta z \hat{Z} dW(t) + 2\nabla \Delta z D\rho \cdot \nabla \Delta z dt - 2d(\rho \cdot \nabla \Delta z \hat{z}) + 2\rho_t \cdot \nabla \Delta z \hat{z} dt \\
&\quad + 2\rho \cdot \nabla \Delta (Z - a_5 z) \hat{Z} dt + 2\rho \cdot \nabla \Delta (Z - a_5 z) \hat{z} dW(t) + 2\operatorname{div}(\hat{z} \rho \nabla^2 \hat{z}) dt \\
&\quad - 2\operatorname{div}(\hat{z} \nabla \hat{z} D\rho) dt + 2\Delta \rho \cdot \nabla \hat{z} \hat{z} dt + 2\nabla \hat{z} D\rho \cdot \nabla \hat{z} dt - \operatorname{div}(\rho |\nabla \hat{z}|^2) dt + \operatorname{div} \rho |\nabla \hat{z}|^2 dt.
\end{aligned} \tag{2.1}$$

Since $\Gamma \in C^3$, there exists a vector field $\zeta \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ such that $\zeta = \nu$ on Γ (e.g., [13, Lemma 2.1]). Setting $\rho = \zeta$, integrating (2.1) in Q_τ , and taking expectation on Ω , we have

$$\begin{aligned}
& \mathbb{E} \int_{\Sigma_\tau} \left(2 \left| \frac{\partial \Delta z}{\partial \nu} \right|^2 - |\nabla \Delta z|^2 \right) d\Gamma dt \\
&= \mathbb{E} \int_{\Sigma_\tau} \sum_{j=1}^n (2\rho \cdot \nabla \Delta z \Delta z_{x_j} - \rho^j |\nabla \Delta z|^2) \nu^j d\Gamma dt \\
&= -2\mathbb{E} \int_G \rho \cdot \nabla \Delta z^\tau \hat{z}^\tau dx + 2\mathbb{E} \int_G \rho \cdot \nabla \Delta z(0) \hat{z}(0) dx \\
&\quad + \mathbb{E} \int_{Q_\tau} \left\{ -\operatorname{div} \rho |\nabla \Delta z|^2 + 2\rho \cdot \nabla \Delta z [(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2 \cdot \nabla z + a_4 Z - a_3 \hat{Z}] \right. \\
&\quad \quad + 2\nabla \Delta z D\rho \cdot \nabla \Delta z + 2\rho_t \cdot \nabla \Delta z \hat{z} + 2\rho \cdot \nabla \Delta (Z - a_5 z) \hat{Z} + 2\Delta \rho \cdot \nabla \hat{z} \hat{z} \\
&\quad \quad \left. + 2\nabla \hat{z} D\rho \cdot \nabla \hat{z} + \operatorname{div} \rho |\nabla \hat{z}|^2 \right\} dx dt.
\end{aligned}$$

This implies

$$\begin{aligned}
& 2 \left| \frac{\partial \Delta z}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0, \tau; L^2(\Gamma))}^2 - |\nabla \Delta z|_{L^2_{\mathbb{F}}(0, \tau; L^2(\Gamma))}^2 \\
&\leq C(|z|_{L^2_{\mathbb{F}}(\Omega; C([0, \tau]; H^3(G) \cap H_0^2(G)))}^2 + |\hat{z}|_{L^2_{\mathbb{F}}(\Omega; C([0, \tau]; H_0^1(G)))}^2 + |Z|_{L^2_{\mathbb{F}}(0, \tau; H^3(G) \cap H_0^2(G))}^2 \\
&\quad + |\hat{Z}|_{L^2_{\mathbb{F}}(0, \tau; L^2(G))}^2) \\
&\leq C|(z^\tau, \hat{z}^\tau)|_{L^2_{\mathbb{F}_\tau}(\Omega; H^3(G) \cap H_0^2(G)) \times L^2_{\mathbb{F}_\tau}(\Omega; H_0^1(G))}^2.
\end{aligned} \tag{2.2}$$

Denote by ∇_σ the tangential gradient on Γ . We have

$$|\nabla \Delta z|^2 = \left| \frac{\partial \Delta z}{\partial \nu} \right|^2 + |\nabla_\sigma \Delta z|^2,$$

which, together with (2.2), implies that

$$\left| \frac{\partial \Delta z}{\partial \nu} \right|_{L_{\mathbb{F}}^2(0, \tau; L^2(\Gamma))}^2 \leq |\nabla_{\sigma} \Delta z|_{L_{\mathbb{F}}^2(0, \tau; L^2(\Gamma))}^2 + C|(z^{\tau}, \hat{z}^{\tau})|_{L_{\mathcal{F}\tau}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}\tau}^2(\Omega; H_0^1(G))}^2.$$

Now we are going to prove

$$|\nabla_{\sigma} \Delta z|_{L_{\mathbb{F}}^2(0, \tau; L^2(\Gamma))}^2 \leq C|(z^{\tau}, \hat{z}^{\tau})|_{L_{\mathcal{F}\tau}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}\tau}^2(\Omega; H_0^1(G))}^2. \quad (2.3)$$

Using boundary charts and a partition of unity, choose finitely many tangential vector fields $\mathcal{B}_{\alpha}$ spanning the tangent space on each chart. They can be chosen with C^3 coefficients on the C^4 domain. The estimates below apply to each field; for their derivation we write $\mathcal{B}$ for one of them

$$\left\{ \begin{array}{l} \mathcal{B} = \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} = \text{a first-order differential operator (time independent)} \\ \text{with coefficients } b_i \in C^3(\overline{G}) \text{ and such that } \mathcal{B} \text{ is tangential to } \Gamma, \text{ i.e.,} \\ \sum_{i=1}^n b_i(x) \nu^i = 0 \text{ on } \Gamma. \end{array} \right. \quad (2.4)$$

These fields are local pullbacks of tangential coordinate derivatives, multiplied by cutoffs. In the commutator $[\Delta^2, \mathcal{B}]$, derivatives of the coefficients of order four are interpreted weakly: integrating one derivative onto the test function gives the continuous map $H^3(G) \rightarrow H^{-1}(G)$ with only C^3 coefficient bounds.

Define

$$p \triangleq \mathcal{B}z \in C_{\mathbb{F}}([0, \tau]; L^2(\Omega; H^2(G))), \quad \hat{p} \triangleq \mathcal{B}\hat{z} \in C_{\mathbb{F}}([0, \tau]; L^2(\Omega; L^2(G))).$$

From (1.2), we have

$$\left\{ \begin{array}{ll} dp = \hat{p}dt + g_1 dW(t) & \text{in } Q_{\tau}, \\ d\hat{p} + \Delta^2 p dt = f_2 dt + g_2 dW(t) & \text{in } Q_{\tau}, \\ p = 0, \frac{\partial p}{\partial \nu} = \left[\frac{\partial}{\partial \nu}, \mathcal{B} \right] z & \text{on } \Sigma_{\tau}, \\ (p(\tau), \hat{p}(\tau)) = (\mathcal{B}z^{\tau}, \mathcal{B}\hat{z}^{\tau}) & \text{in } G, \end{array} \right. \quad (2.5)$$

where

$$g_1 = \mathcal{B}(Z - a_5 z), \quad g_2 = \mathcal{B}\hat{Z},$$

and

$$f_2 = \mathcal{B}[(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2 \cdot \nabla z - a_3 \hat{Z} + a_4 Z] + [\Delta^2, \mathcal{B}]z,$$

and $[\cdot, \cdot]$ denotes the commutator of two operators.

From (2.4), we get that

$$|[\Delta^2, \mathcal{B}]z|_{L_{\mathbb{F}}^2(0, \tau; H^{-1}(G))} \leq C|(z^{\tau}, \hat{z}^{\tau})|_{L_{\mathcal{F}\tau}^2(\Omega; (H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}\tau}^2(\Omega; H_0^1(G))} \quad (2.6)$$

and that

$$\mathbb{E} \int_{\Sigma_{\tau}} |\nabla_{\sigma} \Delta z|^2 d\Gamma dt \leq C \sum_{\alpha} \mathbb{E} \int_{\Sigma_{\tau}} |\mathcal{B}_{\alpha} \Delta z|^2 d\Gamma dt$$

$$\leq C \sum_{\alpha} \mathbb{E} \int_{\Sigma_{\tau}} (|\Delta(\mathcal{B}_{\alpha} z)|^2 + |[\mathcal{B}_{\alpha}, \Delta]z|^2) d\Gamma dt. \quad (2.7)$$

By (2.4) again, we find that

$$\begin{aligned} \mathbb{E} \int_{\Sigma_{\tau}} |[\mathcal{B}, \Delta]z|^2 d\Gamma dt &\leq C \mathbb{E} \int_0^{\tau} |z|_{H^3(G)}^2 dt \\ &\leq C |(z^{\tau}, \hat{z}^{\tau})|_{L_{\mathcal{F}_{\tau}}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_{\tau}}^2(\Omega; H_0^1(G))}. \end{aligned} \quad (2.8)$$

Combining (2.7) and (2.8), to show (2.3), it suffices to prove the following estimate for each field and then sum over the finite family:

$$\mathbb{E} \int_{\Sigma_{\tau}} |\Delta p|^2 d\Gamma dt \leq C |(z^{\tau}, \hat{z}^{\tau})|_{L_{\mathcal{F}_{\tau}}^2(\Omega; (H^3(G) \cap H_0^2(G))) \times L_{\mathcal{F}_{\tau}}^2(\Omega; H_0^1(G))}. \quad (2.9)$$

For any $\rho \triangleq (\rho^1, \dots, \rho^n) \in C^2(\mathbb{R}^{n+1}; \mathbb{R}^n)$, by Itô's formula and (2.5), we have

$$\begin{aligned} &\sum_{i,k,l=1}^n (\rho^i p_{x_k x_k} p_{x_l x_l})_{x_i} dt \\ &= \sum_{i,k,l=1}^n [\rho_{x_i}^i p_{x_k x_k} p_{x_l x_l} + (2\rho_{x_l}^i p_{x_i x_k x_k} p_{x_l})_{x_l} - (2\rho_{x_l}^i p_{x_i x_k} p_{x_l})_{x_k} + 2\rho_{x_l x_k}^i p_{x_i x_k} p_{x_l} \\ &\quad + 2\rho_{x_l}^i p_{x_i x_k} p_{x_l x_k} - (2\rho_{x_l}^i p_{x_l x_k x_k} p_{x_l})_{x_i} + (2\rho_{x_i}^i p_{x_l x_k} p_{x_l})_{x_k} - 2\rho_{x_i x_k}^i p_{x_l x_k} p_{x_l} \\ &\quad - 2\rho_{x_i}^i p_{x_l x_k} p_{x_l x_k} + (2\rho_{x_l}^i p_{x_l x_k x_k} p_{x_i})_{x_l} - (2\rho_{x_l}^i p_{x_l x_k} p_{x_i})_{x_k} + 2\rho_{x_l x_k}^i p_{x_l x_k} p_{x_i} \\ &\quad + 2\rho_{x_l}^i p_{x_l x_k} p_{x_i x_k}] dt - 2\rho \cdot \nabla p(f_2 dt + g_2 dW(t)) + d(2\rho \cdot \nabla p \hat{p}) - 2\rho_t \cdot \nabla p \hat{p} dt \\ &\quad + \operatorname{div} \rho \hat{p}^2 dt - 2\rho \cdot d\nabla p d\hat{p} - \operatorname{div}(\rho \hat{p}^2) dt - 2\rho \cdot \nabla g_1 \hat{p} dW(t). \end{aligned} \quad (2.10)$$

Setting $\rho = \zeta$, integrating (2.10) in Q_{τ} , and taking expectation on Ω , we have

$$\begin{aligned} &\mathbb{E} \int_{\Sigma_{\tau}} |\Delta p|^2 d\Gamma dt \\ &= \mathbb{E} \int_{\Sigma_{\tau}} \sum_{i,k,l=1}^n \rho^i \nu^i p_{x_k x_k} p_{x_l x_l} d\Gamma dt \\ &= 2\mathbb{E} \int_G \rho \cdot \nabla p^{\tau} \hat{p}^{\tau} dx - 2\mathbb{E} \int_G \rho \cdot \nabla p(0) \hat{p}(0) dx \\ &\quad + \mathbb{E} \int_{Q_{\tau}} (\operatorname{div} \rho \hat{p}^2 - 2\rho \cdot \nabla p f_2 - 2\rho_t \cdot \nabla p \hat{p} - 2\rho \cdot \nabla g_1 g_2) dx dt \\ &\quad + \mathbb{E} \int_{Q_{\tau}} \sum_{i,k,l=1}^n (\rho_{x_i}^i p_{x_k x_k} p_{x_l x_l} + 2\rho_{x_l x_k}^i p_{x_i x_k} p_{x_l} + 2\rho_{x_l}^i p_{x_i x_k} p_{x_l x_k} - 2\rho_{x_i x_k}^i p_{x_l x_k} p_{x_l} \\ &\quad - 2\rho_{x_i}^i p_{x_l x_k} p_{x_l x_k} + 2\rho_{x_l x_k}^i p_{x_l x_k} p_{x_i} + 2\rho_{x_l}^i p_{x_l x_k} p_{x_i x_k}) dx dt \\ &\quad + 2\mathbb{E} \int_{\Sigma_{\tau}} \sum_{i,k,l=1}^n (\rho_{x_i}^i p_{x_l x_k} p_{x_l} \nu^k - \rho_{x_l}^i p_{x_i x_k} p_{x_l} \nu^k + \rho_{x_l}^i p_{x_l x_k} p_{x_i} \nu^l - \rho_{x_l}^i p_{x_l x_k} p_{x_i} \nu^k) d\Gamma dt, \end{aligned} \quad (2.11)$$

which, together with (2.4)–(2.6), implies (2.9). Then we complete the proof. $\square$

Next, we give a pointwise weighted identity, which will play an important role in the proof of [Theorem 3.2](#).

