

The ϕ -null controllability for semi-discrete stochastic semilinear parabolic equations*

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Abstract

The global null controllability of stochastic semilinear parabolic equations with globally Lipschitz nonlinearities has been addressed in recent literature. However, there are no results concerning their numerical approximation and the behavior of discrete controls when the discretization parameter goes to zero. This paper is intended to studying the ϕ -null controllability for semi-discrete stochastic semilinear parabolic equations, where the spatial variable is discretized using a finite difference scheme while the temporal variable remains continuous. The ϕ -null controllability means that terminal state can be dominated by a function that tends to zero as the spatial discretization parameter h approaches zero. The proof is based on a new refined semi-discrete Carleman estimate and Banach fixed point method. The main novelty here is that the Carleman parameters and discretization parameter are made explicit and are then used in a Banach fixed point method.

1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Write $\mathbb{F}$ for the progressive σ -field with respect to $\mathbf{F}$. Let $\mathcal{H}$ be a Banach space. Denote by $L^2_{\mathcal{F}_\tau}(\Omega; \mathcal{H})$ the space of all $\mathcal{F}_\tau$ -measurable random variables ξ such that $\mathbb{E}|\xi|_{\mathcal{H}}^2 < \infty$; by $L^2_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$ -valued $\mathbf{F}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{L^2(0, T; \mathcal{H})}^2) < \infty$, with the canonical norm; by $L^\infty_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$ -valued $\mathbf{F}$ -adapted essentially bounded processes; and by $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathcal{H}))$ the Banach space consisting of all $\mathcal{H}$ -valued $\mathbf{F}$ -adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{C([0, T]; \mathcal{H})}^2) < \infty$.

Let $G = (0, 1)$ and G_0 be a nonempty open subset of G . As usual, χ_{G_0} denotes the characteristic function of G_0 . We consider the following stochastic semilinear parabolic problem in a continuous framework:

$$\begin{cases} dy - y_{xx}dt = (f(\omega, t, x, y) + \chi_{G_0}u)dt + (g(\omega, t, x, y) + v)dW(t) & \text{in } (0, T) \times G, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ y(0) = y_0 & \text{in } G. \end{cases} \quad (1.1)$$

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Here, the initial datum $y_0 \in L^2(G)$ and the controls $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$. The null controllability problem of (1.1) consists of finding controls (u, v) such that $y(T) = 0$ in G , $\mathbb{P}$ -a.s.

As we known, the controllability of deterministic nonlinear parabolic equations has been studied by many authors and the results available in the literature are very rich (see e.g., [2, 9–12, 16] and the references cited therein). A common feature of these results is that they usually use linearization and suitable fixed point methods to address the controllability problem of nonlinear systems. To this end, the important property of compactness is often required.

In recent years, the controllability of stochastic parabolic equations has received much attention. However, most of the existing results are focused only on stochastic linear parabolic equations, with very few results on the nonlinear case. The main reason is that the compactness property, which is one of the key tools in the deterministic setting, is known to be false for the functional spaces related to stochastic setting. To the best of our knowledge, the only known results in this direction are [13, 14, 24, 25]. In [14], the authors prove the global null controllability of forward stochastic semilinear parabolic equations with globally Lipschitz nonlinearities using a refined Carleman estimate and the Banach fixed point theorem in weighted Sobolev spaces. This method effectively circumvents the well-known problem of the lack of compactness embeddings for the solution spaces arising in the study of controllability problems for stochastic parabolic equations.

Specifically, consider the following assumptions for f and g :

(A1). The functions f and g have pointwise representatives measurable with respect to $\mathcal{P} \otimes \mathcal{B}(G) \otimes \mathcal{B}(\mathbb{R})$, where $\mathcal{P}$ is the progressively measurable σ -algebra on $\Omega \times [0, T]$. In particular, their values at every grid point are progressively measurable. For each $\varphi \in L^2(G)$, the compositions $f(\cdot, \cdot, \cdot, \varphi)$ and $g(\cdot, \cdot, \cdot, \varphi)$ are $L^2(G)$ -valued progressively measurable processes;

(A2). There exists $L > 0$, for $(\omega, t, x, s_1, s_2) \in \Omega \times [0, T] \times G \times \mathbb{R} \times \mathbb{R}$, it holds that

$$|f(\omega, t, x, s_1) - f(\omega, t, x, s_2)| \leq L|s_1 - s_2|;$$

(A3). For $(\omega, t, x) \in \Omega \times [0, T] \times G$, $f(\omega, t, x, 0) = 0$;

(A4). There exists $L > 0$ for $(\omega, t, x, s_1, s_2) \in \Omega \times [0, T] \times G \times \mathbb{R} \times \mathbb{R}$, it holds that

$$|g(\omega, t, x, s_1) - g(\omega, t, x, s_2)| \leq L|s_1 - s_2|;$$

(A5). For $(\omega, t, x) \in \Omega \times [0, T] \times G$, $g(\omega, t, x, 0) = 0$.

Assuming (A1)–(A3) and square integrability of $g(\cdot, \cdot, \cdot, y)$ along the trajectory used in the feedback cancellation, the system (1.1) is null controllable, i.e. for every $T > 0$ and for every $y_0 \in L^2(G)$, there exist controls $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$ such that the constructed solution y of (1.1) satisfies $y(T, x) = 0$ in G , $\mathbb{P}$ -a.s. (see [14, Theorem 1.1]).

As the numerical approximation of the control system (1.1), it is natural to consider the null controllability for semi-discrete stochastic semilinear parabolic equations. We shall introduce a finite difference approximation of the system (1.1).

Let $N \in \mathbb{N}$. Define the spatial discretization parameter $h \triangleq 1/(N+1)$. Consider the pairs (t, x_i) with $t \in (0, T)$ and $x_i = ih$ for $i = 0, \dots, N+1$. Applying the centered finite difference method to the spatial variable for the system (1.1), we consider the following semi-discrete stochastic semilinear

system:

$$\begin{cases} dy^i - \frac{1}{h^2}(y^{i+1} - 2y^i + y^{i-1})dt \\ \quad = (f^i(\omega, t, x_i, y^i) + \chi_{G_0}^i u^i)dt + (g^i(\omega, t, x_i, y^i) + v^i)dW(t) & t \in (0, T), \quad i = 1, \dots, N, \\ y^0(t) = y^{N+1}(t) = 0 & t \in (0, T), \\ y^i(0) = y_0^i & i = 1, \dots, N, \end{cases} \quad (1.2)$$

where $y_0^i = h^{-1} \int_{x_i-h/2}^{x_i+h/2} y_0(x) dx$ and $\chi_{G_0}^i = \chi_{G_0}(x_i)$.

To present our main results, we need to introduce some notations. Define the following regular partition of the interval $[0, 1]$ as $\mathcal{K} \triangleq \{x_i \mid i = 0, 1, \dots, N, N+1\}$. For any set of points $\mathcal{W} \subset \mathcal{K}$, we denote the dual meshes $\mathcal{W}'$ and $\mathcal{W}^*$, respectively, by

$$\mathcal{W}' \triangleq \tau_+(\mathcal{W}) \cap \tau_-(\mathcal{W}), \quad \mathcal{W}^* \triangleq \tau_+(\mathcal{W}) \cup \tau_-(\mathcal{W}),$$

where

$$\tau_{\pm}(\mathcal{W}) \triangleq \left\{ x \pm \frac{h}{2} \mid x \in \mathcal{W} \right\}.$$

Define $\overline{\mathcal{W}} \triangleq (\mathcal{W}^*)^*$ and $\mathring{\mathcal{W}} \triangleq (\mathcal{W}')'$. Notice that for two consecutive points $x_i, x_{i+1} \in \mathcal{W}$, we obtain $x_{i+1} - x_i = h$ provided $\overline{\mathcal{W}} = \mathcal{W}$. Such a subset $\mathcal{W} \subset \mathcal{K}$ is called a regular mesh. Denote the boundary of a regular mesh $\mathcal{W}$ as $\partial\mathcal{W} \triangleq \overline{\mathcal{W}} \setminus \mathcal{W}$.

We define the average operator A_h and the difference operator D_h by

$$\begin{aligned} A_h(u)(t, x) &\triangleq \frac{\tau_+ u(t, x) + \tau_- u(t, x)}{2}, \\ D_h(u)(t, x) &\triangleq \frac{\tau_+ u(t, x) - \tau_- u(t, x)}{h}, \end{aligned}$$

where $\tau_{\pm} u(t, x) \triangleq u(t, x \pm \frac{h}{2})$.

Let $L(\mathcal{W})$ represent the set of real-valued functions defined in $\mathcal{W}$. We denote by $L_h^2(\mathcal{W})$ the Hilbert space with the inner product

$$\langle u, v \rangle_{L_h^2(\mathcal{W})} \triangleq \int_{\mathcal{W}} uv \triangleq h \sum_{x \in \mathcal{W}} u(x)v(x)$$

For $u \in L(\mathcal{W})$, its $L_h^\infty(\mathcal{W})$ -norm is defined as

$$\|u\|_{L_h^\infty(\mathcal{W})} \triangleq \max_{x \in \mathcal{W}} \{|u(x)|\}.$$

With above notations, letting $Q \triangleq (0, T) \times \mathcal{M}$ where $\mathcal{M} \triangleq \mathring{\mathcal{K}}$, the controlled semi-discrete stochastic semilinear system (1.2) can be written as

$$\begin{cases} dy - D_h^2 y dt = (f(\omega, t, x, y) + \chi_{G_0} u)dt + (g(\omega, t, x, y) + v)dW(t) & \text{in } Q, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ y(0, x) = y_0 & \text{in } \mathcal{M}, \end{cases} \quad (1.3)$$

where the initial datum $y_0 \in L_h^2(\mathcal{M})$ and the controls $(u, v) \in L_{\mathbb{F}}^2(0, T; L_h^2(G_0 \cap \mathcal{M})) \times L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M}))$. Utilizing the classical theory of stochastic differential equations (see [20, Theorem 3.2]), there is a unique solution $y \in L_{\mathbb{F}}^2(\Omega; C([0, T]; L_h^2(\mathcal{M})))$ to (1.3). The main purpose of this paper is to study the null controllability for the semi-discrete stochastic semilinear parabolic equation (1.3).

Remark 1.1. It is a technical requirement to introduce two controls u and v in (1.3). This is because, in the Carleman estimate (3.10), the term involving $|g|$ appear on the right-hand side of the inequality. So far, we do not know how to handle this term. In fact, for general stochastic parabolic equations in the continuous setting, this remains an open problem. However, for some special linear systems where the principal coefficients depend only on the spatial variable and the lower order terms are independent of spatial variable, controllability results with control only in the drift term can be obtained (see [19, 23]).

Remark 1.2. The question of whether controllability can be achieved with a scalar control $v(\omega, t)$ in the diffusion term is a fundamental and highly challenging problem. To the best of our knowledge, even in the continuous (non-discrete) setting, this problem remains open and is considered a major obstacle in the control theory of SPDEs. Existing results on scalar controls for parabolic equations are limited to deterministic systems [3], and their extension to the stochastic case is not straightforward. It would likely require the development of entirely new analytical tools.

Remark 1.3. One might consider simplifying the system by absorbing the diffusion term via feedback control, i.e., by taking $v = -g$. However, this does not necessarily reduce the difficulty of proving controllability. In this case, the system becomes a parabolic equation with random coefficients:

$$\begin{cases} dy - D_h^2 y dt = (f(\omega, t, x, y) + \chi_{G_0} u) dt & \text{in } Q, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ y(0, x) = y_0 & \text{in } \mathcal{M}. \end{cases}$$

If we treat the sample point ω as a parameter, then for any fixed $\omega \in \Omega$, the controllability of such a deterministic equation with random coefficients is known (see [5]), implying the existence of a control $u(\omega, \cdot, \cdot)$. However, the crucial difficulty lies in the stochastic setting: there is no guarantee that such a control, constructed for each ω , would be adapted to the filtration $\mathbf{F}$. A non-adapted control would require future information, which is inadmissible. This “adaptedness” constraint is a fundamental challenge in SPDE control theory that is not circumvented by this simplification.

Remark 1.4. The assumption that the control v acts on the whole domain G is an idealization. A more physically relevant and challenging problem is to consider a control localized to a smaller region $G_1 \subset G$. This localized control problem is a major open question in the field, even for continuous linear SPDEs. The primary technical obstacle is the reliance of current methods on global Carleman estimates. A solution would require a new type of Carleman estimate where the right-hand-side integral is restricted to the control region G_1 , i.e., involving the term $\mathbb{E} \int_0^T \int_{G_1} s^2 e^{2s\phi} |g|^2 dt$. Developing such an estimate would be a significant breakthrough. Therefore, our work, which addresses the global control case, should be seen as an essential first step. It establishes a foundational result and develops techniques in the semi-discrete stochastic setting, providing a benchmark for future investigations into more realistic scenarios like the localized control problem.

Remark 1.5. It is an interesting problem to consider the case where $f(\omega, t, x, y, y_x)$ depends on y_x . Although our Carleman estimate (3.10) involves the term $\mathbb{E} \int_{Q^*} s e^{2s\phi} |D_h w|^2 dt$, the main difficulty lies in how to add the term $\mathbb{E} \int_Q e^{-2s\phi} s^{-2} |A_h D_h y|^2 dx$ into the left-hand side of (4.6). Even in the continuous case [14], this problem remains open. At present, we do not know how to overcome this difficulty.

There are numerous studies on the controllability theory of deterministic linear semi-discrete parabolic equations (see [1, 6, 8, 15, 21, 27, 28] and the references cited therein). However, there are fewer studies on the controllability of semi-discrete semilinear systems. In [5], the authors study the null controllability of semi-discrete semilinear parabolic equations, using the linearization and Schauder topological fixed point theorem. In contrast, there are only a few papers which are concerned with controllability of the semi-discrete stochastic parabolic equations, i.e., [26] which announced the null controllability of linear semi-discrete stochastic second order parabolic equations, and [22] which announced the null controllability of linear semi-discrete stochastic fourth order parabolic equations. To the best of our knowledge, there are no results available regarding the controllability of semi-discrete stochastic nonlinear parabolic equations.

For discrete systems, it is well known that uniform null controllability is hard to address, i.e. for any initial condition, there exist controls (u, v) such the solution of system satisfies $y^i(T) = 0$ for $i \in 1, \dots, N$. Thus, we consider the ϕ -null controllability which proposed in [5], i.e. terminal state can be dominated by a function which tends to zero as spatial discretization parameter h tends to zero. This definition is highly useful for characterizing the behavior of discrete controls when the discretization parameter goes to zero.

To overcome the lack of compactness for the solution spaces of stochastic differential equations, we borrow some ideas from [14] for proving the null controllability of stochastic semilinear parabolic equations (1.3). First, we introduce a new weight function that differs from the continuous case in [14], and we obtain a new global Carleman estimate for a linear backward stochastic semi-discrete parabolic equations with a source term in a suitable weighted space. Next, by combining the Carleman estimate with the duality argument and the Hilbert Uniqueness Method introduced in [4], we establish the ϕ -null controllability for a linear system, which Carleman parameters and discretization parameter h are made explicit. At last, the ϕ -null controllability of system (1.3) is ensured through a Banach fixed point method which does not rely on any compactness argument.

The main result of this paper is the following.

Theorem 1.1. *Under assumptions (A1) – (A5), there exist positive constants h_0 and C such that for all $h \leq h_0$, there exist $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2_h(G_0 \cap \mathcal{M})) \times L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$, so that the solution y to (1.3) satisfies*

$$\mathbb{E} \int_Q |y|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} |u|^2 dt + \mathbb{E} \int_Q |v|^2 dt \leq C \int_{\mathcal{M}} |y_0|^2, \quad (1.4)$$

and

$$\mathbb{E} \int_{\mathcal{M}} |y(T)|^2 \leq C e^{-\frac{C}{h}} \int_{\mathcal{M}} |y_0|^2. \quad (1.5)$$

Here and in what follows, unless otherwise stated, C stands for a generic positive constant depending only on G , G_0 , T and the Lipschitz constant L whose value may vary from line to line. The mesh threshold h_0 in the main theorem may also depend on L .

Remark 1.6. For the semi-discrete linear system, we can obtain a controllability result with only one control. In fact, consider the following equation:

$$\begin{cases} dy - D_h^2 y dt = \chi_{G_0} u dt + a(t) y dW(t) & \text{in } Q, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ y(0, x) = y_0 & \text{in } \mathcal{M}, \end{cases} \quad (1.6)$$

where $a(t) \in L^\infty_{\mathbb{F}}(0, T; \mathbb{R})$, the initial datum $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2_h(\mathcal{M}))$ and the control $u \in L^\infty_{\mathbb{F}}(0, T; L^2(\Omega; L^2_h(\mathcal{M})))$. By using the partial spectral inequality for discrete elliptic operators established in [7] and the techniques developed in [19], one can show that the ϕ -null controllability for (1.6) holds. However, the above method is difficult to apply to the semilinear case.

Remark 1.7. We claim that, under assumptions (A1) – (A5), the ϕ -null controllability for (1.2) can be reduced to the ϕ -null controllability of the following equation:

$$\begin{cases} dy - D_h^2 y dt = (f(y) + \chi_{G_0} u) dt + v dW(t) & \text{in } Q, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ y(0, x) = y_0 & \text{in } \mathcal{M}. \end{cases} \quad (1.7)$$

Indeed, assume that the ϕ -null controllability for (1.7) holds. Then there exist $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2_h(G_0 \cap \mathcal{M})) \times L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$ such that the solution to (1.7) satisfies (1.4) and (1.5). Since the control v is distributed in the whole domain Q , the state y satisfies (1.3) with the control $u^* = u$ and $v^* = v - g(y)$. Hence (1.5) holds. Thanks to assumptions (A1), (A4), (A5) and (1.4), one can show that (u^*, v^*) satisfies

$$\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} |u^*|^2 dt + \mathbb{E} \int_Q |v^*|^2 dt \leq C \int_{\mathcal{M}} |y_0|^2,$$

which implies the ϕ -null controllability for (1.3).

Remark 1.8. From Remark 1.7, assumptions (A1)–(A3) suffice for the feedback cancellation argument provided the constructed trajectory satisfies $g(\cdot, \cdot, \cdot, y) \in L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$. Under this additional integrability condition, the feedback defines an admissible control v and (1.5) holds. Without it, even the stochastic integral need not be defined. However, without assumptions (A4) and (A5), it is difficult to show that the control of the diffusion term is uniformly bounded with respect to discretization parameter h . This is why we need these assumptions (A4)–(A5) in Theorem 1.1.

The rest of this paper is organized as follows. In section 2, we give some preliminaries. A new Carleman estimate is proved in section 3. In section 4, we prove the controllability for linear stochastic semi-discrete parabolic equations with a source. Finally, we prove the main controllability result Theorem 1.1 based on the Banach fixed point method in Section 5.

2 Some preliminary results

This section is devoted to presenting some preliminary results we needed.

Lemma 2.1. [8, Lemma 2.1] *For any $u, v \in L(\overline{W})$, we have for the difference operator*

$$D_h(uv) = D_h u A_h v + A_h u D_h v, \text{ on } \mathcal{W}^*. \quad (2.1)$$

Similarly, the average of the product gives

$$A_h(uv) = A_h u A_h v + \frac{h^2}{4} D_h u D_h v, \text{ on } \mathcal{W}^*. \quad (2.2)$$

Finally, on $\mathring{\mathcal{W}}$ we have

$$u = A_h^2 u - \frac{h^2}{4} D_h^2 u. \quad (2.3)$$

Corollary 2.2. [8, Corollary 2.2] Let $\overline{\mathcal{W}} \subseteq \mathcal{M}$ be a regular mesh. For $u \in L(\overline{\mathcal{W}})$,

$$A_h(u^2) = (A_h u)^2 + \frac{h^2}{4}(D_h u)^2, \text{ on } \mathcal{W}^*. \quad (2.4)$$

In particular, for all $u \in L(\overline{\mathcal{W}})$,

$$A_h(u^2) \geq (A_h u)^2, \text{ on } \mathcal{W}^*. \quad (2.5)$$

For $u \in L(\overline{\mathcal{W}})$

$$D_h(u^2) = 2D_h u A_h u. \quad (2.6)$$

To deal with the boundary conditions, we define the outward normal for $x \in \partial\mathcal{W}$ as

$$\nu(x) \triangleq \begin{cases} 1, & \text{if } \tau_-(x) \in \mathcal{W}^* \text{ and } \tau_+(x) \notin \mathcal{W}^*, \\ -1, & \text{if } \tau_-(x) \notin \mathcal{W}^* \text{ and } \tau_+(x) \in \mathcal{W}^*, \\ 0, & \text{otherwise.} \end{cases} \quad (2.7)$$

We also introduce the trace operator for $u \in L(\mathcal{W}^*)$ as

$$\forall x \in \partial\mathcal{W}, \quad t_r(u) \triangleq \begin{cases} \tau_- u(x), & \text{if } \nu(x) = 1, \\ \tau_+ u(x), & \text{if } \nu(x) = -1, \\ 0, & \text{if } \nu(x) = 0. \end{cases}$$

Then by the definition of trace operator, for $x \in \partial\mathcal{W}$ and $u \in L(\overline{\mathcal{W}})$, it is easy to check that

$$t_r(|A_h u|^2)\nu - \frac{h^2}{4}t_r(|D_h u|^2)\nu = |u|^2\nu - h u t_r(D_h u). \quad (2.8)$$

Finally, let us introduce the discrete integration on the boundary for $u \in L(\partial\mathcal{W})$ as

$$\int_{\partial\mathcal{W}} u \triangleq \sum_{x \in \partial\mathcal{W}} u(x).$$

Proposition 2.3. [8, Proposition 2.3] Let $\mathcal{W}$ be a semi-discrete regular mesh. For $u \in L(\overline{\mathcal{W}})$ and $v \in L(\mathcal{W}^*)$ we have

$$\int_{\mathcal{W}} u D_h v = - \int_{\mathcal{W}^*} D_h u v + \int_{\partial\mathcal{W}} u t_r(v) \nu \quad (2.9)$$

and

$$\int_{\mathcal{W}} u A_h v = \int_{\mathcal{W}^*} A_h u v - \frac{h}{2} \int_{\partial\mathcal{W}} u t_r(v). \quad (2.10)$$

Proposition 2.4. [8, Corollary 2.7] Let f be a $(n+4)$ -times differentiable function defined on $\mathbb{R}$ and $m, n \in \mathbb{N}$. Then

$$A_h^m D_h^n f = f^{(n)} + R_{A_h^m}(f^{(n)}) + R_{D_h^n}(f) + R_{A_h^m D_h^n}(f),$$

where

$$R_{D_h^n}(f) := h^2 \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{(n-2k)}{2} \right)^{n+2} \int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!} f^{(n+2)} \left(\cdot + \frac{(n-2k)h}{2} \sigma \right) d\sigma$$

$$R_{A_h^n}(f) := \frac{h^2}{2^{n+2}} \sum_{k=0}^n \binom{n}{k} (n-2k)^2 \int_0^1 (1-\sigma) f^{(2)} \left(\cdot + \frac{(n-2k)h}{2} \sigma \right) d\sigma,$$

and $R_{A_h^m D_h^n} := R_{A_h^m} \circ R_{D_h^n}$. Here, $f^{(n)}$ denotes the n -th derivative of f with respect to x .

3 Carleman estimate

This section is devoted to proving a new Carleman estimate for a backward stochastic semi-discrete parabolic equation. To state our Carleman estimates, we first introduce the weight functions.