We have the following fundamental identity.

Theorem 2.1 Let v be an $H^4(G)$ -valued Itô process and $\hat{v}$ be an $H^2(G)$ -valued Itô process such that

$$dv = (\hat{v} + f_1)dt + g_1 dW(t)$$

for some $f_1 \in L^2_{\mathbb{F}}(0, T; H^2_0(G))$ and $g_1 \in L^2_{\mathbb{F}}(0, T; H^4(G) \cap H^2_0(G))$. Let $\eta \in C^6(\mathbb{R} \times \mathbb{R}^n)$. Set $\theta = e^\ell$, $\ell = s\xi$, $\xi = e^{\lambda\eta}$, $w = \theta v$, and $\hat{w} = \theta\hat{v} + \ell_t w$. Then, for any $t \in [0, T]$ and a.e. $(x, \omega) \in G \times \Omega$,

$$\begin{aligned} & 2\theta I_2(d\hat{v} + \Delta^2 v dt) - 2\operatorname{div}(V_1 + V_2)dt \\ &= 2I_2^2 dt + 2I_2 I_3 + 2(M_1 + M_2)dt + 2 \sum_{i,j,k,l=1}^n \Lambda_1^{ijkl} w_{x_i x_j} w_{x_k x_l} dt + 2 \sum_{i,j=1}^n \Lambda_2^{ij} w_{x_i} w_{x_j} dt \quad (2.12) \\ &+ 2\Lambda_3 w^2 dt + 2\Lambda_4 + 2d(I_2 \hat{w}) - 2 \sum_{i=1}^n (\Phi_1^i \hat{w} d\Delta w)_{x_i}. \end{aligned}$$

Here

$$I_1 = \Delta^2 w dt + \Psi_2 \Delta w dt + \sum_{i,j=1}^n \Psi_3^{ij} w_{x_i x_j} dt + \sum_{i=1}^n \Psi_4^i w_{x_i} dt + \sum_{i=1}^n \Psi_5^i w_{x_i} dt + \Psi_6 w dt + d\hat{w}, \quad (2.13)$$

with

$$\begin{cases} \Psi_2 = 2s^2 \lambda^2 \xi^2 |\nabla \eta|^2, & \Psi_3^{ij} = 4s^2 \lambda^2 \xi^2 \eta_{x_i} \eta_{x_j}, \\ \Psi_4^i = 12s^2 \lambda^3 \xi^2 |\nabla \eta|^2 \eta_{x_i} + 4s^2 \lambda^2 \xi^2 \Delta \eta \eta_{x_i}, \\ \Psi_5^i = \sum_{j=1}^n 8s^2 \lambda^2 \xi^2 \eta_{x_i x_j} \eta_{x_j}, & \Psi_6 = s^4 \lambda^4 \xi^4 |\nabla \eta|^4, \end{cases} \quad i, j = 1, \dots, n \quad (2.14)$$

$$I_2 = \sum_{i=1}^n \Phi_1^i \Delta w_{x_i} + \Phi_2 \Delta w + \sum_{i,j=1}^n \Phi_3^{ij} w_{x_i x_j} + \sum_{i=1}^n \Phi_4^i w_{x_i} + \Phi_5 w, \quad (2.15)$$

with

$$\begin{cases} \Phi_1^i = -4s\lambda\xi\eta_{x_i}, & \Phi_2 = -2s\lambda^2\xi|\nabla\eta|^2 - 2s\lambda\xi\Delta\eta - \lambda, \\ \Phi_3^{ij} = 4s\lambda\xi\eta_{x_i x_j} - 4s\lambda^2\xi\eta_{x_i}\eta_{x_j}, & \Phi_4^i = -4s^3\lambda^3\xi^3|\nabla\eta|^2\eta_{x_i}, \\ \Phi_5 = -6s^3\lambda^4\xi^3|\nabla\eta|^4 - 12s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta) \\ \quad - 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta - s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4, \end{cases} \quad i, j = 1, \dots, n \quad (2.16)$$

$$\begin{aligned} I_3 &= -8s\lambda\xi \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} dt - 4\nabla\Delta\ell \cdot \nabla w dt + 4(\nabla\ell \cdot \nabla\Delta\ell)w dt + 2|\nabla^2\ell|^2 w dt \\ &- \Delta^2 \ell w dt + |\Delta\ell|^2 w dt + 8s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)w dt + s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4 w dt + \lambda\Delta w dt \quad (2.17) \\ &- \ell_t \theta f_1 dt - \ell_t \theta g_1 dW(t) + (\ell_t^2 - \ell_{tt})w dt - 2\ell_t \hat{w} dt, \end{aligned}$$

$$\begin{aligned}
V_1 &= [V_1^1, V_1^2, \dots, V_1^n], \quad V_2 = [V_2^1, V_2^2, \dots, V_2^n], \\
V_1^j &= \sum_{i,k=1}^n \left[\sum_{l=1}^n \Phi_1^l w_{x_k x_k x_l} w_{x_i x_i x_j} - \frac{1}{2} \sum_{l=1}^n \Phi_1^j w_{x_k x_k x_l} w_{x_i x_i x_l} + \frac{1}{2} \Psi_2 \Phi_1^j w_{x_i x_i} w_{x_k x_k} \right. \\
&\quad + \sum_{l=1}^n \Psi_3^{ik} \Phi_1^l w_{x_i x_k} w_{x_l x_j} - \frac{1}{2} \sum_{l=1}^n \Psi_3^{ij} \Phi_1^l w_{x_i x_k} w_{x_k x_l} + \Phi_4^k w_{x_i x_i x_j} w_{x_k} - \Phi_4^k w_{x_i x_j} w_{x_i x_k} \\
&\quad \left. + \frac{1}{2} \Phi_4^j w_{x_i x_k}^2 + \left(\Psi_2 \Phi_4^k \delta_{ij} - \frac{1}{2} \Psi_2 \Phi_4^j \delta_{ik} - \frac{1}{2} \Psi_6 \Phi_1^j \delta_{ik} + \frac{1}{2} \Psi_3^{ij} \Phi_4^k \right) w_{x_i} w_{x_k} \right], \\
V_2^j &= \sum_{i,l,r,m=1}^n \Theta_1^{ijlrm} w_{x_i x_i x_l} w_{x_r x_m} + \sum_{i,k,l,r=1}^n \Theta_2^{ijklr} w_{x_i x_k} w_{x_l x_r} + \sum_{i,k,l=1}^n \Theta_3^{ijkl} w_{x_i x_k} w_{x_l} \\
&\quad + \sum_{i,k=1}^n \Theta_4^{ijk} w_{x_i} w_{x_k} + \sum_{i=1}^n \Theta_5 w_{x_i x_i x_j} w + \sum_{i,k=1}^n \Theta_6^{ijk} w_{x_i x_k} w + \sum_{i=1}^n \Theta_7^{ij} w_{x_i} w + \Theta_8^j w^2 + \Theta_9^j,
\end{aligned}$$

$$\begin{aligned}
M_1 &= 8s\lambda^2 \xi |\nabla \Delta w \cdot \nabla \eta|^2 + 32s^3 \lambda^4 \xi^3 |\nabla^2 w \nabla \eta \nabla \eta|^2 + 48s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 w \nabla \eta) (\nabla^2 w \nabla \eta) \\
&\quad + 16s^3 \lambda^3 \xi^3 (\nabla^2 w \nabla \eta \nabla \eta) \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} - 16s^3 \lambda^4 \xi^3 |\nabla \eta|^2 |\nabla^2 w \nabla \eta|^2 \\
&\quad + 6s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\nabla^2 w|^2 + 4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla^2 w|^2 + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta |\nabla^2 w|^2 \\
&\quad + 2s^3 \lambda^4 \xi^3 |\nabla \eta|^4 |\Delta w|^2 - 4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\Delta w|^2 - 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta |\Delta w|^2 \\
&\quad + 40s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 + 64s^5 \lambda^5 \xi^5 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla w \cdot \nabla \eta|^2 \\
&\quad - 16s^5 \lambda^6 \xi^5 |\nabla \eta|^6 |\nabla w|^2 + 8s^7 \lambda^8 \xi^7 |\nabla \eta|^8 w^2 + \lambda |\nabla \Delta w|^2 - s^3 \lambda^{\frac{7}{2}} \xi^3 |\nabla \eta|^4 |\Delta w|^2 \\
&\quad + 4s^5 \lambda^{\frac{11}{2}} \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 + 2s^5 \lambda^{\frac{11}{2}} \xi^5 |\nabla \eta|^6 |\nabla w|^2 - s^7 \lambda^{\frac{15}{2}} \xi^7 |\nabla \eta|^8 w^2,
\end{aligned}$$

$$\begin{aligned}
M_2 &= \left(-\frac{1}{2} \sum_{i=1}^n \Phi_1^i x_i + \Phi_2 \right) |\nabla \hat{w}|^2 + \sum_{i,j=1}^n (-\Phi_1^i x_j + \Phi_3^{ij}) \hat{w}_{x_i} \hat{w}_{x_j} \\
&\quad + \frac{1}{2} \left[\sum_{i,j=1}^n (\Phi_1^i x_i x_j x_j - \Phi_2 x_i x_i \delta_{ij} - \Phi_3^{ij} x_i x_j + \Phi_4^i \delta_{ij}) - 2\Phi_5 \right] \hat{w}^2 \\
&\quad - \sum_{i,k,j=1}^n (\Psi_5^i \Phi_1^k)_{x_j} w_{x_i} w_{x_k x_j} - 4s^2 \lambda^3 \xi^2 |\nabla^2 w \nabla \eta|^2 - 2s^2 \lambda^3 \xi^2 |\nabla \eta|^2 |\Delta w|^2 \\
&\quad + s^4 \lambda^5 \xi^4 |\nabla \eta|^4 |\nabla w|^2,
\end{aligned}$$