Let $\tilde{G}$ be an open interval contains $[0, 1]$, and G_1, G_2 be given two nonempty open subsets of G satisfying $G_2 \subset \overline{G_2} \subset G_1 \subset \overline{G_1} \subset G_0$ and $\overline{\mathcal{M} \cap G_2} \subset \mathcal{M} \cap G_1$. Thanks to [12, Lemma 1.1], there exists a function $\psi \in C^k(\tilde{G})$ with k large enough such that

$$0 < \psi \leq 1 \text{ in } \tilde{G}, \quad |\partial_x \psi| \geq C_0 > 0 \text{ in } \overline{G \setminus G_2}, \quad \partial_x \psi(0) > 0, \text{ and } \partial_x \psi(1) < 0. \quad (3.1)$$

Without loss of generality, in what follows, we assume that $0 < T < 1$. For some constants $m \geq 1$, $\sigma \geq 2$ and $0 < \delta < \frac{1}{2}$, we define the following weight function

$$\begin{cases} \theta(t) = 1 + \left(1 - \frac{4t}{T}\right)^\sigma, & t \in [0, T/4), \\ \theta(t) = 1, & t \in [T/4, T/2), \\ \theta \text{ is in increasing on } [T/2, 3T/4), \\ \theta(t) = \frac{1}{(T-t+\delta T)^m}, & t \in [3T/4, T], \\ \theta \in C^2([0, T]). \end{cases} \quad (3.2)$$

For any parameters $\lambda \geq 1$ and $\mu \geq 1$, let

$$\begin{cases} \phi(x) = e^{\mu(\psi(x)+6m)} - \mu e^{6\mu(m+1)}, & \varphi(x) = e^{\mu(\psi(x)+6m)}, \\ s(t) = \lambda \theta(t), & r(t, x) = e^{s(t)\phi(x)}, \quad \rho(t, x) = (r(t, x))^{-1}. \end{cases} \quad (3.3)$$

We choose σ as

$$\sigma = \lambda \mu^2 e^{\mu(6m-4)}. \quad (3.4)$$

Remark 3.1. In the continuous setting, the regularity of the weight function ψ can be chosen as $C^2([0, 1]; \mathbb{R})$. Here estimates of discrete derivations of the weight function require knowledge of higher-order derivatives through Taylor formulas. Moreover, to facilitate the Taylor expansion of the weight function at the boundary, the domain of ψ has been slightly extended.

Remark 3.2. The assumption $0 < T < 1$ is only technical. In fact, if $T \geq 1$, one can modify the definition of $\theta(t)$ on the interval $(T/2, T]$ by setting $\theta(t) = 2(T/4 + \delta T)^m (T - t + \delta T)^{-m}$ for $t \in [3T/4, T]$. With this adjustment, the Carleman estimate (3.10) remains valid.

Remark 3.3. It can be seen from the (3.2) that, compared with the weight function in continuous setting (see [14]), the main difference here is that the weight does not singular at times $t = T$.

In the sequel, for any $n \in \mathbb{N}$, we denote by $\mathcal{O}(s^n)$ a function of order s^n for sufficiently large s (which is independent of μ and h), by $\mathcal{O}_\mu(s^n)$ a function of order s^n for fixed μ and sufficiently large s (which is independent of h), and by $\mathcal{O}_\mu((sh)^n)$ a function of order $(sh)^n$ for fixed μ and sufficiently large s , sufficiently small h .

We need some estimates regarding the weighted functions.

Lemma 3.1. [17, Lemma 4.3] For $\alpha, \beta \in \mathbb{N}$, we have

$$\begin{aligned}\partial_x^\beta(r\partial_x^\alpha\rho) &= \alpha^\beta(-s\varphi)^\alpha\mu^{\alpha+\beta}(\partial_x\psi)^{\alpha+\beta} + \alpha\beta(s\varphi)^\alpha\mu^{\alpha+\beta-1}\mathcal{O}(1) + s^{\alpha-1}\alpha(\alpha-1)\mathcal{O}_\mu(1) \\ &= s^\alpha\mathcal{O}_\mu(1).\end{aligned}\tag{3.5}$$

Corollary 3.2. [17, Corollary 4.4] For $\alpha, \beta, \gamma \in \mathbb{N}$, we have

$$\begin{aligned}\partial_x^\gamma(r^2(\partial_x^\alpha\rho)\partial_x^\beta\rho) &= (\alpha+\beta)^\gamma(-s\varphi)^{\alpha+\beta}\mu^{\alpha+\beta+\gamma}(\partial_x\psi)^{\alpha+\beta+\gamma} + \gamma(\alpha+\beta)(s\varphi)^{\alpha+\beta}\mu^{\alpha+\beta+\gamma-1}\mathcal{O}(1) \\ &\quad + s^{\alpha+\beta-1}[\alpha(\alpha-1) + \beta(\beta-1)]\mathcal{O}_\mu(1) \\ &= s^{\alpha+\beta}\mathcal{O}_\mu(1).\end{aligned}\tag{3.6}$$

By (3.3) and (3.5), it easy to see that the following result holds.

Lemma 3.3. For $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, we have

$$\begin{aligned}\partial_x^\sigma(r\partial_x^\alpha\rho\partial_x^\beta(r\partial_x^\gamma\rho)) &= \gamma^\beta(\alpha+\gamma)^\sigma(-s\varphi)^{\alpha+\gamma}\mu^{\alpha+\beta+\gamma+\sigma}(\partial_x\psi)^{\alpha+\beta+\gamma+\sigma} + (s\varphi)^{\alpha+\gamma}\mu^{\alpha+\beta+\gamma+\sigma-1}\mathcal{O}(1) \\ &\quad + s^{\alpha+\gamma-1}\mathcal{O}_\mu(1) \\ &= s^{\alpha+\gamma}\mathcal{O}_\mu(1).\end{aligned}\tag{3.7}$$

Lemma 3.4. [17, Lemma 4.8] Let $\alpha, j, k, m, n \in \mathbb{N}$. Provided $\lambda h(\delta T)^{-m} \leq 1$, we have

$$A_h^j D_h^k \partial_x^\alpha(r A_h^m D_h^n \rho) = \partial_x^k \partial_x^\alpha(r \partial_x^n \rho) + s^n \mathcal{O}_\mu((sh)^2) = s^n \mathcal{O}_\mu(1).\tag{3.8}$$

By Theorem 2.4 and (3.8), it is easy to check that the following estimate holds.

Lemma 3.5. Let $i, j, k, l, m, n, p, q \in \mathbb{N}$. Provided $\lambda h(\delta T)^{-m} \leq 1$, we have

$$\begin{aligned}A_h^i D_h^j(r A_h^k D_h^l \rho A_h^m D_h^n(r A_h^p D_h^q \rho)) &= \partial_x^j(r \partial_x^l \rho \partial_x^n(r \partial_x^q \rho)) + s^{l+q} \mathcal{O}_\mu((sh)^2) \\ &= s^{l+q} \mathcal{O}_\mu(1).\end{aligned}\tag{3.9}$$

Now, we state the main result of this section, which is a Carleman estimate for backward stochastic semi-discrete parabolic equations.

Theorem 3.6. Let the functions s and ϕ be defined in (3.3). For all $m \geq 1$, there exists a constant $\mu_0 \geq 1$ such that for any $\mu \geq \mu_0$, one can find positive constants $C = C(\mu)$, h_0 and ε_0 with $0 < \varepsilon_0 \leq 1$ and $\lambda_0 \geq 1$ so that for all $\lambda \geq \lambda_0$, $0 < h \leq h_0$ and $\lambda h(\delta T)^{-m} \leq \varepsilon_0$, it holds that

$$\begin{aligned}&\mathbb{E} \int_Q s^3 e^{2s\phi} |w|^2 dt + \mathbb{E} \int_{Q^*} s e^{2s\phi} |D_h w|^2 dt + \mathbb{E} \int_{\mathcal{M}} \lambda^2 \mu^3 e^{2\mu(6m+1)} e^{2s\phi} |w|^2 \Big|_{t=0} \\ &\leq C \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} s^3 e^{2s\phi} |w|^2 dt + \mathbb{E} \int_Q e^{2s\phi} |f|^2 dt + \mathbb{E} \int_Q s^2 e^{2s\phi} |g|^2 dt \right)\end{aligned}$$

$$+ h^{-2} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |w|^2 \Big|_{t=T} \Big), \quad (3.10)$$

for all $w \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_h(\mathcal{M})))$ and $f, g \in L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$ satisfying

$$dw + D_h^2 w dt = f dt + g dW(t) \quad (3.11)$$

with $w = 0$ on $\partial\mathcal{M}$.

Remark 3.4. In the semi-discrete stochastic semilinear case, certain terms arise that cannot be directly estimated. The key to handling these terms lies in the careful choice of the auxiliary functions $\Phi(v)$ and $\Psi(v)$ in (3.13). For example, the term $-h^2 \mathbb{E} \int_Q D_h(D_h q^{11} A_h v) dv$ in (3.28) cannot be estimated directly, but it can be absorbed by introducing $\Phi(v)$ and using (3.29). Similarly, the term $2 \mathbb{E} \int_Q \lambda \phi \theta_t \partial_x(r \partial_x \rho) v^2 dt$ in (3.38) is also not directly estimable; by introducing $\Psi(v)$, it is absorbed via I_{33} in (3.44).

Proof of Theorem 3.6. For readability, we have divided the proof into several steps.

Step 1. Letting $v = rw$ and recalling $\rho = r^{-1}$, by (2.1), we have the identity

$$r(dw + D_h^2 w dt) = I_1 dt - \Phi(v) dt - \Psi(v) dt + I_2, \quad (3.12)$$

where

$$\begin{cases} I_1 = r D_h^2 \rho A_h^2 v + r A_h^2 \rho D_h^2 v - \lambda \phi \theta_t v + \Phi(v), \\ I_2 = dv + 2r A_h D_h \rho A_h D_h v dt + \Psi(v) dt, \\ \Phi(v) = \frac{h^2}{2} D_h [D_h (r D_h^2 \rho) A_h v], \\ \Psi(v) = D_h (r D_h \rho) v. \end{cases} \quad (3.13)$$

Here, Φ and Ψ are two auxiliary functions to absorb some of the bad terms. From (3.12), we obtain

$$2r I_1 (dw + D_h^2 w dt) = 2I_1^2 dt - 2I_1 \Phi(v) dt - 2I_1 \Psi(v) dt + 2I_1 I_2,$$

which, combined with Cauchy-Schwarz inequality, and noting (3.11), implies

$$\mathbb{E} \int_Q |rf|^2 dt \geq 2 \mathbb{E} \int_Q I_1 I_2 - 2 \mathbb{E} \int_Q |\Phi(v)|^2 dt - 2 \mathbb{E} \int_Q |\Psi(v)|^2 dt. \quad (3.14)$$

In the following steps, we estimate the right-hand side terms of (3.14). Hereinafter, for convenience, we put

$$\mathbb{E} \int_Q I_1 I_2 = \sum_{i=1}^4 \sum_{j=1}^3 I_{ij}, \quad (3.15)$$

where I_{ij} stands for the inner product in $L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$ between the i -th term of I_1 and j -th term of I_2 .

In order to shorten the formulas used in the sequel, we define the following

$$\mathcal{A}_1 = \mathbb{E} \int_Q [s^2 \mathcal{O}_{\mu}(1) + s^3 \mathcal{O}_{\mu}((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |v|^2 dt$$

$$\begin{aligned}
& + \mathbb{E} \int_{Q^*} [\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2) + s\mu\varphi\mathcal{O}(1)] |D_h v|^2 dt, \\
\mathcal{A}_2 &= \mathbb{E} \int_Q s^2 \mathcal{O}_\mu(1) |dv|^2 + \mathbb{E} \int_{Q^*} \mathcal{O}_\mu((sh)^2) |D_h(dv)|^2, \\
\mathcal{A}_3 &= \mathbb{E} \int_{\mathcal{M}} s^2 \mathcal{O}_\mu(1) |v|^2 \Big|_{t=T} + \mathbb{E} \int_{\mathcal{M}^*} \mathcal{O}(1) |D_h v|^2 \Big|_{t=T} \\
& + \mathbb{E} \int_{\mathcal{M}} (s^2 \mu^2 e^{2\mu(6m+1)} \mathcal{O}(1) + s\mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) |v|^2 \Big|_{t=0}, \\
\mathcal{B} &= \mathbb{E} \int_{\partial Q} [\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2)] t_r (|D_h v|^2) dt.
\end{aligned}$$

Step 2. We compute $\sum_{i=1}^4 I_{i1}$. Combining (2.3) and (3.13), letting $q^{11} = rD_h^2 \rho$, we obtain

$$\begin{aligned}
2 \sum_{i=1}^4 I_{i1} &= 2\mathbb{E} \int_Q (rD_h^2 \rho A_h^2 v + rA_h^2 \rho D_h^2 v - \lambda\phi\theta_t v + \Phi(v)) dv \\
&= 2\mathbb{E} \int_Q \left(rD_h^2 \rho v + D_h^2 v + \frac{h^2}{2} rD_h^2 \rho D_h^2 v - \lambda\phi\theta_t v + \Phi(v) \right) dv \\
&= 2\mathbb{E} \int_Q \left(q^{11} v + D_h^2 v + \frac{h^2}{2} q^{11} D_h^2 v - \lambda\phi\theta_t v + \Phi(v) \right) dv. \tag{3.16}
\end{aligned}$$

Thanks to Itô's formula, we have

$$\begin{aligned}
2\mathbb{E} \int_Q (q^{11} - \lambda\phi\theta_t) v dv &= \mathbb{E} \int_{\mathcal{M}} (q^{11} - \lambda\phi\theta_t) v^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M}} (q^{11} - \lambda\phi\theta_t) v^2 \Big|_{t=0} \\
&\quad - \mathbb{E} \int_Q \partial_t (q^{11} - \lambda\phi\theta_t) v^2 dt - \mathbb{E} \int_Q (q^{11} - \lambda\phi\theta_t) |dv|^2. \tag{3.17}
\end{aligned}$$

Combining (3.5) and (3.8), we conclude that

$$q^{11} = s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 + s\mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2). \tag{3.18}$$

Using definition of θ in (3.2), we obtain $\theta_t(T) = m(\delta T)^{-m-1}$. Thus, we get from (3.3) and (3.18) that

$$\mathbb{E} \int_{\mathcal{M}} (q^{11} - \lambda\phi\theta_t) v^2 \Big|_{t=T} \geq -\mathbb{E} \int_{\mathcal{M}} s^2 \mathcal{O}_\mu(1) |v|^2 \Big|_{t=T} \geq -\mathcal{A}_3. \tag{3.19}$$

Noting $\theta_t(0) = -\frac{4\sigma}{T}$, and from (3.3) and (3.4), a direct computation yields

$$\lambda\phi\theta_t(0) = \frac{4}{T} \lambda^2 \mu^2 e^{\mu(6m-4)} \left(\mu e^{6\mu(m+1)} - e^{\mu(\psi(x)+6m)} \right) \geq c \lambda^2 \mu^3 e^{2\mu(6m+1)},$$

for all $\mu > 1$ and some constant $c > 0$ uniform with respect to T . Hence, using (3.18) it can be readily seen that

$$-\mathbb{E} \int_{\mathcal{M}} (q^{11} - \lambda\phi\theta_t) v^2 \Big|_{t=0} \geq c \mathbb{E} \int_{\mathcal{M}} \lambda^2 \mu^3 e^{2\mu(6m+1)} |v(0)|^2 - \mathcal{A}_3. \tag{3.20}$$

Thanks to (3.3) and (3.18), we see that

$$\partial_t q^{11} = \frac{\theta_t}{\theta} (2s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 + s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)), \quad \partial_t(\lambda \phi \theta_t) = s \phi \frac{\theta_{tt}}{\theta},$$

which implies that

$$\begin{aligned} & -\mathbb{E} \int_Q \partial_t(q^{11} - \lambda \phi \theta_t) v^2 dt \\ &= -\mathbb{E} \int_Q \frac{\theta_t}{\theta} (2s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 + s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) v^2 dt + \mathbb{E} \int_Q s \phi \frac{\theta_{tt}}{\theta} v^2 dt. \end{aligned} \quad (3.21)$$

Using the definition of θ in (3.2), it is not difficult to see that for $t \in [0, \frac{T}{4}]$, $\theta \in [1, 2]$, and $|\theta_{tt}| \leq C \lambda^2 \mu^4 e^{2\mu(6m-4)}$. Noting there exists $\mu_1 \geq 1$ such that for all $\mu \geq \mu_1$, it holds that $\mu^2 e^{-2\mu} \leq 1$. Thus, by the definitions of ϕ, φ in (3.3), for $t \in [0, \frac{T}{4}]$, we end up with

$$s \frac{|\phi \theta_{tt}|}{\theta} \leq C s^3 \mu^5 e^{2\mu(6m-4)} e^{6\mu(m+1)} = C s^3 \mu^5 e^{\mu(18m-2)} \leq C s^3 \mu^3 \varphi^3.$$

Moreover, for $t \in [\frac{T}{4}, T]$, we have $|\theta_{tt}| \leq C \theta^3$, thus it is easy to check that

$$s \frac{|\phi \theta_{tt}|}{\theta} \leq s \mu \theta^2 e^{6\mu(m+1)} \leq s^3 \mu \varphi^3.$$

Therefore, by (3.21), we conclude that

$$\begin{aligned} & -\mathbb{E} \int_Q \partial_t(q^{11} - \lambda \phi \theta_t) v^2 dt \\ & \geq -\mathbb{E} \int_Q \frac{\theta_t}{\theta} (2s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 + s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) v^2 dt - \mathcal{A}_1. \end{aligned} \quad (3.22)$$

From (3.2), it is easy to check that $|\theta_t| \leq 4\sigma/T = \lambda \mathcal{O}_\mu(1)$ for $t \in [0, \frac{T}{4}]$, and $|\theta_t| \leq C \theta^2$ for $t \in [\frac{T}{4}, T]$. Then $|\lambda \phi \theta_t| \leq s^2 \mathcal{O}_\mu(1)$ for $t \in [0, T]$. And noting (3.18), we deduce that

$$-\mathbb{E} \int_Q (q^{11} - \lambda \phi \theta_t) |dv|^2 \geq -\mathbb{E} \int_Q s^2 \mathcal{O}_\mu(1) |dv|^2 \geq -\mathcal{A}_2. \quad (3.23)$$

Combining (3.17), (3.19), (3.20), (3.22) and (3.23), for $\mu \geq \mu_1$ and $\lambda h(\delta T)^{-m} \leq 1$, we conclude that

$$\begin{aligned} 2\mathbb{E} \int_Q (q^{11} - \lambda \phi \theta_t) v dv & \geq -\mathbb{E} \int_Q \frac{\theta_t}{\theta} (2s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 + s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) v^2 dt \\ & \quad + c \mathbb{E} \int_{\mathcal{M}} \lambda^2 \mu^3 e^{2\mu(6m+1)} |v(0)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \end{aligned} \quad (3.24)$$

From (2.9) and Itô's formula, and noting $v = 0$ on ∂Q , we obtain

$$\begin{aligned} 2\mathbb{E} \int_Q D_h^2 v dv &= -2\mathbb{E} \int_{Q^*} D_h v D_h(dv) + 2\mathbb{E} \int_{\partial Q} dv t_r(D_h v) \nu \\ &= -\mathbb{E} \int_{\mathcal{M}^*} |D_h v(T)|^2 + \mathbb{E} \int_{\mathcal{M}^*} |D_h v(0)|^2 + \mathbb{E} \int_{Q^*} |D_h(dv)|^2 \end{aligned}$$

$$\geq \mathbb{E} \int_{\mathcal{M}^*} |D_h v(0)|^2 + \mathbb{E} \int_{Q^*} |D_h(dv)|^2 - \mathcal{A}_3. \quad (3.25)$$

Thanks to (2.1) and (2.9) and $v = 0$ on ∂Q , we have

$$h^2 \mathbb{E} \int_Q q^{11} D_h^2 v dv = -h^2 \mathbb{E} \int_{Q^*} A_h q^{11} D_h(dv) D_h v - h^2 \mathbb{E} \int_{Q^*} D_h q^{11} A_h(dv) D_h v. \quad (3.26)$$

From Itô's formula, we obtain

$$\begin{aligned} -h^2 \mathbb{E} \int_{Q^*} A_h q^{11} D_h(dv) D_h v &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} d(A_h q^{11} |D_h v|^2) + \frac{h^2}{2} \mathbb{E} \int_{Q^*} \partial_t(A_h q^{11}) |D_h v|^2 \\ &\quad + \frac{h^2}{2} \mathbb{E} \int_{Q^*} A_h q^{11} |D_h(dv)|^2. \end{aligned} \quad (3.27)$$

By Itô's formula and (2.1), a direct computation shows that

$$A_h(dv) D_h v = \frac{1}{2} d(D_h(v^2)) - A_h v D_h(dv) - \frac{1}{2} D_h((dv)^2),$$

which combined with (3.13), (2.9) and $t_r(v) = 0$ on ∂Q^* implies

$$\begin{aligned} &-h^2 \mathbb{E} \int_{Q^*} D_h q^{11} A_h(dv) D_h v \\ &= -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q^{11} d(D_h(v^2)) + h^2 \mathbb{E} \int_{Q^*} D_h q^{11} A_h v D_h(dv) + \frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q^{11} D_h((dv)^2) \\ &= \frac{h^2}{2} \mathbb{E} \int_Q D_h^2 q^{11} d(v^2) - h^2 \mathbb{E} \int_Q D_h(D_h q^{11} A_h v) dv + \frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q^{11} D_h((dv)^2) \\ &= \frac{h^2}{2} \mathbb{E} \int_Q d(D_h^2 q^{11} v^2) - \frac{h^2}{2} \mathbb{E} \int_Q \partial_t(D_h^2 q^{11}) v^2 dt - h^2 \mathbb{E} \int_Q D_h(D_h q^{11} A_h v) dv - \frac{h^2}{2} \mathbb{E} \int_Q D_h^2 q^{11} |dv|^2. \end{aligned} \quad (3.28)$$

Combining (3.26)–(3.28), and noting (3.13), $q^{11} = r D_h^2 \rho$, it holds that

$$\begin{aligned} &2 \mathbb{E} \int_Q \left(\frac{h^2}{2} q^{11} D_h^2 v + \Phi(v) \right) dv \\ &= -\frac{h^2}{2} \left[\mathbb{E} \int_{Q^*} d(A_h q^{11} |D_h v|^2) - \mathbb{E} \int_{Q^*} \partial_t(A_h q^{11}) |D_h v|^2 - \mathbb{E} \int_{Q^*} A_h q^{11} |D_h(dv)|^2 \right. \\ &\quad \left. - \mathbb{E} \int_Q d(D_h^2 q^{11} v^2) + \mathbb{E} \int_Q \partial_t(D_h^2 q^{11}) v^2 dt + \mathbb{E} \int_Q D_h^2 q^{11} |dv|^2 \right]. \end{aligned} \quad (3.29)$$

By $q^{11} = r D_h^2 \rho$, from (3.5) and (3.8) and $|D_h v|^2 \leq 2h^{-2}(|\tau_+ v|^2 + |\tau_- v|^2)$, we get

$$\begin{aligned} &-\frac{h^2}{2} \mathbb{E} \int_{Q^*} d(A_h q^{11} |D_h v|^2) \\ &= -\frac{h^2}{2} \mathbb{E} \int_{\mathcal{M}^*} A_h q^{11} |D_h v|^2 \Big|_{t=T} + \frac{h^2}{2} \mathbb{E} \int_{\mathcal{M}^*} A_h q^{11} |D_h v|^2 \Big|_{t=0} \end{aligned}$$

$$\begin{aligned}
&\geq -\mathbb{E} \int_{\mathcal{M}^*} \mathcal{O}_\mu((sh)^2) |D_h v|^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M}} (s^2 \mu^2 e^{2\mu(6m+1)} \mathcal{O}(1) + s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) |v|^2 \Big|_{t=0} \\
&\geq -\mathcal{A}_3,
\end{aligned} \tag{3.30}$$

and

$$\frac{h^2}{2} \mathbb{E} \int_Q d(D_h^2 q^{11} v^2) \geq -\mathbb{E} \int_{\mathcal{M}} \mathcal{O}_\mu((sh)^2) |v|^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M}} \mathcal{O}_\mu((sh)^2) |v|^2 \Big|_{t=0} \geq -\mathcal{A}_3. \tag{3.31}$$