$$\begin{aligned}
\Lambda_1^{ijkl} &= \Phi_{3x_i x_j}^{kl} + \Phi_{3x_i x_l}^{kj} + \sum_{r=1}^n \left(\frac{1}{2} \Phi_{2x_r x_r} \delta_{ij} \delta_{kl} - \Phi_{3x_j x_r}^{kr} \delta_{il} - \Phi_{3x_i x_r}^{kr} \delta_{lj} + \frac{1}{2} \sum_{m=1}^n \Phi_{3x_r x_m}^{rm} \delta_{ik} \delta_{lj} \right), \\
\Lambda_2^{ij} &= \sum_{k=1}^n \left[-\Phi_{4x_i x_k x_k}^j + \frac{1}{2} \Phi_{4x_i x_j x_k}^k + (\Psi_2 \Phi_3^{jk})_{x_i x_k} - \frac{1}{2} (\Psi_2 \Phi_3^{ij})_{x_k x_k} \right. \\
&\quad - \sum_{l=1}^n \frac{1}{2} (\Psi_2 \Phi_3^{kl} \delta_{ij})_{x_k x_l} + (\Psi_3^{ik} \Phi_2)_{x_k x_j} - \sum_{l=1}^n \frac{1}{2} (\Psi_3^{kl} \Phi_2 \delta_{ij})_{x_k x_l} - \frac{1}{2} (\Psi_3^{ij} \Phi_2)_{x_k x_k} \\
&\quad \left. + \sum_{l=1}^n (\Psi_3^{ik} \Phi_3^{jl})_{x_k x_l} - \sum_{l=1}^n \frac{1}{2} (\Psi_3^{ij} \Phi_3^{kl})_{x_k x_l} - \sum_{l=1}^n \frac{1}{2} (\Psi_3^{kl} \Phi_3^{ij})_{x_k x_l} + \frac{1}{2} (\Psi_4^i \Phi_1^j)_{x_k x_k} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(\Psi_4^k \Phi_2 \delta_{ij})_{x_k} - (\Psi_4^i \Phi_3^{jk})_{x_k} + \frac{1}{2}(\Psi_4^k \Phi_3^{ij})_{x_k} \\
& + \frac{1}{2}(\Psi_5^k \Phi_2 \delta_{ij})_{x_k} - (\Psi_5^i \Phi_3^{jk})_{x_k} + \frac{1}{2}(\Psi_5^k \Phi_3^{ij})_{x_k} \Big] - 2\Phi_{5x_i x_j} - (\Psi_4^i \Phi_2)_{x_j} - (\Psi_5^i \Phi_2)_{x_j}, \\
\Lambda_3 = & \sum_{i=1}^n \left[\sum_{j=1}^n \frac{1}{2} \Phi_{5x_i x_i x_j x_j} + \frac{1}{2}(\Psi_2 \Phi_5)_{x_i x_i} + \sum_{j=1}^n \frac{1}{2}(\Psi_3^{ij} \Phi_5)_{x_i x_j} - \frac{1}{2}(\Psi_4^i \Phi_5)_{x_i} - \frac{1}{2}(\Psi_5^i \Phi_5)_{x_i} \right. \\
& \left. - \sum_{j=1}^n \frac{1}{2}(\Psi_6 \Phi_1^i)_{x_i x_j x_j} + \frac{1}{2}(\Psi_6 \Phi_2)_{x_i x_i} + \sum_{j=1}^n \frac{1}{2}(\Psi_6 \Phi_3^{ij})_{x_i x_j} \right], \\
\Lambda_4 = & \sum_{i,j=1}^n \left[-\Phi_{1t}^i w_{x_i x_j x_j} + (-\Phi_{2t} \delta_{ij} - \Phi_{3t}^{ij}) w_{x_i x_j} \right] \hat{w} dt - \sum_{i=1}^n \Phi_{4t}^i w_{x_i} \hat{w} dt - \Phi_{5t} w \hat{w} dt \\
& + \sum_{i,j=1}^n (\Phi_{1x_i}^i \hat{w} + \Phi_1^i \hat{w}_{x_i} - \Phi_2 \hat{w} \delta_{ij}) [(\theta f_1)_{x_j x_j} dt + (\theta g_1)_{x_j x_j} dW(t)] - \Phi_5 \hat{w} (\theta f_1 dt + \theta g_1 dW(t)) \\
& - \sum_{i,j=1}^n \Phi_3^{ij} \hat{w} [(\theta f_1)_{x_i x_j} dt + (\theta g_1)_{x_i x_j} dW(t)] - \sum_{i=1}^n \Phi_4^i \hat{w} [(\theta f_1)_{x_i} dt + (\theta g_1)_{x_i} dW(t)] \\
& - \sum_{i,j=1}^n (\Phi_1^i dw_{x_i x_j x_j} + \Phi_2 \delta_{ij} dw_{x_i x_i} + \Phi_3^{ij} dw_{x_i x_j} + \Phi_4^i \delta_{ij} dw_{x_i}) d\hat{w} - \Phi_5 dw d\hat{w},
\end{aligned}$$

and

$$\begin{aligned}
\Theta_1^{ijklm} &= \Phi_2 \delta_{lj} \delta_{rm} + \Phi_3^{rm} \delta_{lj} - \Phi_3^{mj} \delta_{lr} + \Phi_3^{lr} \delta_{mj}, \\
\Theta_2^{ijklr} &= -\frac{1}{2} \Phi_{2x_j} \delta_{ik} \delta_{lr} - \Phi_{3x_i}^{lr} \delta_{kj} + \Phi_{3x_k}^{rj} \delta_{il} + \sum_{m=1}^n \Phi_{3x_m}^{rm} \delta_{kj} \delta_{il} - \frac{1}{2} \sum_{m=1}^n \Phi_{3x_m}^{jm} \delta_{il} \delta_{kr} - \Phi_{3x_i}^{kl} \delta_{rj}, \\
\Theta_3^{ijkl} &= -\Phi_{4x_i}^l \delta_{kj} - \Phi_5 \delta_{ik} \delta_{lj} + \Psi_2 \Phi_3^{ik} \delta_{lj} - \Psi_2 \Phi_3^{lj} \delta_{kl} + \Psi_3^{lj} \Phi_2 \delta_{ik} - \Psi_3^{lk} \Phi_2 \delta_{ij} + \Psi_3^{ik} \Phi_3^{lj} \\
& - \Psi_3^{ij} \Phi_3^{kl} + \Psi_4^l \Phi_1^k \delta_{ij} + \Psi_5^l \Phi_1^k \delta_{ij}, \\
\Theta_4^{ijk} &= \Phi_{4x_i x_j}^k - \frac{1}{2} \Phi_{4x_i x_k}^j + 2\Phi_{5x_i} \delta_{kj} - \Phi_{5x_j} \delta_{ik} - (\Psi_2 \Phi_3^{jk})_{x_i} + \frac{1}{2}(\Psi_2 \Phi_3^{ik})_{x_j} \\
& + \sum_{l=1}^n \frac{1}{2}(\Psi_2 \Phi_3^{jl})_{x_l} \delta_{ik} - \sum_{l=1}^n (\Psi_3^{il} \Phi_2)_{x_l} \delta_{kj} + \frac{1}{2} \sum_{l=1}^n (\Psi_3^{lj} \Phi_2)_{x_l} \delta_{ik} + \frac{1}{2}(\Psi_3^{ik} \Phi_2)_{x_j} \\
& - \sum_{l=1}^n (\Psi_3^{ij} \Phi_3^{kl})_{x_l} + \frac{1}{2} \sum_{l=1}^n (\Psi_3^{ik} \Phi_3^{jl})_{x_l} + \frac{1}{2} \sum_{l=1}^n (\Psi_3^{lj} \Phi_3^{ik})_{x_l} - \frac{1}{2}(\Psi_4^i \Phi_1^k)_{x_j} + \Psi_4^i \Phi_2 \delta_{kj} \\
& - \frac{1}{2} \Psi_4^j \Phi_2 \delta_{ik} + \Psi_4^i \Phi_3^{kj} - \frac{1}{2} \Psi_4^j \Phi_3^{ik} + \Psi_5^i \Phi_2 \delta_{kj} - \frac{1}{2} \Psi_5^j \Phi_2 \delta_{ik} + \Psi_5^i \Phi_3^{kj} - \frac{1}{2} \Psi_5^j \Phi_3^{ik}, \\
\Theta_5 &= \Phi_5, \\
\Theta_6^{ijk} &= -\Phi_{5x_j} \delta_{ik} + \Psi_6 \Phi_1^k \delta_{ij}, \\
\Theta_7^{ij} &= \sum_{k=1}^n \Phi_{5x_k x_k} \delta_{ij} + \Psi_2 \Phi_5 \delta_{ij} + \Psi_3^{ij} \Phi_5 - (\Psi_6 \Phi_1^i)_{x_j} + \Psi_6 \Phi_2 \delta_{ij} + \Psi_6 \Phi_3^{ij}, \\
\Theta_8^j &= -\frac{1}{2} \sum_{i=1}^n \Phi_{5x_i x_i x_j} - \frac{1}{2}(\Psi_2 \Phi_5)_{x_j} - \frac{1}{2} \sum_{i=1}^n (\Psi_3^{ij} \Phi_5)_{x_i} + \frac{1}{2} \Psi_4^j \Phi_5 + \frac{1}{2} \Psi_5^j \Phi_5 \\
& + \frac{1}{2} \sum_{i=1}^n (\Psi_6 \Phi_1^j)_{x_i x_i} - \frac{1}{2}(\Psi_6 \Phi_2)_{x_j} - \frac{1}{2} \sum_{i=1}^n (\Psi_6 \Phi_3^{ij})_{x_i} + \frac{1}{2} \Psi_6 \Phi_4^j,
\end{aligned}$$

$$\begin{aligned} \Theta_9^j = & \sum_{i=1}^n \left(\Phi_{1x_i}^i \hat{w}_{x_j} \hat{w} + \Phi_1^i \hat{w}_{x_i} \hat{w}_{x_j} - \frac{1}{2} \Phi_{1x_i x_j}^i \hat{w}^2 - \frac{1}{2} \Phi_1^j \hat{w}_{x_i}^2 - \Phi_3^{ij} \hat{w} \hat{w}_{x_i} + \frac{1}{2} \Phi_{3x_i}^{ij} \hat{w}^2 \right) \\ & - \Phi_2 \hat{w} \hat{w}_{x_j} + \frac{1}{2} \Phi_{2x_j} \hat{w}^2 - \frac{1}{2} \Phi_4^j \hat{w}^2. \end{aligned}$$

Remark 2.1. Since we do not put any further assumptions on v and $\hat{v}$, the identity (2.12) seems to be very complicated. For solutions to (1.2) or (1.5), many terms, such as V_1 and V_2 , will merge or vanish by means of the boundary conditions. Furthermore, compared with energy terms, such as $8s\lambda^2\xi|\nabla\Delta w \cdot \nabla\eta|^2$, $2s^3\lambda^4\xi^3|\nabla\eta|^4|\Delta w|^2$, $40s^5\lambda^6\xi^5|\nabla\eta|^4|\nabla w \cdot \nabla\eta|^2$, $\lambda|\nabla\Delta w|^2$, many terms in (2.12) is of no great importance. We only need to estimate their orders with respect to s and λ . Hence, an effective way to simplify (2.12) is that we do not write these terms explicitly and just claim that the order of s and λ for them are lower than the terms with a “good sign”. However, we do not do this since we want to provide the full details for readers, and particularly for beginners.

Although the form of the identity (2.12) is very complex, its proof follows from some basic computations. To avoid defocusing the main theme of this paper, we put the proof of Theorem 2.1 in the appendix.

3 Carleman estimate for the adjoint equation

This section is devoted to establishing a Carleman estimate for a backward stochastic plate equation. To this end, we first introduce the weight function η .

For any $\delta > 0$, $T > 0$ and $0 < \varepsilon_1 < \frac{1}{2}$, we choose $x_0 \in \mathbb{R}^n \setminus \overline{G}$ such that

$$R_0 \triangleq \min_{x \in \overline{G}} |x - x_0|^2 > 2\delta, \quad (3.1)$$

and choose sufficiently large β satisfying

$$R_1 \triangleq \max_{x \in \overline{G}} |x - x_0|^2 \leq \beta \varepsilon_1^2 T^2 - \delta. \quad (3.2)$$

We also choose sufficiently small ε_0 with $0 < \varepsilon_0 < \varepsilon_1$ such that

$$R_0 - \beta \varepsilon_0^2 T^2 \geq \delta. \quad (3.3)$$

Let

$$\eta(t, x) = |x - x_0|^2 - \beta \left(t - \frac{T}{2} \right)^2. \quad (3.4)$$

From (3.1)–(3.4), it is easy to see that η satisfies the following conditions.

Condition 3.1.

- (1). $|\eta(t, x)|_{C^2(\overline{Q})} \leq C_1$.
- (2). $|\nabla\eta(t, x)| \geq C_2 > 0$, $\forall (t, x) \in \overline{Q}$.
- (3). For all (t, x) in $J_1 \triangleq [(0, T/2 - \varepsilon_1 T) \cup (T/2 + \varepsilon_1 T, T)] \times G$, it holds that $\eta(t, x) \leq -\delta$.
- (4). For all (t, x) in $J_2 \triangleq (T/2 - \varepsilon_0 T, T/2 + \varepsilon_0 T) \times G$, it holds that $\eta(t, x) \geq \delta$.

Recall that $\theta = e^\ell$, $\ell = s\xi$ and $\xi = e^{\lambda\eta}$. With η given by (3.4), the functions ℓ and θ are also defined.

We also need the following known result.

Lemma 3.1 [7, Theorem 2.1] Let $q \in H_0^2(G)$. Then there exists a constant $C > 0$ independent of s and λ , and parameter $\widehat{\lambda} > 1$ and $\widehat{s} > 1$ such that, for all $\lambda \geq \widehat{\lambda}$ and $s \geq \widehat{s}$,

$$s^4 \lambda^6 \int_Q \xi^4 \theta^2 (s^2 \lambda^2 \xi^2 |q|^2 + |\nabla q|^2) dx dt \leq C \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\Delta q|^2 dx dt.$$

We have the following Carleman estimate.

Theorem 3.2 There exist constants $C > 0$ and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, one can find $s_0 = s_0(\lambda) > 0$ so that for any $s \geq s_0$,

$$(v, \hat{v}) \in L_{\mathbb{F}}^2(\Omega; C([0, T]; H^4(G) \cap H_0^2(G))) \times L_{\mathbb{F}}^2(\Omega; C([0, T]; H_0^2(G))),$$

and

$$f_1, f_2, g_2 \in L_{\mathbb{F}}^2(0, T; H_0^2(G)), \quad g_1 \in L_{\mathbb{F}}^2(0, T; H^4(G) \cap H_0^2(G))$$

satisfying

$$\begin{cases} dv = (\hat{v} + f_1)dt + g_1 dW(t) & \text{in } Q, \\ d\hat{v} + \Delta^2 v dt = f_2 dt + g_2 dW(t) & \text{in } Q, \\ v = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \end{cases} \quad (3.5)$$

and

$$v(0, \cdot) = v(T, \cdot) = \hat{v}(0, \cdot) = \hat{v}(T, \cdot) = 0 \quad \text{in } G, \quad \mathbb{P}\text{-a.s.}, \quad (3.6)$$

it holds that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 (s \lambda \xi |\nabla \hat{v}|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 |\hat{v}|^2 + \lambda |\nabla \Delta v|^2 + s^2 \lambda^4 \xi^2 |\nabla^2 v|^2 + s^3 \lambda^4 \xi^3 |\Delta v|^2 \\ & \quad + s^4 \lambda^6 \xi^4 |\nabla v|^2 + s^6 \lambda^8 \xi^6 |v|^2) dx dt \\ & \leq C \mathbb{E} \int_Q \theta^2 (s^6 \lambda^6 \xi^6 f_1^2 + s^4 \lambda^4 \xi^4 |\nabla f_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 f_1|^2 + f_2^2 + s^6 \lambda^6 \xi^6 g_1^2 \\ & \quad + s^4 \lambda^4 \xi^4 |\nabla g_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 g_1|^2 + |\nabla \Delta g_1|^2 + s^2 \lambda^2 \xi^2 |g_2|^2) dx dt \\ & \quad + C \mathbb{E} \int_{\Sigma} \theta^2 (s \lambda \xi |\nabla \Delta v|^2 + s^3 \lambda^3 \xi^3 |\Delta v|^2) d\Gamma dt. \end{aligned} \quad (3.7)$$

Proof of Theorem 3.2. In what follows, for a positive integer r , we denote by $O(\lambda^r)$ a function of order λ^r for large λ . Similarly, we use the notation $O(e^{C\lambda})$.

In order to shorten the formulae, we define

$$\begin{aligned} \mathcal{A}_1 & \triangleq \mathbb{E} \int_Q (s^5 O(e^{C\lambda}) + s^6 \xi^6 O(\lambda^7)) |w|^2 dx dt, \\ \mathcal{A}_2 & \triangleq \mathbb{E} \int_Q (s^3 O(e^{C\lambda}) + s^4 \xi^4 O(\lambda^5)) |\nabla w|^2 dx dt, \\ \mathcal{A}_3 & \triangleq \mathbb{E} \int_Q (s O(e^{C\lambda}) + s^2 \xi^2 O(\lambda^3)) |\nabla^2 w|^2 dx dt, \end{aligned}$$

$$\begin{aligned}
\mathcal{A}_4 &\triangleq \mathbb{E} \int_Q O(1) |\nabla \Delta w|^2 dx dt, \\
\mathcal{B}_2 &\triangleq \mathbb{E} \int_{\Sigma} (s^2 O(e^{C\lambda}) + s^3 \xi^3 O(\lambda^2)) |\nabla^2 w|^2 d\Gamma dt, \\
\hat{\mathcal{A}}_1 &\triangleq \mathbb{E} \int_Q (s^2 O(e^{C\lambda}) + s^3 \xi^3 O(\lambda^3)) |\hat{w}|^2 dx dt, \\
\hat{\mathcal{A}}_2 &\triangleq \mathbb{E} \int_Q (O(\lambda) + s \xi O(1)) |\nabla \hat{w}|^2 dx dt,
\end{aligned} \tag{3.8}$$

and

$$\mathcal{A} \triangleq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \hat{\mathcal{A}}_1 + \hat{\mathcal{A}}_2.$$

Integrating the equality (2.12) on Q , taking mathematical expectation in both sides, and noting (3.6), we obtain

$$\begin{aligned}
&2\mathbb{E} \int_Q \theta I_2 (d\hat{v} + \Delta^2 v dt) dx - 2\mathbb{E} \int_Q \operatorname{div}(V_1 + V_2) dx dt \\
&= 2\mathbb{E} \int_Q I_2^2 dx dt + 2\mathbb{E} \int_Q I_2 I_3 dx + 2\mathbb{E} \int_Q (M_1 + M_2) dx dt \\
&\quad + 2 \sum_{i,j,k,l=1}^n \mathbb{E} \int_Q \Lambda_1^{ijkl} w_{x_i x_j} w_{x_k x_l} dx dt + 2 \sum_{i,j=1}^n \mathbb{E} \int_Q \Lambda_2^{ij} w_{x_i} w_{x_j} dx dt \\
&\quad + 2\mathbb{E} \int_Q \Lambda_3 w^2 dx dt + 2\mathbb{E} \int_Q \Lambda_4 dx.
\end{aligned} \tag{3.9}$$

By Condition 3.1, (2.14), (2.16) and (3.8), we have

$$\sum_{i,j,k,l=1}^n \mathbb{E} \int_Q \Lambda_1^{ijkl} w_{x_i x_j} w_{x_k x_l} dx dt \geq -C\mathbb{E} \int_Q s \lambda^4 \xi |\nabla^2 w|^2 dx dt \geq -\mathcal{A}_3, \tag{3.10}$$

$$\sum_{i,j=1}^n \mathbb{E} \int_Q \Lambda_2^{ij} w_{x_i} w_{x_j} dx dt \geq -C\mathbb{E} \int_Q (s^3 \lambda^6 \xi^3 + s^2 \lambda^6 \xi^2) |\nabla w|^2 dx dt \geq -\mathcal{A}_2, \tag{3.11}$$

$$\mathbb{E} \int_Q \Lambda_3 w^2 dx dt \geq -C\mathbb{E} \int_Q s^3 \lambda^8 \xi^3 (1 + s \xi + s^2 \xi^2) |w|^2 dx dt \geq -\mathcal{A}_1 \tag{3.12}$$

and

$$\begin{aligned}
\mathbb{E} \int_Q \Lambda_4 dx &\geq -C\mathbb{E} \int_Q [|\nabla \Delta w|^2 + s O(e^{C\lambda}) |\nabla^2 w|^2 + s^3 O(e^{C\lambda}) |\nabla w|^2 + s^5 O(e^{C\lambda}) |w|^2 \\
&\quad + s^2 O(e^{C\lambda}) |\hat{w}|^2 + s^3 \lambda^3 \xi^3 |\hat{w}|^2 + (s \xi + \lambda) |\nabla \hat{w}|^2] dx dt \\
&\quad - C\mathbb{E} \int_Q \theta^2 (s^6 \lambda^6 \xi^6 f_1^2 + s^4 \lambda^4 \xi^4 |\nabla f_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 f_1|^2 + s^6 \lambda^6 \xi^6 g_1^2 \\
&\quad + s^4 \lambda^4 \xi^4 |\nabla g_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 g_1|^2 + |\nabla \Delta g_1|^2 + s^2 \lambda^2 \xi^2 |g_2|^2) dx dt \\
&\geq -C\mathbb{E} \int_Q \theta^2 (s^6 \lambda^6 \xi^6 f_1^2 + s^4 \lambda^4 \xi^4 |\nabla f_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 f_1|^2 + s^6 \lambda^6 \xi^6 g_1^2 \\
&\quad + s^4 \lambda^4 \xi^4 |\nabla g_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 g_1|^2 + |\nabla \Delta g_1|^2 + s^2 \lambda^2 \xi^2 |g_2|^2) dx dt - \mathcal{A}.
\end{aligned} \tag{3.13}$$

It follows from (3.9)–(3.13) that

$$\begin{aligned}
& 2\mathbb{E} \int_Q \theta I_2(d\hat{v} + \Delta^2 v dt) dx - 2\mathbb{E} \int_Q \operatorname{div}(V_1 + V_2) dx dt \\
& \geq 2\mathbb{E} \int_Q I_2^2 dx dt + 2\mathbb{E} \int_Q I_2 I_3 dx + 2\mathbb{E} \int_Q (M_1 + M_2) dx dt \\
& \quad - C\mathbb{E} \int_Q \theta^2 (s^6 \lambda^6 \xi^6 f_1^2 + s^4 \lambda^4 \xi^4 |\nabla f_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 f_1|^2 + s^6 \lambda^6 \xi^6 g_1^2 + s^4 \lambda^4 \xi^4 |\nabla g_1|^2 \\
& \quad + s^2 \lambda^2 \xi^2 |\nabla^2 g_1|^2 + |\nabla \Delta g_1|^2 + s^2 \lambda^2 \xi^2 |g_2|^2) dx dt - \mathcal{A}.
\end{aligned} \tag{3.14}$$

We will estimate the terms in (3.14) one by one. The procedure is divided into three steps.

Step 1. In this step, we consider the divergence terms.

Thanks to the boundary conditions satisfied by v , it is easy to check that

$$w = 0, \quad \nabla w = 0, \quad \nabla^2 w = \theta \nabla^2 v \quad \text{on } \Sigma. \tag{3.15}$$

We also have

$$\frac{\partial w_{x_i}}{\partial \nu} \nu^j = w_{x_i x_j} = w_{x_j x_i} = \frac{\partial w_{x_j}}{\partial \nu} \nu^i, \tag{3.16}$$

which implies

$$\begin{aligned}
|\Delta w|^2 &= \sum_{i,j=1}^n w_{x_i x_i} w_{x_j x_j} = \sum_{i,j=1}^n \frac{\partial w_{x_i}}{\partial \nu} \nu^i \frac{\partial w_{x_j}}{\partial \nu} \nu^j \\
&= \sum_{i,j=1}^n \frac{\partial w_{x_i}}{\partial \nu} \nu^j \frac{\partial w_{x_j}}{\partial \nu} \nu^i = \sum_{i,j=1}^n w_{x_i x_j}^2 \\
&= |\nabla^2 w|^2 = \sum_{i=1}^n \left(\frac{\partial w_i}{\partial \nu} \right)^2 \quad \text{on } \Sigma.
\end{aligned} \tag{3.17}$$

Thanks to (3.15), we obtain that

$$\begin{aligned}
V_1 \cdot \nu &= \sum_{i,j,k,l=1}^n \Phi_1^l w_{x_k x_k x_l} w_{x_i x_i x_j} \nu^j - \frac{1}{2} \sum_{i,j,k,l=1}^n \Phi_1^j w_{x_k x_k x_l} w_{x_i x_i x_l} \nu^j \\
&+ \frac{1}{2} \sum_{i,j,k=1}^n \Psi_2 \Phi_1^j w_{x_i x_i} w_{x_k x_k} \nu^j + \sum_{i,j,k,l=1}^n \Psi_3^{ik} \Phi_1^l w_{x_i x_k} w_{x_l x_j} \nu^j \\
&- \frac{1}{2} \sum_{i,j,k,l=1}^n \Psi_3^{ij} \Phi_1^l w_{x_i x_k} w_{x_k x_l} \nu^j - \sum_{i,j,k=1}^n \Phi_4^k w_{x_i x_j} w_{x_i x_k} \nu^j \\
&+ \frac{1}{2} \sum_{i,j,k=1}^n \Phi_4^j w_{x_i x_k}^2 \nu^j \quad \text{on } \Sigma.
\end{aligned} \tag{3.18}$$

By Condition 3.1 and (2.16), we have that

$$\sum_{i,j,k,l=1}^n \Phi_1^l w_{x_k x_k x_l} w_{x_i x_i x_j} \nu^j - \frac{1}{2} \sum_{i,j,k,l=1}^n \Phi_1^j w_{x_k x_k x_l} w_{x_i x_i x_l} \geq -Cs\lambda\xi |\nabla \Delta w|^2 \quad \text{on } \Sigma. \tag{3.19}$$

From (2.14) and (2.16), we get that

$$\frac{1}{2} \sum_{i,j,k=1}^n \Psi_2 \Phi_1^j \nu^j w_{x_i x_i} w_{x_k x_k} = -4s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \frac{\partial \eta}{\partial \nu} |\Delta w|^2 \quad \text{on } \Sigma. \quad (3.20)$$