Thanks to (3.3), (3.5) and (3.8), noting $q^{11} = r D_h^2 \rho$, we see that

$$\partial_t(A_h q^{11}) = \frac{\theta_t}{\theta} s^2 \mathcal{O}_\mu(1), \quad \partial_t(D_h^2 q^{11}) = \frac{\theta_t}{\theta} s^2 \mathcal{O}_\mu(1),$$

and noting $|\theta_t| \leq \lambda \theta^2 \mathcal{O}_\mu(1)$, which implies that, for $\lambda h(\delta T)^{-m} \leq 1$,

$$\begin{aligned}
&-\frac{h^2}{2} \left[-\mathbb{E} \int_{Q^*} \partial_t(A_h q^{11}) |D_h v|^2 + \mathbb{E} \int_Q \partial_t(D_h^2 q^{11}) v^2 dt \right] \\
&\geq -\mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt - \mathbb{E} \int_Q s \mathcal{O}_\mu((sh)^2) |v|^2 dt \\
&\geq -\mathcal{A}_1.
\end{aligned} \tag{3.32}$$

It follows from (3.5), (3.8) and $q^{11} = r D_h^2 \rho$ that, for $\lambda h(\delta T)^{-m} \leq 1$,

$$\begin{aligned}
&-\frac{h^2}{2} \left[-\mathbb{E} \int_{Q^*} A_h q^{11} |D_h(dv)|^2 + \mathbb{E} \int_Q D_h^2 q^{11} |dv|^2 \right] \\
&\geq -\mathbb{E} \int_Q \mathcal{O}_\mu((sh)^2) |dv|^2 - \mathbb{E} \int_{Q^*} \mathcal{O}_\mu((sh)^2) |D_h(dv)|^2 \\
&\geq -\mathcal{A}_2.
\end{aligned} \tag{3.33}$$

Combining (3.29)–(3.33), we conclude that, for $\lambda h(\delta T)^{-m} \leq 1$,

$$2\mathbb{E} \int_Q \left(\frac{h^2}{2} q^{11} D_h^2 v + \Phi(v) \right) dv \geq -\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \tag{3.34}$$

Thus, from (3.16), (3.24), (3.25) and (3.34), for $\mu \geq \mu_1$ and $\lambda h(\delta T)^{-m} \leq 1$, we see that

$$\begin{aligned}
2 \sum_{i=1}^4 I_{i1} &\geq c \mathbb{E} \int_{\mathcal{M}} \lambda^2 \mu^3 e^{2\mu(6m+1)} |v(0)|^2 + \mathbb{E} \int_{\mathcal{M}^*} |D_h v(0)|^2 + \mathbb{E} \int_{Q^*} |D_h(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 \\
&\quad - \mathbb{E} \int_Q \frac{\theta_t}{\theta} (2s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 + s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) v^2 dt.
\end{aligned} \tag{3.35}$$

Step 3. We compute $\sum_{i=1}^4 I_{i2}$.

From (2.4), (2.6), (2.8)–(2.10), (3.6), (3.9) and (3.13), letting $q^{12} = 2r^2 D_h^2 \rho A_h D_h \rho$, for $\lambda h(\delta T)^{-m} \leq 1$, we arrive at the following estimate

$$2I_{12} = \mathbb{E} \int_Q q^{12} D_h((A_h v)^2) dt$$

$$\begin{aligned}
&= -\mathbb{E} \int_{Q^*} D_h q^{12} (A_h v)^2 dt + \mathbb{E} \int_{\partial Q} q^{12} t_r ((A_h v)^2) \nu dt \\
&= -\mathbb{E} \int_{Q^*} D_h q^{12} A_h (v^2) dt + \frac{h^2}{4} \mathbb{E} \int_{Q^*} D_h q^{12} |D_h v|^2 dt + \frac{h^2}{4} \mathbb{E} \int_{\partial Q} q^{12} t_r (|D_h v|^2) \nu dt \\
&= -\mathbb{E} \int_Q A_h D_h q^{12} |v|^2 dt + \frac{h^2}{4} \mathbb{E} \int_{Q^*} D_h q^{12} |D_h v|^2 dt + \frac{h^2}{4} \mathbb{E} \int_{\partial Q} q^{12} t_r (|D_h v|^2) \nu dt \\
&\geq 6\mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |v|^2 dt - \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2) + s^3 \mu^3 \varphi^3 \mathcal{O}(1)] |v|^2 dt \\
&\quad - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^2) t_r (|D_h v|^2) dt \\
&\geq 6\mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \tag{3.36}
\end{aligned}$$

Thanks to (2.6), (2.9), (3.5) and (3.9), letting $q^{22} = 2r^2 A_h^2 \rho A_h D_h \rho$, for $\lambda h(\delta T)^{-m} \leq 1$, a direct computation gives

$$\begin{aligned}
2I_{22} &= \mathbb{E} \int_Q q^{22} D_h ((D_h v)^2) dt \\
&= -\mathbb{E} \int_{Q^*} D_h q^{22} |D_h v|^2 dt + \mathbb{E} \int_{\partial Q} q^{22} t_r (|D_h v|^2) \nu dt \\
&\geq 2\mathbb{E} \int_{Q^*} s \mu^2 \varphi |\partial_x \psi|^2 |D_h v|^2 dt - 2\mathbb{E} \int_{\partial Q} s \mu \varphi \partial_x \psi t_r (|D_h v|^2) \nu dt \\
&\quad - \mathbb{E} \int_{Q^*} [s \mathcal{O}_\mu((sh)^2) + s \mu \varphi \mathcal{O}(1)] |D_h v|^2 dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^2) t_r (|D_h v|^2) dt \\
&\geq 2\mathbb{E} \int_{Q^*} s \mu^2 \varphi |\partial_x \psi|^2 |D_h v|^2 dt - 2\mathbb{E} \int_{\partial Q} s \mu \varphi \partial_x \psi t_r (|D_h v|^2) \nu dt - \mathcal{A}_1 - \mathcal{B}. \tag{3.37}
\end{aligned}$$

Combining (2.1), (2.4), (2.6), (2.9) and (2.10), letting $q^{32} = 2\lambda \phi \theta_t r A_h D_h \rho$, we see that

$$\begin{aligned}
2I_{32} &= -2\mathbb{E} \int_Q q^{32} A_h D_h v v dt \\
&= 2\mathbb{E} \int_{Q^*} D_h q^{32} |A_h v|^2 dt + 2\mathbb{E} \int_{Q^*} A_h q^{32} A_h v D_h v dt \\
&= 2\mathbb{E} \int_{Q^*} D_h q^{32} |A_h v|^2 dt - \mathbb{E} \int_Q D_h A_h q^{32} v^2 dt \\
&= \mathbb{E} \int_Q A_h D_h q^{32} |v|^2 dt - \frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q^{32} |D_h v|^2 dt.
\end{aligned}$$

By Theorem 2.4, (3.5), (3.8), and noting $|\theta_t| \leq \lambda \theta^2 \mathcal{O}_\mu(1)$, we obtain

$$\begin{aligned}
A_h D_h q^{32} &= \partial_x (2\lambda \phi \theta_t r A_h D_h \rho) + \frac{\theta_t}{\theta} s \mathcal{O}_\mu((sh)^2) \\
&= 2\lambda \phi \theta_t \partial_x (r A_h D_h \rho) + 2\partial_x (\lambda \phi \theta_t) r A_h D_h \rho + \frac{\theta_t}{\theta} s \mathcal{O}_\mu((sh)^2) \\
&= 2\lambda \phi \theta_t \partial_x (r \partial_x \rho) + \frac{\theta_t}{\theta} s^2 \mathcal{O}_\mu((sh)^2) - 2\frac{\theta_t}{\theta} s^2 \mu^2 \varphi^2 (\partial_x \psi)^2 + \frac{\theta_t}{\theta} s \mathcal{O}_\mu((sh)^2)
\end{aligned}$$

$$= 2\lambda\phi\theta_t\partial_x(r\partial_x\rho) - 2\frac{\theta_t}{\theta}s^2\mu^2\varphi^2(\partial_x\psi)^2 + \frac{\theta_t}{\theta}s^2\mathcal{O}_\mu((sh)^2),$$

and $D_h q^{32} = s^3\mathcal{O}_\mu(1)$, which implies that, for $\lambda h(\delta T)^{-m} \leq 1$,

$$2I_{32} \geq 2\mathbb{E} \int_Q \left[\lambda\phi\theta_t\partial_x(r\partial_x\rho) - \frac{\theta_t}{\theta}s^2\mu^2\varphi^2(\partial_x\psi)^2 + \frac{\theta_t}{\theta}s^2\mathcal{O}_\mu((sh)^2) \right] v^2 dt - \mathcal{A}_1. \quad (3.38)$$

From (2.1), (2.5), (2.6), (2.8), (2.9), (3.9) and (3.13), for $\lambda h(\delta T)^{-m} \leq 1$, we obtain

$$\begin{aligned} 2I_{42} &= 2h^2\mathbb{E} \int_Q D_h[D_h(rD_h^2\rho)A_h v]rA_h D_h\rho A_h D_h v dt \\ &= 2h^2\mathbb{E} \int_Q D_h^2(rD_h^2\rho)rA_h D_h\rho A_h^2 v A_h D_h v dt + 2h^2\mathbb{E} \int_Q A_h D_h(rD_h^2\rho)rA_h D_h\rho |A_h D_h v|^2 dt \\ &= -h^2\mathbb{E} \int_{Q^*} D_h[D_h^2(rD_h^2\rho)rA_h D_h\rho] |A_h v|^2 dt + \frac{h^4}{4}\mathbb{E} \int_{\partial Q} D_h^2(rD_h^2\rho)rA_h D_h\rho t_r(|D_h v|^2)\nu dt \\ &\quad + 2h^2\mathbb{E} \int_Q A_h D_h(rD_h^2\rho)rA_h D_h\rho |A_h D_h v|^2 dt. \end{aligned} \quad (3.39)$$

From (2.5), (2.10) and (3.9), it is easy to check that the first term of the right hand of (3.39) can be bound by

$$-h^2\mathbb{E} \int_{Q^*} D_h[D_h^2(rD_h^2\rho)rA_h D_h\rho] |A_h v|^2 dt \geq -\mathbb{E} \int_{Q^*} s\mathcal{O}_\mu((sh)^2)A_h(v^2)dt \geq -\mathbb{E} \int_Q s\mathcal{O}_\mu((sh)^2)v^2 dt.$$

Similarity, the second term of the right hand of (3.39) can be estimated by

$$2h^2\mathbb{E} \int_Q A_h D_h(rD_h^2\rho)rA_h D_h\rho |A_h D_h v|^2 dt \geq -\mathbb{E} \int_{Q^*} s\mathcal{O}_\mu((sh)^2)|D_h v|^2 dt.$$

By (3.9), and substituting the above two estimates into (3.39) yields

$$2I_{42} \geq -\mathcal{A}_1 - \mathcal{B}. \quad (3.40)$$

Combining (3.36)–(3.38) and (3.40), for $\lambda h(\delta T)^{-m} \leq 1$, we conclude that

$$\begin{aligned} 2\sum_{i=1}^4 I_{i2} &\geq 6\mathbb{E} \int_Q s^3\mu^4\varphi^3|\partial_x\psi|^4|v|^2 dt + 2\mathbb{E} \int_{Q^*} s\mu^2\varphi|\partial_x\psi|^2|D_h v|^2 dt - 2\mathbb{E} \int_{\partial Q} s\mu\varphi\partial_x\psi t_r(|D_h v|^2)\nu dt \\ &\quad + 2\mathbb{E} \int_Q \left[\lambda\phi\theta_t\partial_x(r\partial_x\rho) - \frac{\theta_t}{\theta}s^2\mu^2\varphi^2|\partial_x\psi|^2 + \frac{\theta_t}{\theta}s^2\mathcal{O}_\mu((sh)^2) \right] v^2 dt - \mathcal{A}_1 - \mathcal{B}. \end{aligned} \quad (3.41)$$