Combining (2.14), (2.16), (3.16) and (3.17), we find that

$$\begin{aligned} \sum_{i,j,k,l=1}^n \Psi_3^{ik} \Phi_1^l w_{x_i x_k} w_{x_l x_j} \nu^j &= -16 \sum_{i,j,k,l=1}^n s^3 \lambda^3 \xi^3 \eta_{x_i} \eta_{x_k} \eta_{x_l} \nu^j w_{x_i x_k} w_{x_l x_j} \\ &= -16 \sum_{i,j,k=1}^n s^3 \lambda^3 \xi^3 \eta_{x_i} \eta_{x_k} \nu^j w_{x_i x_k} \frac{\partial w_j}{\partial \nu} \frac{\partial \eta}{\partial \nu} \\ &= -16 \sum_{j=1}^n s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 \left(\frac{\partial w_j}{\partial \nu} \right)^2 \\ &= -16 s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\Delta w|^2 \quad \text{on } \Sigma \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} & -\frac{1}{2} \sum_{i,j,k,l=1}^n \Psi_3^{ij} \Phi_1^l w_{x_i x_k} w_{x_k x_l} \nu^j - \sum_{i,j,k=1}^n \Phi_4^k w_{x_i x_j} w_{x_i x_k} \nu^j + \frac{1}{2} \sum_{i,j,k=1}^n \Phi_4^j w_{x_i x_k}^2 \nu^j \\ &= 8s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\Delta w|^2 + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \frac{\partial \eta}{\partial \nu} |\Delta w|^2 \quad \text{on } \Sigma. \end{aligned} \quad (3.22)$$

Hence, combining Condition 3.1 and (3.18)–(3.22), we obtain that

$$\begin{aligned} V_1 \cdot \nu &\geq -Cs \lambda \xi |\nabla \Delta w|^2 - 8s^3 \lambda^3 \xi^3 \left(\frac{\partial \eta}{\partial \nu} \right)^3 |\Delta w|^2 - 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \frac{\partial \eta}{\partial \nu} |\Delta w|^2 \\ &\geq -Cs \lambda \xi |\nabla \Delta w|^2 - Cs^3 \lambda^3 \xi^3 |\Delta w|^2 \quad \text{on } \Sigma. \end{aligned}$$

This implies that

$$\mathbb{E} \int_Q \operatorname{div} V_1 dx dt \geq -C \mathbb{E} \int_{\Sigma} s^3 \lambda^3 \xi^3 |\Delta w|^2 d\Gamma dt - C \mathbb{E} \int_{\Sigma} s \lambda \xi |\nabla \Delta w|^2 d\Gamma dt. \quad (3.23)$$

Thanks to Condition 3.1, (2.16), (3.8) and (3.15), we have

$$\begin{aligned} \mathbb{E} \int_Q \operatorname{div} V_2 dx dt &= \mathbb{E} \int_{\Sigma} V_2 \cdot \nu d\Gamma dt \\ &= \mathbb{E} \int_{\Sigma} \sum_{j=1}^n \left(\sum_{i,l,r,m=1}^n \Theta_1^{ijlrm} w_{x_i x_i x_l} w_{x_r x_m} + \sum_{i,k,l,r=1}^n \Theta_2^{ijklr} w_{x_i x_k} w_{x_l x_r} + \Theta_9^j \right) \nu^j d\Gamma dt \\ &\geq \mathbb{E} \int_{\Sigma} \left[s \xi O(\lambda) |\nabla \Delta w|^2 + s O(e^{C\lambda}) |\nabla^2 w|^2 \right] d\Gamma dt \\ &\geq -C \mathbb{E} \int_{\Sigma} s \lambda \xi |\nabla \Delta w|^2 d\Gamma dt + \mathcal{B}_2. \end{aligned} \quad (3.24)$$

Combining (3.14), (3.23) and (3.24), we obtain that

$$2\mathbb{E} \int_Q \theta I_2(d\hat{v} + \Delta^2 v dt) dx + C \mathbb{E} \int_{\Sigma} s^3 \lambda^3 \xi^3 |\Delta w|^2 d\Gamma dt$$

$$\begin{aligned}
& + C\mathbb{E} \int_Q \theta^2 (s^6 \lambda^6 \xi^6 f_1^2 + s^4 \lambda^4 \xi^4 |\nabla f_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 f_1|^2 + s^6 \lambda^6 \xi^6 g_1^2 + s^4 \lambda^4 \xi^4 |\nabla g_1|^2 \\
& \quad + s^2 \lambda^2 \xi^2 |\nabla^2 g_1|^2 + |\nabla \Delta g_1|^2 + s^2 \lambda^2 \xi^2 |g_2|^2) dxdt \\
& + C\mathbb{E} \int_{\Sigma} s \lambda \xi |\nabla \Delta w|^2 d\Gamma dt + \mathcal{A} + \mathcal{B}_2 \\
& \geq 2\mathbb{E} \int_Q I_2^2 dxdt + 2\mathbb{E} \int_Q I_2 I_3 dx + 2\mathbb{E} \int_Q (M_1 + M_2) dxdt.
\end{aligned} \tag{3.25}$$

Step 2. In this step, we study $2\mathbb{E} \int_Q (M_1 + M_2) dxdt$ via integration by parts. From [Condition 3.1](#), [\(3.8\)](#) and [\(3.15\)](#), we get that

$$\begin{aligned}
& -\mathbb{E} \int_Q 16s^3 \lambda^4 \xi^3 |\nabla \eta|^2 |\nabla^2 w \nabla \eta|^2 dxdt \\
& = -\mathbb{E} \int_Q 16s^3 \lambda^4 \xi^3 \sum_{i,j,k,l=1}^n \eta_{x_i}^2 \eta_{x_k} \eta_{x_l} w_{x_k x_j} w_{x_l x_j} dxdt \\
& = \mathbb{E} \int_{\Sigma} 8s^3 \lambda^4 \sum_{i,j,k,l=1}^n [-2\xi^3 \eta_{x_i}^2 \eta_{x_k} \eta_{x_l} w_{x_k x_j} w_{x_l} + (\xi^3 \eta_{x_i}^2 \eta_{x_k} \eta_{x_l})_{x_j} w_{x_k} w_{x_l}] \nu^j d\Gamma dt \\
& \quad + \mathbb{E} \int_Q 8s^3 \lambda^4 \sum_{i,j,k,l=1}^n [2\xi^3 \eta_{x_i}^2 \eta_{x_k} \eta_{x_l} w_{x_k x_j x_j} w_{x_l} - (\xi^3 \eta_{x_i}^2 \eta_{x_k} \eta_{x_l})_{x_j x_j} w_{x_k} w_{x_l}] dxdt \\
& \geq -4\mathbb{E} \int_Q s \lambda^2 \xi |\nabla \Delta w \nabla \eta|^2 dxdt - 16\mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \nabla \eta|^2 dxdt - \mathcal{A}_2.
\end{aligned} \tag{3.26}$$

Thanks to [\(3.4\)](#) and [Condition 3.1](#), we know that there exists $\lambda_1 > 0$ such that for all $\lambda \geq \lambda_1$, it holds that

$$\begin{aligned}
& 64\mathbb{E} \int_Q s^5 \lambda^5 \xi^5 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla w \cdot \nabla \eta|^2 dxdt \\
& = 128\mathbb{E} \int_Q s^5 \lambda^5 \xi^5 |\nabla \eta|^2 |\nabla w \cdot \nabla \eta|^2 dxdt \\
& \geq -\mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 dxdt.
\end{aligned} \tag{3.27}$$

Combining [Condition 3.1](#), [\(3.4\)](#), [\(3.8\)](#) and [\(3.15\)](#), we get that

$$\begin{aligned}
& \mathbb{E} \int_Q [4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta] (|\nabla^2 w|^2 - |\Delta w|^2) dxdt \\
& = \frac{1}{2}\mathbb{E} \int_Q \Delta [4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta] |\nabla w|^2 dxdt \\
& \quad + \mathbb{E} \int_Q \nabla [4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) + 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta] \cdot \nabla w \Delta w dxdt \\
& \geq -\mathbb{E} \int_Q s^2 \xi^2 O(\lambda^3) |\nabla^2 w|^2 dxdt - \mathbb{E} \int_Q (s^3 O(e^{C\lambda}) + s^4 \xi^4 O(\lambda^5)) |\nabla w|^2 dxdt \\
& \geq -\mathcal{A}_2 - \mathcal{A}_3,
\end{aligned} \tag{3.28}$$

and that

$$\begin{aligned}
& 32\mathbb{E} \int_Q s^3 \lambda^3 \xi^3 (\nabla^2 w \nabla \eta \nabla \eta) \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} dx dt \\
&= 32 \sum_{i,j,k,l=1}^n \mathbb{E} \int_Q s^3 \lambda^3 \xi^3 \eta_{x_k x_l} \eta_{x_i} \eta_{x_j} w_{x_i x_j} w_{x_k x_l} dx dt \\
&= 32 \sum_{i,j,k,l=1}^n \mathbb{E} \int_{\Sigma} s^3 \lambda^3 \xi^3 \eta_{x_k x_l} \eta_{x_i} \eta_{x_j} (w_{x_i} w_{x_k x_l} \nu^j - w_{x_i} w_{x_j x_l} \nu^k) d\Gamma dt \\
&\quad - 32 \sum_{i,j,k,l=1}^n \mathbb{E} \int_Q s^3 \lambda^3 (\xi^3 \eta_{x_k x_l} \eta_{x_i} \eta_{x_j})_{x_j} w_{x_i} w_{x_k x_l} dx dt \\
&\quad + 32 \sum_{i,j,k,l=1}^n \mathbb{E} \int_Q s^3 \lambda^3 (\xi^3 \eta_{x_k x_l} \eta_{x_i} \eta_{x_j})_{x_k} w_{x_i} w_{x_j x_l} dx dt \\
&\quad + 32\mathbb{E} \int_Q s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 w \nabla \eta) (\nabla^2 w \nabla \eta) dx dt \\
&\geq 32\mathbb{E} \int_Q s^3 \lambda^3 \xi^3 \nabla^2 \eta (\nabla^2 w \nabla \eta) (\nabla^2 w \nabla \eta) dx dt - \mathcal{A}_2 - \mathcal{A}_3 \\
&= 64\mathbb{E} \int_Q s^3 \lambda^3 \xi^3 |\nabla^2 w \nabla \eta|^2 dx dt - \mathcal{A}_2 - \mathcal{A}_3.
\end{aligned} \tag{3.29}$$

From (3.8) and (3.15), we see that

$$\begin{aligned}
& \mathbb{E} \int_Q [(8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^4 |\Delta w|^2 - 2(8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^6 s^2 \lambda^2 \xi^2 |\nabla w|^2 \\
&\quad + (8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^8 s^4 \lambda^4 \xi^4 |w|^2] dx dt \\
&= \mathbb{E} \int_Q [(8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^4 (\Delta w + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 w)^2 \\
&\quad - 2(8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^6 s^2 \lambda^2 \xi^2 (|\nabla w|^2 + w \Delta w)] dx dt \\
&\geq \mathbb{E} \int_Q (8s^3 \lambda^4 \xi^3 - s^3 \lambda^{\frac{7}{2}} \xi^3) |\nabla \eta|^4 (\Delta w + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 w)^2 dx dt - \mathcal{A}_1.
\end{aligned} \tag{3.30}$$

Noting that $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ for $a, b, c \in \mathbb{R}$, we find that

$$\begin{aligned}
& \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\nabla \eta|^4 |\Delta v|^2 dx dt \\
&= \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 |\nabla \eta|^4 (\Delta w - 2s\lambda\xi \nabla \eta \nabla w + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 w - s\lambda^2 \xi |\nabla \eta|^2 w - s\lambda\xi |\Delta \eta| w)^2 dx dt \\
&\leq 3\mathbb{E} \int_Q s^3 \lambda^4 \xi^3 |\nabla \eta|^4 (\Delta w + s^2 \lambda^2 \xi^2 |\nabla \eta|^2 w)^2 dx dt + 12\mathbb{E} \int_Q s^5 \lambda^6 \xi^5 |\nabla \eta|^4 |\nabla w \cdot \nabla \eta|^2 dx dt + \mathcal{A}_1.
\end{aligned} \tag{3.31}$$

Combining (3.26)–(3.31), we know there exists $\lambda_2 \geq \lambda_1$ such that for all $\lambda \geq \lambda_2$, it holds that

$$\mathbb{E} \int_Q M_1 dx dt + \mathcal{A} \geq \mathbb{E} \int_Q \lambda |\nabla \Delta w|^2 dx dt + \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\nabla \eta|^4 |\Delta v|^2 dx dt. \tag{3.32}$$

It follows from [Condition 3.1](#), [\(2.14\)](#), [\(2.16\)](#), [\(3.4\)](#) and [\(3.8\)](#) that

$$\mathbb{E} \int_Q M_2 dx dt + \mathcal{A} \geq \mathbb{E} \int_Q (16s\lambda\xi|\nabla\hat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4\hat{w}^2) dx dt. \quad (3.33)$$

Thanks to [\(3.25\)](#), [\(3.32\)](#) and [\(3.33\)](#), for $\lambda \geq \lambda_2$, we have that

$$\begin{aligned} & 2\mathbb{E} \int_Q \theta I_2(d\hat{v} + \Delta^2 v dt) dx + C\mathbb{E} \int_{\Sigma} s^3\lambda^3\xi^3|\Delta w|^2 d\Gamma dt \\ & + C\mathbb{E} \int_{\Sigma} s\lambda\xi|\nabla\Delta w|^2 d\Gamma dt + \mathcal{A} + \mathcal{B}_2 \\ & + C\mathbb{E} \int_Q \theta^2 (s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + s^6\lambda^6\xi^6 g_1^2 \\ & \quad + s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2) dx dt \\ & \geq 2\mathbb{E} \int_Q I_2^2 dx dt + 2\mathbb{E} \int_Q I_2 I_3 dx \\ & \quad + \mathbb{E} \int_Q (\lambda|\nabla\Delta w|^2 + s^3\lambda^4\xi^3\theta^2|\Delta v|^2 + s\lambda\xi|\nabla\hat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\hat{w}^2) dx dt. \end{aligned} \quad (3.34)$$

Step 3. In this step, we get the estimate of v .

From [\(3.5\)](#), we have that

$$2\mathbb{E} \int_Q \theta I_2(d\hat{v} + \Delta^2 v dt) dx \leq \mathbb{E} \int_Q I_2^2 dx dt + \mathbb{E} \int_Q \theta^2 f_2^2 dx dt. \quad (3.35)$$

Thanks to [Condition 3.1](#), [\(2.17\)](#) and [\(3.8\)](#), we find that

$$2\mathbb{E} \int_Q I_2 I_3 dx \geq -\mathbb{E} \int_Q I_2^2 dx dt - \mathcal{A} - C\mathbb{E} \int_Q s^2\lambda^2\xi^2\theta^2 f_1^2 dx dt. \quad (3.36)$$

Noting that

$$|\nabla\Delta w|^2 \leq C\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\nabla^2 v|^2) \quad \text{on } \Sigma,$$

by [\(3.15\)](#) and [\(3.17\)](#), there exists $\lambda_3 > 0$ such that for all $\lambda \geq \lambda_3$, there is $s_1 = s_1(\lambda) > 0$, such that for all $s \geq s_1$, we have that

$$\begin{aligned} & \mathcal{B}_2 + \mathbb{E} \int_{\Sigma} s^3\lambda^3\xi^3|\Delta w|^2 d\Gamma dt + \mathbb{E} \int_{\Sigma} s\lambda\xi|\nabla\Delta w|^2 d\Gamma dt \\ & \leq C\mathbb{E} \int_{\Sigma} s\lambda\xi\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\Delta v|^2) d\Gamma dt. \end{aligned} \quad (3.37)$$

Thanks to [\(3.34\)](#)–[\(3.37\)](#), for $\lambda \geq \lambda_3$ and $s \geq s_1$, we get that

$$\begin{aligned} & C\mathbb{E} \int_{\Sigma} s\lambda\xi\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\Delta v|^2) d\Gamma dt + \mathcal{A} \\ & + C\mathbb{E} \int_Q \theta^2 (s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + f_2^2 + s^6\lambda^6\xi^6 g_1^2 \\ & \quad + s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2) dx dt \end{aligned} \quad (3.38)$$

$$\geq \mathbb{E} \int_Q (\lambda |\nabla \Delta w|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta v|^2 + s \lambda \xi |\nabla \hat{w}|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 \hat{w}^2) dx dt.$$

By [Theorem 3.1](#), for $\lambda \geq \max\{\lambda_3, \hat{\lambda}\}$ and $s \geq \max\{s_1, \hat{s}\}$, we obtain that

$$\mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |v|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla v|^2) dx dt \leq C \mathbb{E} \int_Q s^3 \lambda^4 \xi^3 \theta^2 |\Delta v|^2 dx dt,$$

which, together with [\(3.38\)](#), implies that

$$\begin{aligned} & C \mathbb{E} \int_{\Sigma} s \lambda \xi \theta^2 (|\nabla \Delta v|^2 + s^2 \lambda^2 \xi^2 |\Delta v|^2) d\Gamma dt + \mathcal{A} \\ & + C \mathbb{E} \int_Q \theta^2 (s^6 \lambda^6 \xi^6 f_1^2 + s^4 \lambda^4 \xi^4 |\nabla f_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 f_1|^2 + f_2^2 + s^6 \lambda^6 \xi^6 g_1^2 \\ & \quad + s^4 \lambda^4 \xi^4 |\nabla g_1|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 g_1|^2 + |\nabla \Delta g_1|^2 + s^2 \lambda^2 \xi^2 |g_2|^2) dx dt \\ & \geq \mathbb{E} \int_Q (\lambda |\nabla \Delta w|^2 + s^3 \lambda^4 \xi^3 \theta^2 |\Delta v|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla v|^2 + s^6 \lambda^8 \xi^6 \theta^2 |v|^2 \\ & \quad + s \lambda \xi |\nabla \hat{w}|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 \hat{w}^2) dx dt. \end{aligned} \quad (3.39)$$

Let $\tilde{v} = s \lambda^2 \xi e^{s\xi} v$. Then, we have that

$$\begin{aligned} & \mathbb{E} \int_Q s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 v|^2 dx dt \\ & = \mathbb{E} \int_Q s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 (s^{-1} \lambda^{-2} \xi^{-1} \theta^{-1} \tilde{v})|^2 dx dt \\ & \leq C \mathbb{E} \int_Q s^2 \lambda^4 \xi^2 (s^{-2} \lambda^{-4} \xi^{-2} |\nabla^2 \tilde{v}|^2 + \lambda^{-2} |\nabla \tilde{v}|^2 + s^2 \xi^2 |\tilde{v}|^2) dx dt \\ & \leq C |\tilde{v}|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 + C \mathbb{E} \int_Q s^2 \lambda^2 \xi^2 |\nabla \tilde{v}|^2 dx dt + C \mathbb{E} \int_Q s^4 \lambda^4 \xi^4 |\tilde{v}|^2 dx dt \\ & \leq C |\tilde{v}|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 + C \mathbb{E} \int_Q s^2 \lambda^2 \xi^2 (s^4 \lambda^6 \xi^4 \theta^2 v^2 + s^2 \lambda^4 \xi^2 \theta^2 |\nabla v|^2) dx dt \\ & \quad + C \mathbb{E} \int_Q s^6 \lambda^8 \xi^6 \theta^2 |v|^2 dx dt \\ & \leq C |\tilde{v}|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 + C \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |v|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla v|^2) dx dt. \end{aligned} \quad (3.40)$$

It follows from $\tilde{v} = 0$ on Σ that

$$\begin{aligned} & |\tilde{v}|_{L_{\mathbb{F}}^2(0,T;H^2(G))}^2 \\ & \leq C |\Delta \tilde{v}|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 \\ & \leq C \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |v|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla v|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\Delta v|^2) dx dt. \end{aligned} \quad (3.41)$$

Combining [\(3.40\)](#) and [\(3.41\)](#), we obtain that

$$\mathbb{E} \int_Q s^2 \lambda^4 \xi^2 \theta^2 |\nabla^2 v|^2 dx dt \leq C \mathbb{E} \int_Q (s^6 \lambda^8 \xi^6 \theta^2 |v|^2 + s^4 \lambda^6 \xi^4 \theta^2 |\nabla v|^2 + s^2 \lambda^4 \xi^2 \theta^2 |\Delta v|^2) dx dt. \quad (3.42)$$

From (3.39) and (3.42), there exists $\lambda_4 \geq \max\{\lambda_3, \widehat{\lambda}\}$ such that for all $\lambda \geq \lambda_4$, there is an $s_2 = s_2(\lambda) > \max\{s_1, \widehat{s}\}$, such that for all $s \geq s_2$, we have that

$$\begin{aligned}
& C\mathbb{E} \int_{\Sigma} s\lambda\xi\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\Delta v|^2)d\Gamma dt + \mathcal{A} \\
& + C\mathbb{E} \int_Q \theta^2(s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + f_2^2 + s^6\lambda^6\xi^6 g_1^2 \\
& \quad + s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)dxdt \\
& \geq \mathbb{E} \int_Q (\lambda|\nabla\Delta w|^2 + s^3\lambda^4\xi^3\theta^2|\Delta v|^2 + s^2\lambda^4\xi^2\theta^2|\nabla^2 v|^2 + s^4\lambda^6\xi^4\theta^2|\nabla v|^2 + s^6\lambda^8\xi^6\theta^2|v|^2 \\
& \quad + s\lambda\xi|\nabla\hat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\hat{w}^2)dxdt.
\end{aligned} \tag{3.43}$$

Recalling $v = \theta^{-1}w$ and $\hat{v} = \theta^{-1}(\hat{w} - \ell_t w)$, we get that

$$\begin{aligned}
& \mathbb{E} \int_Q \theta^2(\lambda|\nabla\Delta v|^2 + s\lambda\xi|\nabla\hat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\hat{v}|^2)dxdt \\
& = \mathbb{E} \int_Q \theta^2(\lambda|\nabla\Delta(\theta^{-1}w)|^2 + s\lambda\xi|\nabla[\theta^{-1}(\hat{w} - \ell_t w)]|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\theta^{-2}|\hat{w} - \ell_t w|^2)dxdt \\
& \leq C\mathbb{E} \int_Q [\lambda(|\nabla\Delta w|^2 + s^2\lambda^2\xi^2|\nabla^2 w|^2 + s^4\lambda^4\xi^4|\nabla w|^2 + s^6\lambda^6\xi^6|w|^2) + s^3\lambda^{\frac{7}{2}}\xi^3(|\hat{w}|^2 + s^2\lambda^2\xi^2|w|^2) \\
& \quad + s\lambda\xi(|\nabla\hat{w}|^2 + s^2\lambda^2\xi^2|\hat{w}|^2 + s^2\lambda^2\xi^2|\nabla w|^2 + s^4\lambda^4\xi^4|w|^2)]dxdt \\
& \leq C\mathbb{E} \int_Q (\lambda|\nabla\Delta w|^2 + s\lambda\xi|\nabla\hat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\hat{w}|^2)dxdt + \mathcal{A}.
\end{aligned} \tag{3.44}$$

Thanks to (3.8) and Condition 3.1, there exists $\lambda_5 > 0$ such that for all $\lambda \geq \lambda_5$, there is an $s_3 = s_3(\lambda) > 0$, such that for all $s \geq s_3$, we have

$$\begin{aligned}
\mathcal{A} & \leq \frac{1}{C}\mathbb{E} \int_Q \theta^2(\lambda|\nabla\Delta v|^2 + s\lambda\xi|\nabla\hat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\hat{v}|^2 + s^2\lambda^4\xi^2|\nabla^2 v|^2 + s^3\lambda^4\xi^3|\Delta v|^2 \\
& \quad + s^4\lambda^6\xi^4|\nabla v|^2 + s^6\lambda^8\xi^6|v|^2)dxdt.
\end{aligned} \tag{3.45}$$

Let us choose $\lambda_0 \geq \max\{\lambda_4, \lambda_5\}$. Combining (3.43)–(3.45), for all $\lambda \geq \lambda_0$, one can find $s_0 = s_0(\lambda) \geq \max\{s_2, s_3\}$ so that for any $s \geq s_0$, inequality (3.7) holds. $\square$

4 Proof of the observability estimate

Proof of Theorem 1.5. Let $\chi \in C_0^\infty([0, T])$ satisfy

$$\chi = 1 \text{ in } \left(\frac{T}{2} - \varepsilon_1 T, \frac{T}{2} + \varepsilon_1 T\right).$$

Put $v = \chi z$ and $\hat{v} = \chi\hat{z} + \chi_t z$ for $(z, \hat{z})$ satisfying (1.2), then $(v, \hat{v})$ fulfills $v(0, \cdot) = v(T, \cdot) = \hat{v}(0, \cdot) = \hat{v}(T, \cdot) = 0$ in G , and solves

$$\begin{cases} dv = \hat{v}dt + \chi(Z - a_5 z)dW(t) & \text{in } Q, \\ d\hat{v} + \Delta^2 vdt = \tilde{f}_2 dt + \tilde{g}_2 dW(t) & \text{in } Q, \\ v = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \end{cases} \tag{4.1}$$

where

$$\tilde{f}_2 = \chi[(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2 \nabla z - a_3 \hat{Z} + a_4 Z] + 2\chi_t \hat{z} + \chi_{tt} z$$

and

$$\tilde{g}_2 = \chi \hat{Z} + \chi_t (Z - a_5 z).$$

We justify applying the estimate at the regularity of the adjoint solution. Let $A = \Delta^2$ with $D(A) = H^4(G) \cap H_0^2(G)$ be the positive clamped realization, and let P_N be its first N spectral projections. Its fractional domains are $D(A^{3/4}) = H^3(G) \cap H_0^2(G)$ and $D(A^{1/4}) = H_0^1(G)$, with equivalent norms. For $g_1 = \chi(Z - a_5 z)$, the coefficient assumptions give $g_1 \in L_{\mathbb{F}}^2(0, T; D(A^{3/4}))$ and $\tilde{f}_2, \tilde{g}_2 \in L_{\mathbb{F}}^2(0, T; D(A^{1/4}))$. Apply P_N to both equations in (4.1). The resulting processes $(v_N, \hat{v}_N) = (P_N v, P_N \hat{v})$, with the three projected sources, satisfy the hypotheses and end-point conditions of Theorem 3.2. They converge in the state spaces $L^2(\Omega; C([0, T]; D(A^{3/4}) \times D(A^{1/4})))$, and the sources converge in the spaces just displayed.