Step 4. We compute $\sum_{i=1}^4 I_{i3}$. By (2.1), (2.3), (2.6), (2.9), (3.7), (3.9) and (3.13), letting $q^{13} = rD_h^2\rho D_h(rD_h\rho)$, for $\lambda h(\delta T)^{-m} \leq 1$, we have

$$\begin{aligned} 2I_{13} &= 2\mathbb{E} \int_Q q^{13}A_h^2 v v dt \\ &= 2\mathbb{E} \int_Q q^{13}|v|^2 dt + \frac{h^2}{2}\mathbb{E} \int_Q q^{13}D_h^2 v v dt \end{aligned}$$

$$\begin{aligned}
&= 2\mathbb{E} \int_Q q^{13} |v|^2 dt - \frac{h^2}{2} \mathbb{E} \int_{Q^*} A_h q^{13} |D_h v|^2 dt + \frac{h^2}{4} \mathbb{E} \int_Q D_h^2 q^{13} |v|^2 dt \\
&\geq -2\mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |v|^2 dt - \mathcal{A}_1.
\end{aligned} \tag{3.42}$$

From (2.1), (2.6), (2.9), (3.5), (3.9) and (3.13), letting $q^{23} = r A_h^2 \rho D_h(r D_h \rho)$, for $\lambda h(\delta T)^{-m} \leq 1$, we see that

$$\begin{aligned}
2I_{23} &= 2\mathbb{E} \int_Q q^{23} D_h^2 v v dt \\
&= -2\mathbb{E} \int_{Q^*} A_h q^{23} |D_h v|^2 dt + \mathbb{E} \int_Q D_h^2 q^{23} |v|^2 dt \\
&\geq 2\mathbb{E} \int_{Q^*} s \mu^2 \varphi |\partial_x \psi|^2 |D_h v|^2 dt - \mathcal{A}_1.
\end{aligned} \tag{3.43}$$

Thanks to (3.8) and (3.13), we deduce that

$$\begin{aligned}
2I_{33} &= -2\mathbb{E} \int_Q \lambda \phi \theta_t D_h(r D_h \rho) v^2 dt \\
&\geq -2\mathbb{E} \int_Q \lambda \phi \theta_t \partial_x(r \partial_x \rho) v^2 dt + \mathbb{E} \int_Q \frac{\theta_t}{\theta} s^2 \mathcal{O}_\mu((sh)^2) v^2 dt.
\end{aligned} \tag{3.44}$$

From Cauchy-Schwarz inequality and (3.13), we have

$$2I_{43} = 2\mathbb{E} \int_Q \Phi(v) \Psi(v) dt \geq -\mathbb{E} \int_Q |\Phi(v)|^2 dt - \mathbb{E} \int_Q |\Psi(v)|^2 dt. \tag{3.45}$$

By (2.1), (2.5), (2.10), (3.9) and (3.13), for $\lambda h(\delta T)^{-m} \leq 1$, a direct computation shows that

$$\begin{aligned}
-\mathbb{E} \int_Q |\Phi(v)|^2 dt &\geq -\mathbb{E} \int_Q \mathcal{O}_\mu((sh)^2) |A_h^2 v|^2 dt - \mathbb{E} \int_Q \mathcal{O}_\mu((sh)^2) |A_h D_h v|^2 dt \\
&\geq -\mathbb{E} \int_Q \mathcal{O}_\mu((sh)^2) |v|^2 dt - \mathbb{E} \int_{Q^*} \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt,
\end{aligned} \tag{3.46}$$

and

$$-\mathbb{E} \int_Q |\Psi(v)|^2 dt \geq -\mathbb{E} \int_Q s^2 \mathcal{O}_\mu(1) |v|^2 dt. \tag{3.47}$$

Substituting (3.46) and (3.47) into (3.45), it follows that

$$2I_{43} \geq -\mathcal{A}_1. \tag{3.48}$$

Hence, combining (3.42)–(3.44) and (3.48), for $\lambda h(\delta T)^{-m} \leq 1$, we conclude that

$$2 \sum_{i=1}^4 I_{i3} \geq -2\mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |v|^2 dt + 2\mathbb{E} \int_{Q^*} s \mu^2 \varphi |\partial_x \psi|^2 |D_h v|^2 dt$$

$$-2\mathbb{E} \int_Q \lambda \phi \theta_t \partial_x (r \partial_x \rho) v^2 dt + \mathbb{E} \int_Q \frac{\theta_t}{\theta} s^2 \mathcal{O}_\mu((sh)^2) v^2 dt - \mathcal{A}_1. \quad (3.49)$$

Step 5. In this step, we summarize the I_{ij} . Combining (3.15), (3.35), (3.41) and (3.49), for $\mu \geq \mu_1$ and $\lambda h(\delta T)^{-m} \leq 1$, we obtain

$$\begin{aligned} 2\mathbb{E} \int_Q I_1 I_2 &\geq \mathbb{E} \int_{\mathcal{M}} c \lambda^2 \mu^3 e^{2\mu(6m+1)} |v(0)|^2 + \mathbb{E} \int_{\mathcal{M}^*} |D_h v(0)|^2 + \mathbb{E} \int_{Q^*} |D_h(dv)|^2 \\ &\quad - \mathbb{E} \int_Q \frac{\theta_t}{\theta} (4s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 + s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) v^2 dt \\ &\quad + 4\mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |v|^2 dt + 4\mathbb{E} \int_{Q^*} s \mu^2 \varphi |\partial_x \psi|^2 |D_h v|^2 dt \\ &\quad - 2\mathbb{E} \int_{\partial Q} s \mu \varphi \partial_x \psi t_r (|D_h v|^2) \nu dt - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 - \mathcal{B}. \end{aligned} \quad (3.50)$$

From (3.1) and (3.2), noting $|\theta_t| \leq C\theta^2$ in $(T/4, T)$, $\theta \in [1, 2]$, $|\theta_t| \leq \lambda \mathcal{O}_\mu(1)$ in $(0, T/4)$, we have

$$\mathbb{E} \int_Q \frac{\theta_t}{\theta} (s \mathcal{O}_\mu(1) + s^2 \mathcal{O}_\mu((sh)^2)) v^2 dt \leq \mathbb{E} \int_Q (s^2 \mathcal{O}_\mu(1) + s^3 \mathcal{O}_\mu((sh)^2)) v^2 dt \leq \mathcal{A}_1. \quad (3.51)$$

Further, noting $\theta_t < 0$ in $(0, T/4)$, we have

$$-4\mathbb{E} \int_Q \frac{\theta_t}{\theta} s^2 \mu^2 \varphi^2 |\partial_x \psi|^2 |v|^2 dt \geq -\mathbb{E} \int_Q s^3 \mu^2 \varphi^2 \mathcal{O}(1) |v|^2 dt \geq -\mathcal{A}_1. \quad (3.52)$$

Hence, from (3.14) and (3.50)–(3.52) and noting (3.46) and (3.47), there exists $\mu_2 \geq \mu_1$ such that for all $\mu \geq \mu_2$, one can find three constants $1 \geq \varepsilon_1 > 0$, $h_1 > 0$ and $\lambda_1 \geq 1$, so that for all $\lambda \geq \lambda_1$, $h \leq h_1$ and $\lambda h(\delta T)^{-m} \leq \varepsilon_1$, we have

$$\begin{aligned} &\mathbb{E} \int_{\mathcal{M}} \lambda^2 \mu^3 e^{2\mu(6m+1)} |v(0)|^2 + \mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |v|^2 dt + \mathbb{E} \int_{Q^*} s \mu^2 \varphi |D_h v|^2 dt + \mathbb{E} \int_{Q^*} |D_h(dv)|^2 \\ &\leq C\mathbb{E} \int_Q |rf|^2 dt + C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_2} s^3 \mu^4 \varphi^3 |v|^2 dt + C\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_2} s \mu^2 \varphi |D_h v|^2 dt \\ &\quad + C(\mu) \mathbb{E} \int_Q s^2 |dv|^2 + C(\mu) h^{-2} \mathbb{E} \int_{\mathcal{M}} |v(T)|^2. \end{aligned} \quad (3.53)$$

Step 6. In this step, we return to the variable w .

Noting $w = \rho v$, combining (2.1), (3.6) and (3.8), for $\lambda h(\delta T)^{-m} \leq 1$, we obtain

$$\begin{aligned} &\mathbb{E} \int_{Q^*} s \mu^2 \varphi r^2 |D_h w|^2 dt = \mathbb{E} \int_{Q^*} s \mu^2 \varphi r^2 |D_h(\rho v)|^2 dt \\ &\leq 2\mathbb{E} \int_{Q^*} s \mu^2 \varphi r^2 (|D_h \rho A_h v|^2 + |A_h \rho D_h v|^2) dt \\ &\leq C\mathbb{E} \int_{Q^*} s^3 \mu^4 \varphi^3 |A_h v|^2 dt + 2\mathbb{E} \int_{Q^*} s \mu^2 \varphi |D_h v|^2 dt + \mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) |v|^2 dt \\ &\quad + \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt. \end{aligned} \quad (3.54)$$

From (2.5) and (2.10) and Theorem 2.4, we get

$$\mathbb{E} \int_{Q^*} s^3 \mu^4 \varphi^3 |A_h v|^2 dt \leq \mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |v|^2 dt + \mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) |v|^2 dt. \quad (3.55)$$

Combining (3.54) and (3.55), for $\lambda h(\delta T)^{-m} \leq 1$, we have

$$\mathbb{E} \int_{Q^*} s \mu^2 \varphi r^2 |D_h w|^2 dt \leq C \mathbb{E} \int_Q s^3 \mu^4 \varphi^3 |v|^2 dt + 2 \mathbb{E} \int_{Q^*} s \mu^2 \varphi |D_h v|^2 dt + \mathcal{A}_1. \quad (3.56)$$

From (2.1) and (3.8), for $\lambda h(\delta T)^{-m} \leq 1$, we obtain

$$r D_h w = r D_h \rho A_h v + r A_h \rho D_h v = r D_h \rho A_h v + D_h v + \mathcal{O}_\mu((sh)^2) D_h v,$$

which implies

$$\begin{aligned} & \mathbb{E} \int_0^T \int_{M^* \cap G_2} s \mu^2 \varphi |D_h v|^2 dt \\ & \leq C \mathbb{E} \int_0^T \int_{M^* \cap G_2} s \mu^2 \varphi (r D_h \rho)^2 |A_h v|^2 dt + C \mathbb{E} \int_0^T \int_{M^* \cap G_2} s \mu^2 \varphi r^2 |D_h w|^2 dt \\ & \quad + \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt \\ & \leq C \mathbb{E} \int_0^T \int_{M \cap G_1} s^3 \mu^4 \varphi^3 r^2 |w|^2 dt + C \mathbb{E} \int_0^T \int_{M^* \cap G_1} s \mu^2 \varphi r^2 |D_h w|^2 dt \\ & \quad + \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt + \mathbb{E} \int_Q s^3 \mathcal{O}_\mu((sh)^2) |v|^2 dt \\ & \leq C \mathbb{E} \int_0^T \int_{M \cap G_1} s^3 \mu^4 \varphi^3 r^2 |w|^2 dt + C \mathbb{E} \int_0^T \int_{M^* \cap G_1} s \mu^2 \varphi r^2 |D_h w|^2 dt + \mathcal{A}_1, \end{aligned} \quad (3.57)$$

where we have used $|A_h v|^2 \leq C(|\tau_+ v|^2 + |\tau_- v|^2)$.

Combining (3.53), (3.56) and (3.57), there exists $\mu_3 \geq \mu_2$ such that for all $\mu \geq \mu_3$, one can find three positive constants $\varepsilon_2 \leq \varepsilon_1$, $h_2 \leq h_1$ and $\lambda_2 \geq \lambda_1$, so that for all $\lambda \geq \lambda_2$, $h \leq h_2$ and $\lambda h(\delta T)^{-m} \leq \varepsilon_2$, we have

$$\begin{aligned} & \mathbb{E} \int_{\mathcal{M}} \lambda^2 \mu^3 e^{2\mu(6m+1)} e^{2s\phi} |w|^2 \Big|_{t=0} + \mathbb{E} \int_Q s^3 \mu^4 \varphi^3 e^{2s\phi} |w|^2 dt + \mathbb{E} \int_{Q^*} s \mu^2 \varphi e^{2s\phi} |D_h w|^2 dt \\ & \leq C \mathbb{E} \int_Q e^{2s\phi} |f|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_1} s^3 \mu^4 \varphi^3 e^{2s\phi} |w|^2 dt + C \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s \mu^2 \varphi e^{2s\phi} |D_h w|^2 dt \\ & \quad + C(\mu) \mathbb{E} \int_Q s^2 e^{2s\phi} |g|^2 dt + C(\mu) h^{-2} \mathbb{E} \int_{\mathcal{M}} e^{2s\phi} |w|^2 \Big|_{t=T}. \end{aligned} \quad (3.58)$$

Step 7. In this step, we estimate the local terms.