Boundary convergence follows from the same multiplier proof used for Theorem 1.1, before substituting the particular lower order terms. More precisely, for the clamped system $du = \hat{u} dt + G_1 dW$, $d\hat{u} + Au dt = F dt + G_2 dW$, it gives

$$\begin{aligned} \mathbb{E} \int_{\Sigma} |\nabla \Delta u|^2 d\Gamma dt &\leq C \mathbb{E} \sup_{0 \leq t \leq T} (|u(t)|_{H^3}^2 + |\hat{u}(t)|_{H^1}^2) \\ &\quad + C(|F|_{L_{\mathbb{F}}^2(0, T; H^1)}^2 + |G_1|_{L_{\mathbb{F}}^2(0, T; H^3)}^2 + |G_2|_{L_{\mathbb{F}}^2(0, T; H^1)}^2). \end{aligned}$$

Indeed, in (2.1) the new drift is paired with $\rho \cdot \nabla \Delta u$, while the quadratic variation is $2\rho \cdot \nabla \Delta G_1 G_2$; both have the displayed bounds. For each tangential field $\mathcal{B}_\alpha$, (2.10) has drift $\mathcal{B}_\alpha F + [A, \mathcal{B}_\alpha]u \in H^{-1}$ paired with $\rho \cdot \nabla(\mathcal{B}_\alpha u) \in H_0^1$, and noise $\mathcal{B}_\alpha G_1 \in H^2$, $\mathcal{B}_\alpha G_2 \in L^2$. Thus the tangential and normal estimates close exactly as above, with constants independent of N . Applying this bound to spectral differences proves convergence of $\nabla \Delta v_N$ in $L^2(\Omega \times \Sigma)$; the trace theorem gives convergence of Δv_N there. For fixed s, λ , the smooth weights are bounded on $\bar{Q}$, so passing to the limit in (3.7) is justified.

By Theorem 3.2, for $\lambda \geq \lambda_0$ and $s \geq s_0$, we have

$$\begin{aligned} &\mathbb{E} \int_Q \theta^2 \chi^2 (\lambda |\nabla \Delta z|^2 + s^2 \lambda^4 \xi^2 |\nabla^2 z|^2 + s^3 \lambda^4 \xi^3 |\Delta z|^2 + s^4 \lambda^6 \xi^4 |\nabla z|^2 + s^6 \lambda^8 \xi^6 |z|^2) dx dt \\ &\leq \mathbb{E} \int_Q \theta^2 (s \lambda \xi |\nabla \hat{v}|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 |\hat{v}|^2 + \lambda |\nabla \Delta v|^2 + s^2 \lambda^4 \xi^2 |\nabla^2 v|^2 + s^3 \lambda^4 \xi^3 |\Delta v|^2 \\ &\quad + s^4 \lambda^6 \xi^4 |\nabla v|^2 + s^6 \lambda^8 \xi^6 |v|^2) dx dt \\ &\leq C \mathbb{E} \int_Q \theta^2 \chi^2 (s^6 \lambda^6 \xi^6 z^2 + s^4 \lambda^4 \xi^4 |\nabla z|^2 + s^2 \lambda^2 \xi^2 |\nabla^2 z|^2 + |\nabla \Delta z|^2 + |z|^2 + |\nabla z|^2) dx dt \\ &\quad + C \mathbb{E} \int_{J_1} \theta^2 (|\hat{z}|^2 + z^2 + s^2 \lambda^2 \xi^2 z^2) dx dt + C(s, \lambda) |(Z, \hat{Z})|_{L_{\mathbb{F}}^2(0, T; H^3(G)) \times L_{\mathbb{F}}^2(0, T; L^2(G))}^2 \\ &\quad + C \mathbb{E} \int_{\Sigma} \theta^2 (s \lambda \xi |\nabla \Delta z|^2 + s^3 \lambda^3 \xi^3 |\Delta z|^2) d\Gamma dt. \end{aligned}$$

This, together with Condition 3.1, implies that there exists $\tilde{\lambda}_1 \geq \lambda_0$ such that for all $\lambda \geq \tilde{\lambda}_1$, there is $\tilde{s}_1 = \tilde{s}_1(\lambda) \geq s_0$, so that for any $s \geq \tilde{s}_1$, it holds that

$$\mathbb{E} \int_Q \theta^2 (s \lambda \xi |\nabla \hat{v}|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 |\hat{v}|^2 + \lambda |\nabla \Delta v|^2 + s^2 \lambda^4 \xi^2 |\nabla^2 v|^2 + s^3 \lambda^4 \xi^3 |\Delta v|^2$$

$$\begin{aligned}
& + s^4 \lambda^6 \xi^4 |\nabla v|^2 + s^6 \lambda^8 \xi^6 |v|^2) dxdt \\
& \leq C \mathbb{E} \int_{J_1} \theta^2 (|\hat{z}|^2 + z^2 + s^2 \lambda^2 \xi^2 z^2) dxdt + C \mathbb{E} \int_{\Sigma} \theta^2 (s \lambda \xi |\nabla \Delta z|^2 + s^3 \lambda^3 \xi^3 |\Delta z|^2) d\Gamma dt \\
& \quad + C(s, \lambda) |(Z, \hat{Z})|_{L_{\mathbb{F}}^2(0, T; H^3(G)) \times L_{\mathbb{F}}^2(0, T; H^1(G))}.
\end{aligned} \tag{4.2}$$

Thanks to [Condition 3.1](#), we obtain

$$\begin{aligned}
& e^{2se^{\lambda\delta} - C\lambda - 6\ln s} \mathbb{E} \int_{J_2} (|\nabla \Delta z|^2 + |\nabla^2 z|^2 + |\nabla z|^2 + z^2 + |\nabla \hat{z}|^2 + |\hat{z}|^2) dxdt \\
& \leq e^{2se^{\lambda\delta}} \mathbb{E} \int_{J_2} (s \lambda \xi |\nabla \hat{v}|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 |\hat{v}|^2 + \lambda |\nabla \Delta v|^2 + s^2 \lambda^4 \xi^2 |\nabla^2 v|^2 + s^3 \lambda^4 \xi^3 |\Delta v|^2 \\
& \quad + s^4 \lambda^6 \xi^4 |\nabla v|^2 + s^6 \lambda^8 \xi^6 |v|^2) dxdt \\
& \leq \mathbb{E} \int_Q \theta^2 (s \lambda \xi |\nabla \hat{v}|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 |\hat{v}|^2 + \lambda |\nabla \Delta v|^2 + s^2 \lambda^4 \xi^2 |\nabla^2 v|^2 + s^3 \lambda^4 \xi^3 |\Delta v|^2 \\
& \quad + s^4 \lambda^6 \xi^4 |\nabla v|^2 + s^6 \lambda^8 \xi^6 |v|^2) dxdt.
\end{aligned} \tag{4.3}$$

From [\(1.3\)](#) and [Condition 3.1](#), we see that

$$\begin{aligned}
& |(z^T, \hat{z}^T)|_{L_{\mathcal{F}_T}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_T}^2(\Omega; H_0^1(G))} \\
& \leq C \mathbb{E} \int_{J_2} (|\nabla \Delta z|^2 + |\nabla^2 z|^2 + |\nabla z|^2 + z^2 + |\nabla \hat{z}|^2 + |\hat{z}|^2) dxdt \\
& \quad + C(s, \lambda) |(Z, \hat{Z})|_{L_{\mathbb{F}}^2(0, T; H^3(G) \cap H_0^2(G)) \times L_{\mathbb{F}}^2(0, T; H_0^1(G))},
\end{aligned} \tag{4.4}$$

and that

$$\begin{aligned}
& \mathbb{E} \int_{J_1} \theta^2 (|\hat{z}|^2 + z^2 + s^2 \lambda^2 \xi^2 z^2) dxdt \\
& \leq e^{2se^{-\lambda\delta} + C\lambda + 2\ln s} \mathbb{E} \int_{J_1} (|\hat{z}|^2 + z^2) dxdt \\
& \leq C e^{2se^{-\lambda\delta} + C\lambda + 2\ln s} |(z^T, \hat{z}^T)|_{L_{\mathcal{F}_T}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_T}^2(\Omega; H_0^1(G))} \\
& \quad + C(s, \lambda) |(Z, \hat{Z})|_{L_{\mathbb{F}}^2(0, T; H^3(G) \cap H_0^2(G)) \times L_{\mathbb{F}}^2(0, T; H_0^1(G))}.
\end{aligned} \tag{4.5}$$

Combining [\(4.2\)](#)–[\(4.5\)](#) choosing $\lambda \geq \tilde{\lambda}_1$ and $s \geq \tilde{s}_1$ such that

$$C \exp(2se^{-\lambda\delta} - 2se^{\lambda\delta} + C\lambda + 8\ln s) \leq \frac{1}{2},$$

we get the desired observability estimate. $\square$

5 Proof of Theorem 1.4

The proof of [Theorem 1.4](#) is similar to that for [\[23, Theorem 2.3\]](#). We provide it here for the convenience of the readers. To begin with, we recall the following result.

Lemma 5.1 [\[25, Lemma 2.1\]](#) There exists a random variable $\zeta \in L_{\mathcal{F}_T}^2(\Omega)$ such that it is impossible to find $\varsigma_1, \varsigma_2 \in L_{\mathbb{F}}^2(0, T) \times C_{\mathbb{F}}([0, T]; L^2(\Omega))$ and $\alpha \in \mathbb{R}$ satisfying

$$\zeta = \alpha + \int_0^T \varsigma_1(t) dt + \int_0^T \varsigma_2(t) dW(t).$$

Proof of Theorem 1.4. We employ a contradiction argument and divide the proof into three cases.

Case 1: $a_3 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$, $G \setminus \overline{G_0} \neq \emptyset$ and $\text{supp } f \subset G_0$.

Let $\rho \in C_0^\infty(G \setminus G_0)$ satisfying $|\rho|_{L^2(G)} = 1$. Suppose that (1.1) was exactly controllable. By Definition 1.2, for $(y_0, \hat{y}_0) = (0, 0)$, there exist controls (f, g, h_1, h_2) with $\text{supp } f \subset G_0$ a.e. $(t, \omega) \in (0, T) \times \Omega$ such that the solution to (1.1) fulfills $(y(T), \hat{y}(T)) = (\rho\zeta, 0)$, where ζ is given in Theorem 5.1. Hence,

$$\rho\zeta = \int_0^T \hat{y} dt + \int_0^T (a_3 y + f) dW(t). \quad (5.1)$$

Multiplying (5.1) by ρ and integrating it in G , we arrive that

$$\zeta = \int_0^T \langle \hat{y}, \rho \rangle_{(H^3(G) \cap H_0^2(G))^*, H^3(G) \cap H_0^2(G)} dt + \int_0^T \langle a_3 y, \rho \rangle_{H^{-1}(G), H_0^1(G)} dW(t),$$

which contradicts Theorem 5.1.

Case 2: $a_4 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$, $G \setminus \overline{G_0} \neq \emptyset$ and $\text{supp } g \subset G_0$.