Choose a function $\xi \in C_0^\infty(G_0; [0, 1])$ such that $\xi = 1$ in G_1 . Thanks to Itô's formula and (3.11), we have

$$d(s\varphi \xi^2 e^{2s\phi} w^2) = 2s\varphi \xi^2 e^{2s\phi} w dw + \varphi \xi^2 \partial_t (s e^{2s\phi}) w^2 dt + s\varphi \xi^2 e^{2s\phi} (dw)^2.$$

Hence, we have

$$\begin{aligned}
& \mathbb{E} \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} |w|^2 \Big|_{t=T} - \mathbb{E} \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} |w|^2 \Big|_{t=0} \\
&= \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \varphi \xi^2 \partial_t (s e^{2s\phi}) w^2 dt + 2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} w (-D_h^2 w + f) dt \\
&\quad + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} |g|^2 dt.
\end{aligned} \tag{3.59}$$

From (2.1), (2.9) and (3.59), a direct computation shows that

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \varphi \xi^2 \partial_t (s e^{2s\phi}) w^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} 2s A_h (\xi^2 \varphi e^{2s\phi}) |D_h w|^2 dt \\
&= -2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} w f dt - \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} |g|^2 dt + \mathbb{E} \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} |w|^2 \Big|_{t=T} \\
&\quad - \mathbb{E} \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} |w|^2 \Big|_{t=0} - \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} 2s D_h (\xi^2 \varphi e^{2s\phi}) A_h w D_h w dt
\end{aligned} \tag{3.60}$$

Thanks to (3.2) and (3.3), noting that $\theta_t < 0$ in $(0, T/4)$, $\theta_t < C\theta^2$ in $(T/4, T)$ and $\phi < 0$, there exist $\mu_4 \geq 1$ and $\lambda_3 \geq 1$ such that for all $\mu \geq \mu_4$ and $\lambda \geq \lambda_3$, we have

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \varphi \xi^2 \partial_t (s e^{2s\phi}) |w|^2 dt \\
&= \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \varphi \xi^2 (\lambda \theta_t + 2\lambda s \theta_t \phi) e^{2s\phi} |w|^2 dt \\
&\geq -\mathbb{E} \int_{\frac{T}{4}}^T \int_{\mathcal{M} \cap G_0} [s^2 \mathcal{O}_\mu(1) + \lambda^{-1} s^3 \mathcal{O}_\mu(1)] e^{2s\phi} |w|^2 dt + \mathbb{E} \int_0^{\frac{T}{4}} \int_{\mathcal{M} \cap G_0} \lambda \theta_t \varphi \xi^2 e^{2s\phi} |w|^2 dt.
\end{aligned} \tag{3.61}$$

Noting $|\theta_t| \leq C\lambda\mu^2 e^{\mu(6m-4)} \leq C\lambda\mu^2 \varphi$, then from (3.61) we have

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \varphi \xi^2 \partial_t (s e^{2s\phi}) |w|^2 dt \\
&\geq -\mathbb{E} \int_{\frac{T}{4}}^T \int_{\mathcal{M} \cap G_0} [s^2 \mathcal{O}_\mu(1) + \lambda^{-1} s^3 \mathcal{O}_\mu(1)] e^{2s\phi} |w|^2 dt - \mathbb{E} \int_0^{\frac{T}{4}} \int_{\mathcal{M} \cap G_0} s^2 \mu^2 \varphi^2 \mathcal{O}(1) e^{2s\phi} |w|^2 dt \\
&\geq -\mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + \lambda^{-1} s^3 \mathcal{O}_\mu(1)] e^{2s\phi} |w|^2 dt.
\end{aligned} \tag{3.62}$$

By Theorem 2.4, it holds that

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} 2s A_h (\xi^2 \varphi e^{2s\phi}) |D_h w|^2 dt \\
&\geq 2\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\varphi \xi^2 e^{2s\phi} |D_h w|^2 dt - C\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} s\mathcal{O}_\mu((sh)^2) e^{2s\phi} |D_h w|^2 dt.
\end{aligned} \tag{3.63}$$

Thanks to Cauchy-Schwarz inequality, we have

$$-2\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} w f dt \leq C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^2 \mu^2 \varphi^2 e^{2s\phi} |w|^2 dt + C\mathbb{E} \int_Q \mu^{-2} e^{2s\phi} |f|^2 dt. \quad (3.64)$$

For $\lambda h(\delta T)^{-m} \leq 1$, we obtain

$$\mathbb{E} \int_{\mathcal{M} \cap G_0} s\varphi \xi^2 e^{2s\phi} |w|^2 \Big|_{t=T} \leq C(\mu)h^{-1} \mathbb{E} \int_{\mathcal{M} \cap G_0} e^{2s\phi} |w|^2 \Big|_{t=T}. \quad (3.65)$$

By (2.6) and (2.9) and Theorem 2.4, we get

$$\begin{aligned} & -\mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_0} 2sD_h(\xi^2 \varphi e^{2s\phi}) A_h w D_h w dt \\ &= \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} sD_h^2(\xi^2 \varphi e^{2s\phi}) |w|^2 dt \\ &\leq C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \mu^2 \varphi^3 e^{2s\phi} |w|^2 dt + C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \mathcal{O}_\mu((sh)^2) e^{2s\phi} |w|^2 dt. \end{aligned} \quad (3.66)$$

Hence, combining (3.60) and (3.62)–(3.66), there exists $\mu_5 \geq \max\{\mu_3, \mu_4\}$ such that for all $\mu \geq \mu_5$, one can find three positive constants $\varepsilon_3 \leq \varepsilon_2$, $h_3 \leq h_2$ and $\lambda_4 \geq \max\{\lambda_2, \lambda_3\}$, so that for all $\lambda \geq \lambda_4$, $h \leq h_3$ and $\lambda h(\delta T)^{-m} \leq \varepsilon_3$, we have

$$\begin{aligned} & \mathbb{E} \int_0^T \int_{\mathcal{M}^* \cap G_1} s\mu^2 \varphi e^{2s\phi} |D_h w|^2 dt \\ &\leq C\mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 \mu^4 \varphi^3 e^{2s\phi} |w|^2 dt + C(\mu)h^{-1} \mathbb{E} \int_{\mathcal{M} \cap G_0} e^{2s\phi} |w|^2 \Big|_{t=T} + C\mathbb{E} \int_Q e^{2s\phi} |f|^2 dt \\ &\quad + \mathbb{E} \int_Q [s^2 \mathcal{O}_\mu(1) + \lambda^{-1} s^3 \mathcal{O}_\mu(1)] e^{2s\phi} |w|^2 dt. \end{aligned}$$

Combining it with (3.58), we complete the proof. $\square$

4 The controllability for linear stochastic semi-discrete parabolic equations

Consider the following linear forward stochastic semi-discrete parabolic equation, which is the special case for (1.7):

$$\begin{cases} dy - D_h^2 y dt = (F + \chi_{G_0} u) dt + v dW(t) & \text{in } Q, \\ y(t, 0) = y(t, 1) = 0 & \text{on } (0, T), \\ y(0, x) = y_0 & \text{in } \mathcal{M}. \end{cases} \quad (4.1)$$

Recalling (3.3), we fix parameters λ, μ and h such that the inequality (3.10) holds true. We define the space

$$\mathfrak{S}_{\lambda, \mu, h} = \left\{ F \in L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M})) \mid \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt < \infty \right\}, \quad (4.2)$$

endowed with the norm

$$|F|_{\mathfrak{S}_{\lambda,\mu,h}}^2 = \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt. \quad (4.3)$$

Here, although the space $\mathfrak{S}_{\lambda,\mu,h}$ does not explicitly contain h , both λ and μ have a certain matching relationship with h , making it related to h as well.

We have the following null controllability for (4.1).

Theorem 4.1. *For all μ, λ and h such that (3.10) holds, there exists a positive constant $C = C(\mu)$ such that for any $y_0 \in L_{\mathcal{F}_0}^2(\Omega; L_h^2(\mathcal{M}))$ and any source term $F \in \mathfrak{S}_{\lambda,\mu,h}$, there exist $(u, v) \in L_{\mathbb{F}}^2(0, T; L_h^2(G_0 \cap \mathcal{M})) \times L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M}))$, so that the solution to (4.1) satisfies*

$$\mathbb{E} \int_{\mathcal{M}} |y(T)|^2 \leq C \mathcal{E}_{\lambda,\mu,h} \left(\mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0} + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right), \quad (4.4)$$

where

$$\mathcal{E}_{\lambda,\mu,h} = C h^{-2} e^{-\lambda\theta(T)\mu e^{6\mu(m+1)}}. \quad (4.5)$$

Moreover, the following estimate holds:

$$\begin{aligned} & \mathbb{E} \int_Q e^{-2s\phi} |y|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^{-3} e^{-2s\phi} |u|^2 dt + \mathbb{E} \int_Q s^{-2} e^{-2s\phi} |v|^2 dt \\ & \leq C \left(\mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0} + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right). \end{aligned} \quad (4.6)$$

Proof. We apply the penalized Hilbert Uniqueness Method.

Step 1. For any $\varepsilon > 0$, we introduce the functional

$$\begin{aligned} J_{\varepsilon}(u, v) &= \frac{1}{2} \left(\mathbb{E} \int_Q e^{-2s\phi} |y|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^{-3} e^{-2s\phi} |u|^2 dt + \mathbb{E} \int_Q s^{-2} e^{-2s\phi} |v|^2 dt \right) \\ &+ \frac{1}{2\varepsilon} \mathbb{E} \int_{\mathcal{M}} |y(T)|^2 \end{aligned} \quad (4.7)$$

and the optimal control problem:

$$\begin{cases} \min_{(u,v) \in \mathcal{H}} J_{\varepsilon}(u, v) \\ \text{subject to equation (4.1),} \end{cases} \quad (4.8)$$

where

$$\mathcal{H} = \left\{ (u, v) \in L_{\mathbb{F}}^2(0, T; L_h^2(G_0 \cap \mathcal{M})) \times L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M})) \mid \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^{-3} e^{-2s\phi} |u|^2 dt < \infty, \right. \\ \left. \mathbb{E} \int_Q s^{-2} e^{-2s\phi} |v|^2 dt < \infty \right\}.$$

It is easily seen that for any $\varepsilon > 0$, the problem (4.8) admits a unique optimal solution $(u_{\varepsilon}, v_{\varepsilon})$. By the standard variational method (see [18, Chapter 3, Section 2]), the pair $(u_{\varepsilon}, v_{\varepsilon})$ can be

characterized as

$$u_\varepsilon = -\chi_{G_0} s^3 e^{2s\phi} z_\varepsilon, \quad v_\varepsilon = -s^2 e^{2s\phi} Z_\varepsilon \quad \text{in } Q, \mathbb{P}\text{-a.s.}, \quad (4.9)$$

where $(z_\varepsilon, Z_\varepsilon)$ solves

$$\begin{cases} dz_\varepsilon + D_h^2 z_\varepsilon dt = -e^{-2s\phi} y_\varepsilon dt + Z_\varepsilon dW(t) & \text{in } Q, \\ z_\varepsilon(t, 0) = z_\varepsilon(t, 1) = 0 & \text{on } (0, T), \\ z_\varepsilon(T, x) = \frac{1}{\varepsilon} y_\varepsilon(T, x) & \text{in } \mathcal{M}, \end{cases} \quad (4.10)$$

and where y_ε is the solution of (4.1) with the controls $(u, v) = (u_\varepsilon, v_\varepsilon)$.