Choose $\rho \in C_c^\infty(G \setminus \overline{G_0})$ as in Case 1, normalized by $\int_G \rho^2 dx = 1$. Suppose that (1.1) is exactly controllable. From zero initial data choose controls with $\text{supp } g \subset G_0$ so that $(y(T), \hat{y}(T)) = (0, \rho\zeta)$, where ζ is given by Theorem 5.1. Pairing the second equation directly with ρ gives

$$\zeta = \int_0^T [-\langle y, \Delta^2 \rho \rangle + \langle a_1 y + a_2 \cdot \nabla y, \rho \rangle] dt + \int_0^T a_4(t) \langle y, \rho \rangle dW(t).$$

All pairings are distributional pairings against smooth compactly supported functions. The terms containing g and $a_5 g$ vanish because their supports are disjoint from that of ρ . The drift integrand is square integrable, and the diffusion integrand is mean-square continuous because $a_4 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$ and y is mean-square continuous in $H^{-1}(G)$. This contradicts Theorem 5.1.

Case 3: $h_1 = h_2 = 0$.

Assume that (1.1) was exactly controllable. Then, from the equivalence between the exact controllability of (1.1) and the observability estimate of (1.2), we get that for any $(z^T, \hat{z}^T) \in L_{\mathcal{F}_T}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_T}^2(\Omega; H_0^1(G))$, the solution $(z, Z, \hat{z}, \hat{Z})$ to (1.2) (with $\tau = T$ and $(z(T), \hat{z}(T)) = (z^T, \hat{z}^T)$) satisfies

$$|(z^T, \hat{z}^T)|_{L_{\mathcal{F}_T}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_T}^2(\Omega; H_0^1(G))} \leq C(|Z|_{L_{\mathbb{F}}^2(0, T; H^3(G) \cap H_0^2(G))} + |\hat{Z}|_{L_{\mathbb{F}}^2(0, T; H_0^1(G))}). \quad (5.2)$$

For any nonzero $(\Phi_0, \Phi_1) \in (H^3(G) \cap H_0^2(G)) \times H_0^1(G)$, let $(\Phi, \hat{\Phi})$ solve the equation

$$\begin{cases} d\Phi = \hat{\Phi} dt - a_5 \Phi dW(t) & \text{in } Q, \\ d\hat{\Phi} + \Delta^2 \Phi dt = [(a_1 - \text{div } a_2 - a_4 a_5) \Phi - a_2 \cdot \nabla \Phi] dt & \text{in } Q, \\ \Phi = \frac{\partial \Phi}{\partial \nu} = 0 & \text{on } \Sigma, \\ (\Phi(0), \hat{\Phi}(0)) = (\Phi_0, \Phi_1) & \text{in } G. \end{cases}$$

Clearly, $(\Phi, 0, \hat{\Phi}, 0)$ solves (1.2) with the final datum $(z^T, \hat{z}^T) = (\Phi(T), \hat{\Phi}(T))$, a contradiction to the inequality (5.2). $\square$

Appendix A Proof of the weighted identity

Proof of Theorem 2.1. It is clear that

$$dw = d(\theta v) = \theta dv + \ell_t \theta v dt = \hat{w} dt + \theta f_1 dt + \theta g_1 dW(t),$$

and

$$\begin{aligned} \theta d\hat{v} &= \theta d[\theta^{-1}(\hat{w} - \ell_t w)] = d\hat{w} - \ell_t dw - \ell_{tt} w dt - \ell_t \hat{w} dt + \ell_t^2 w dt \\ &= d\hat{w} - 2\ell_t \hat{w} dt + (\ell_t^2 - \ell_{tt}) w dt - \ell_t \theta f_1 dt - \ell_t \theta g_1 dW(t). \end{aligned} \quad (\text{A.1})$$

We also have

$$\begin{aligned} \theta \Delta^2 v &= \Delta^2 w - 4s\lambda\xi \nabla \eta \cdot \nabla \Delta w - 4s\lambda^2\xi (\nabla^2 w \nabla \eta \nabla \eta) - 4s\lambda\xi \sum_{i,j=1}^n \eta_{x_i x_j} w_{x_i x_j} \\ &\quad + 2s^2\lambda^2\xi^2 |\nabla \eta|^2 \Delta w - 2s\lambda^2\xi |\nabla \eta|^2 \Delta w - 2s\lambda\xi \Delta \eta \Delta w + 4s^2\lambda^2\xi^2 (\nabla^2 w \nabla \eta \nabla \eta) \\ &\quad - 4\nabla \Delta \ell \cdot \nabla w + 12s^2\lambda^3\xi^2 |\nabla \eta|^2 \nabla \eta \nabla w + 8s^2\lambda^2\xi^2 (\nabla^2 \eta \nabla \eta \nabla w) \\ &\quad - 4s^3\lambda^3\xi^3 |\nabla \eta|^2 \nabla \eta \nabla w + 4s^2\lambda^2\xi^2 \Delta \eta \nabla \eta \nabla w + 4(\nabla \ell \cdot \nabla \Delta \ell) w + 2|\nabla^2 \ell|^2 w - \Delta^2 \ell w \\ &\quad - 6s^3\lambda^4\xi^3 |\nabla \eta|^4 w - 4s^3\lambda^3\xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) w + s^4\lambda^4\xi^4 |\nabla \eta|^4 w - 2s^3\lambda^3\xi^3 |\nabla \eta|^2 \Delta \eta w + |\Delta \ell|^2 w. \end{aligned} \quad (\text{A.2})$$

From (A.1), (A.2) and (2.13)–(2.17), we have

$$2\theta I_2(d\hat{v} + \Delta^2 v dt) = 2I_2(I_1 + I_2 dt + I_3).$$

We will compute $I_1 I_2$ under the form $\sum_{i=1}^7 \sum_{j=1}^5 I_{ij}$, where I_{ij} is the product of the i -th term of I_1 with the j -th term of I_2 . Note that I_{ij} are the same as Appendix A in [22] for $i = 1, \dots, 6$ and $j = 1, \dots, 5$, except for I_{13} .

We have

$$\begin{aligned} I_{13} &= \sum_{i,j,k,l=1}^n \Phi_3^{kl} w_{x_i x_i x_j x_j} w_{x_k x_l} dt \\ &= \sum_{i,j,k,l=1}^n \left(\Phi_3^{kl} w_{x_i x_i x_j} w_{x_k x_l} - \Phi_3^{kj} w_{x_i x_i x_l} w_{x_k x_l} + \Phi_3^{kl} w_{x_i x_i x_l} w_{x_k x_j} - \Phi_{3_{x_i}}^{kl} w_{x_i x_j} w_{x_k x_l} \right. \\ &\quad \left. + \Phi_{3_{x_l}}^{kj} w_{x_i x_l} w_{x_i x_k} + \Phi_{3_{x_l}}^{kl} w_{x_i x_j} w_{x_i x_k} - \frac{1}{2} \Phi_{3_{x_l}}^{jl} w_{x_i x_k}^2 - \Phi_{3_{x_i}}^{kl} w_{x_j x_l} w_{x_i x_k} \right)_{x_j} dt \\ &\quad - \sum_{i,j,k,l=1}^n \Phi_3^{kl} w_{x_i x_i x_l} w_{x_k x_j x_j} dt + \sum_{i,j,k,l=1}^n \Phi_{3_{x_i x_j}}^{kl} w_{x_i x_j} w_{x_k x_l} dt - \sum_{i,j,k,l=1}^n \Phi_{3_{x_l x_j}}^{kl} w_{x_i x_j} w_{x_i x_k} dt \\ &\quad - \sum_{i,j,k,l=1}^n \Phi_{3_{x_i x_l}}^{kl} w_{x_i x_j} w_{x_k x_j} dt + \sum_{i,j,k,l=1}^n \Phi_{3_{x_i x_j}}^{kl} w_{x_i x_l} w_{x_k x_j} dt + \frac{1}{2} \sum_{i,j,k,l=1}^n \Phi_{3_{x_k x_l}}^{kl} w_{x_i x_j}^2 dt. \end{aligned}$$

We have

$$I_{71} = \sum_{i,j=1}^n \Phi_1^i w_{x_i x_j x_j} d\hat{w}$$

$$\begin{aligned}
&= \sum_{i,j=1}^n \left(\Phi_{1x_i}^i \hat{w} \hat{w}_{x_j} - \frac{1}{2} \Phi_{1x_i x_j}^i \hat{w}^2 + \Phi_1^i \hat{w}_{x_i} \hat{w}_{x_j} - \frac{1}{2} \Phi_1^j \hat{w}_{x_i}^2 \right)_{x_j} dt - \sum_{i,j=1}^n (\Phi_1^j \hat{w} dw_{x_i x_i})_{x_j} \\
&\quad + \sum_{i,j=1}^n d(\Phi_1^i w_{x_i x_j x_j} \hat{w}) + \frac{1}{2} \sum_{i,j=1}^n \Phi_{1x_i x_j x_j}^i \hat{w}^2 dt - \frac{1}{2} \sum_{i,j=1}^n \Phi_{1x_i}^i \hat{w}_{x_j}^2 dt - \sum_{i,j=1}^n \Phi_{1x_j}^i \hat{w}_{x_i} \hat{w}_{x_j} dt \\
&\quad - \sum_{i,j=1}^n \Phi_{1t}^i w_{x_i x_j x_j} \hat{w} dt + \sum_{i,j=1}^n (\Phi_{1x_i}^i \hat{w} + \Phi_1^i \hat{w}_{x_i}) [(\theta f_1)_{x_j x_j} dt + (\theta g_1)_{x_j x_j} dW(t)] \\
&\quad - \sum_{i,j=1}^n \Phi_1^i dw_{x_i x_j x_j} d\hat{w}, \\
I_{72} &= \sum_{i=1}^n \Phi_2 w_{x_i x_i} d\hat{w} \\
&= \sum_{j=1}^n \left(-\Phi_2 \hat{w} \hat{w}_{x_j} + \frac{1}{2} \Phi_{2x_j} \hat{w}^2 \right)_{x_j} dt + \sum_{i=1}^n d(\Phi_2 w_{x_i x_i} \hat{w}) - \sum_{i=1}^n \Phi_{2t} w_{x_i x_i} \hat{w} dt \\
&\quad - \frac{1}{2} \sum_{i=1}^n \Phi_{2x_i x_i} \hat{w}^2 dt + \sum_{i=1}^n \Phi_2 \hat{w}_{x_i}^2 dt - \sum_{i=1}^n \Phi_2 dw_{x_i x_i} d\hat{w} \\
&\quad - \sum_{i=1}^n \Phi_2 \hat{w} [(\theta f_1)_{x_i x_i} dt + (\theta g_1)_{x_i x_i} dW(t)], \\
I_{73} &= \sum_{i,j=1}^n \Phi_3^{ij} w_{x_i x_j} d\hat{w} \\
&= \sum_{i,j=1}^n \left(-\Phi_3^{ij} \hat{w} \hat{w}_{x_i} + \frac{1}{2} \Phi_{3x_i}^{ij} \hat{w}^2 \right)_{x_j} dt + \sum_{i,j=1}^n d(\Phi_3^{ij} w_{x_i x_j} \hat{w}) - \sum_{i,j=1}^n \Phi_{3t}^{ij} w_{x_i x_j} \hat{w} dt \\
&\quad - \frac{1}{2} \sum_{i,j=1}^n \Phi_{3x_i x_j}^{ij} \hat{w}^2 dt - \sum_{i,j=1}^n \Phi_3^{ij} dw_{x_i x_j} d\hat{w} + \sum_{i,j=1}^n \Phi_3^{ij} \hat{w}_{x_i} \hat{w}_{x_j} dt \\
&\quad - \sum_{i,j=1}^n \Phi_3^{ij} \hat{w} [(\theta f_1)_{x_i x_j} dt + (\theta g_1)_{x_i x_j} dW(t)], \\
I_{74} &= \sum_{i=1}^n \Phi_4^i w_{x_i} d\hat{w} \\
&= \sum_{j=1}^n \left(-\frac{1}{2} \Phi_4^j \hat{w}^2 \right)_{x_j} dt + \sum_{i=1}^n d(\Phi_4^i w_{x_i} \hat{w}) - \sum_{i=1}^n \Phi_{4t}^i w_{x_i} \hat{w} dt + \frac{1}{2} \sum_{i=1}^n \Phi_{4x_i}^i \hat{w}^2 dt \\
&\quad - \sum_{i=1}^n \Phi_4^i dw_{x_i} d\hat{w} - \sum_{i=1}^n \Phi_4^i \hat{w} [(\theta f_1)_{x_i} dt + (\theta g_1)_{x_i} dW(t)], \\
I_{75} &= \Phi_5 w d\hat{w} = d(\Phi_5 w \hat{w}) - \Phi_{5t} w \hat{w} dt - \Phi_5 \hat{w}^2 dt - \Phi_5 dw d\hat{w} - \Phi_5 \hat{w} (\theta f_1 dt + \theta g_1 dW(t)).
\end{aligned}$$

By summing all the I_{ij} , we get (2.12). $\square$

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