Step 2. Thanks to Itô's formula, (4.1) and (4.10), we deduce

$$\begin{aligned} \mathbb{E} \int_{\mathcal{M}} y_\varepsilon(T) z_\varepsilon(T) &= \mathbb{E} \int_{\mathcal{M}} y_\varepsilon(0) z_\varepsilon(0) + \mathbb{E} \int_Q y_\varepsilon (-D_h^2 z_\varepsilon - e^{-2s\phi} y_\varepsilon) dt \\ &\quad + \mathbb{E} \int_Q z_\varepsilon (D_h^2 y_\varepsilon + F + \chi_{G_0} u_\varepsilon) dt + \mathbb{E} \int_Q v_\varepsilon Z_\varepsilon dt. \end{aligned} \quad (4.11)$$

Combining (2.9), (4.9) and (4.11), we have

$$\begin{aligned} &\mathbb{E} \int_Q e^{-2s\phi} |y_\varepsilon|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 e^{2s\phi} |z_\varepsilon|^2 dt + \mathbb{E} \int_Q s^2 e^{2s\phi} |Z_\varepsilon|^2 dt + \frac{1}{\varepsilon} \mathbb{E} \int_{\mathcal{M}} |y_\varepsilon(T)|^2 \\ &= \mathbb{E} \int_{\mathcal{M}} y_0 z_\varepsilon(0) + \mathbb{E} \int_Q z_\varepsilon F dt. \end{aligned} \quad (4.12)$$

Applying Carleman estimate Theorem 3.6 with $w = z_\varepsilon$, $f = -e^{-2s\phi} y_\varepsilon$ and $g = Z_\varepsilon$, we obtain

$$\begin{aligned} &\mathbb{E} \int_{\mathcal{M}} s^2(0) e^{2s(0)\phi} |z_\varepsilon(0)|^2 + \mathbb{E} \int_Q s^3 e^{2s\phi} |z_\varepsilon|^2 dt \\ &\leq C \left(\mathbb{E} \int_0^T \int_{G_0 \cap \mathcal{M}} s^3 e^{2s\phi} |z_\varepsilon|^2 dt + \mathbb{E} \int_Q e^{-2s\phi} |y_\varepsilon|^2 dt + \mathbb{E} \int_Q s^2 e^{2s\phi} |Z_\varepsilon|^2 dt \right. \\ &\quad \left. + h^{-2} \varepsilon^{-2} \mathbb{E} \int_{\mathcal{M}} e^{2s(T)\phi} |y_\varepsilon(T)|^2 \right). \end{aligned} \quad (4.13)$$

From Cauchy-Schwarz inequality, for any $\kappa > 0$, we have

$$\begin{aligned} &\mathbb{E} \int_{\mathcal{M}} y_0 z_\varepsilon(0) + \mathbb{E} \int_Q z_\varepsilon F dt \\ &\leq \kappa \left(\mathbb{E} \int_{\mathcal{M}} s^2(0) e^{2s(0)\phi} |z_\varepsilon(0)|^2 + \mathbb{E} \int_Q s^3 e^{2s\phi} |z_\varepsilon|^2 dt \right) \\ &\quad + C(\kappa) \left(\mathbb{E} \int_{\mathcal{M}} s^{-2}(0) e^{-2s(0)\phi} |y_0|^2 + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right). \end{aligned} \quad (4.14)$$

Letting $\kappa > 0$ be small enough, from (4.12)–(4.14), we conclude that

$$\mathbb{E} \int_Q e^{-2s\phi} |y_\varepsilon|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^3 e^{2s\phi} |z_\varepsilon|^2 dt + \mathbb{E} \int_Q s^2 e^{2s\phi} |Z_\varepsilon|^2 dt + \frac{1}{\varepsilon} \mathbb{E} \int_{\mathcal{M}} |y_\varepsilon(T)|^2$$

$$\leq C \left(\mathbb{E} \int_{\mathcal{M}} s^{-2}(0) e^{-2s(0)\phi} |y_0|^2 + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right) + Ch^{-2} \varepsilon^{-2} \mathbb{E} \int_{\mathcal{M}} e^{2s(T)\phi} |y_\varepsilon(T)|^2.$$

By the above inequality and (4.9), it holds that

$$\begin{aligned} & \mathbb{E} \int_Q e^{-2s\phi} |y_\varepsilon|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^{-3} e^{-2s\phi} |u_\varepsilon|^2 dt + \mathbb{E} \int_Q s^{-2} e^{-2s\phi} |v_\varepsilon|^2 dt + \frac{1}{\varepsilon} \mathbb{E} \int_{\mathcal{M}} |y_\varepsilon(T)|^2 \\ & \leq C \left(\mathbb{E} \int_{\mathcal{M}} s^{-2}(0) e^{-2s(0)\phi} |y_0|^2 + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right) + Ch^{-2} \varepsilon^{-2} \mathbb{E} \int_{\mathcal{M}} e^{2s(T)\phi} |y_\varepsilon(T)|^2. \end{aligned} \quad (4.15)$$

From (3.3) and (4.5), for $\mu > 2$, we have

$$e^{2s(T)\phi} \leq e^{-\lambda\theta(T)\mu e^{6\mu(m+1)}},$$

which yields that, for $\varepsilon \geq \mathcal{E}_{\lambda,\mu,h}$,

$$\begin{aligned} Ch^{-2} \varepsilon^{-2} \mathbb{E} \int_{\mathcal{M}} e^{2s(T)\phi} |y_\varepsilon(T)|^2 & \leq Ch^{-2} \varepsilon^{-2} e^{-\lambda\theta(T)\mu e^{6\mu(m+1)}} \mathbb{E} \int_{\mathcal{M}} |y_\varepsilon(T)|^2 \\ & \leq \frac{1}{2\varepsilon} \mathbb{E} \int_{\mathcal{M}} |y_\varepsilon(T)|^2. \end{aligned} \quad (4.16)$$

By (4.15) and (4.16), for $\varepsilon \geq \mathcal{E}_{\lambda,\mu,h}$, we see that

$$\begin{aligned} & \mathbb{E} \int_Q e^{-2s\phi} |y_\varepsilon|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^{-3} e^{-2s\phi} |u_\varepsilon|^2 dt + \mathbb{E} \int_Q s^{-2} e^{-2s\phi} |v_\varepsilon|^2 dt + \frac{1}{\varepsilon} \mathbb{E} \int_{\mathcal{M}} |y_\varepsilon(T)|^2 \\ & \leq C \left(\mathbb{E} \int_{\mathcal{M}} s^{-2}(0) e^{-2s(0)\phi} |y_0|^2 + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right). \end{aligned} \quad (4.17)$$

Step 3. Let $\varepsilon_k = \mathcal{E}_{\lambda,\mu,h} + \frac{1}{k}$, where $k \in \mathbb{N}^+$. Since the right-hand side of (4.17) is uniform with respect to ε , there exists $(\hat{u}, \hat{v}, \hat{y}, \hat{y}_T)$ such that

$$\begin{cases} u_{\varepsilon_k} \rightharpoonup \hat{u} & \text{weakly in } L_{\mathbb{F}}^2(0, T; L_h^2(G_0 \cap \mathcal{M})), \\ v_{\varepsilon_k} \rightharpoonup \hat{v} & \text{weakly in } L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M})), \\ y_{\varepsilon_k} \rightharpoonup \hat{y} & \text{weakly in } L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M})), \\ y_{\varepsilon_k}(T) \rightharpoonup \hat{y}_T & \text{weakly in } L_{\mathcal{F}_T}^2(\Omega; L_h^2(\mathcal{M})). \end{cases} \quad (4.18)$$

We claim that $\hat{y}$ is the solution to (4.1) associated to $(\hat{u}, \hat{v})$ and $\hat{y}(T) = \hat{y}_T$. In fact, let $\tilde{y}$ be the solution to (4.1) with controls $(\hat{u}, \hat{v})$. For any $p \in L_{\mathbb{F}}^2(0, T; L_h^2(\mathcal{M}))$ and $q \in L_{\mathcal{F}_T}^2(\Omega; L_h^2(\mathcal{M}))$, consider (z, Z) solves

$$\begin{cases} dz + D_h^2 z dt = p dt + Z dW(t) & \text{in } Q, \\ z(t, 0) = z(t, 1) = 0 & \text{on } (0, T), \\ z(T, x) = q(x) & \text{in } \mathcal{M}. \end{cases} \quad (4.19)$$

Thanks to Itô's formula and (4.1), we obtain

$$\mathbb{E} \int_{\mathcal{M}} y_{\varepsilon_k}(T) q - \mathbb{E} \int_{\mathcal{M}} y_0 z(0) = \mathbb{E} \int_Q p y_{\varepsilon_k} dt + \mathbb{E} \int_Q F z dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} u_{\varepsilon_k} z dt + \mathbb{E} \int_Q v_{\varepsilon_k} Z dt,$$

and

$$\mathbb{E} \int_{\mathcal{M}} \tilde{y}(T)q - \mathbb{E} \int_{\mathcal{M}} y_0 z(0) = \mathbb{E} \int_Q p \tilde{y} dt + \mathbb{E} \int_Q F z dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} \hat{u} z dt + \mathbb{E} \int_Q \hat{v} Z dt.$$

From this and (4.18), we get $\hat{y} = \tilde{y}$ and $\tilde{y}(T) = \hat{y}_T$ $\mathbb{P}$ -a.s. By the weak convergence (4.18), weak lower semicontinuity and (4.17), we deduce (4.4) and (4.6). $\square$

5 Proof of the semilinear result

Proof of Theorem 1.1. Let us consider f fulfilling assumptions (A1)-(A3). Recalling (4.2), for all μ, λ and h such that (3.10) holds, define the nonlinear map

$$\mathcal{N} : F \in \mathfrak{S}_{\lambda, \mu, h} \mapsto f(\hat{y}) \in \mathfrak{S}_{\lambda, \mu, h},$$

where $\hat{y}$ solves (4.1) with the initial datum y_0 and the source term F .

We fix μ and $m = 1$. We can show that $\mathcal{N}$ is well defined. In face, noting $\theta \geq 1$, by (3.2), (3.3), (4.3) and (4.6), we have

$$\begin{aligned} |\mathcal{N}(F)|_{\mathfrak{S}_{\lambda, \mu, h}}^2 &= \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |f(\hat{y})|^2 dt \\ &\leq \lambda^{-3} L^2 \mathbb{E} \int_Q e^{-2s\phi} |\hat{y}|^2 dt \\ &\leq C \lambda^{-3} L^2 \left(\mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0} + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right) \\ &< \infty. \end{aligned}$$

This shows that $\mathcal{N}$ is well defined.

Next, we choose λ such that $\mathcal{N}$ is strictly contractive. Consider two source terms $F_1, F_2 \in \mathfrak{S}_{\lambda, \mu, h}$. We denote the solutions of the corresponding equations by $\hat{y}_1$ and $\hat{y}_2$, respectively. From (3.3), (4.3) and (4.6), we have

$$\begin{aligned} |\mathcal{N}(F_1) - \mathcal{N}(F_2)|_{\mathfrak{S}_{\lambda, \mu, h}}^2 &= \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |f(\hat{y}_1) - f(\hat{y}_2)|^2 dt \\ &\leq L^2 \lambda^{-3} \mathbb{E} \int_Q e^{-2s\phi} |\hat{y}_1 - \hat{y}_2|^2 dt \\ &\leq CL^2 \lambda^{-3} |F_1 - F_2|_{\mathfrak{S}_{\lambda, \mu, h}}^2. \end{aligned}$$

Choosing λ sufficiently large such that

$$CL^2 \lambda^{-3} < 1, \tag{5.1}$$

the map $\mathcal{N}$ is strictly contractive.

By the Banach fixed point theorem, we conclude that $\mathcal{N}$ has a unique fixed point F in $\mathfrak{S}_{\lambda, \mu, h}$. Let y be the solution associated to this F with the control (u, v) . Then, we have $F = f(y)$.

From (4.6), it follows that

$$\mathbb{E} \int_Q e^{-2s\phi} |y|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^{-3} e^{-2s\phi} |u|^2 dt + \mathbb{E} \int_Q s^{-2} e^{-2s\phi} |v|^2 dt$$

$$\leq C\mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0} + CL^2 \lambda^{-3} \mathbb{E} \int_Q e^{-2s\phi} |y|^2 dt.$$

By the above inequality and (5.1), we get

$$\begin{aligned} & \mathbb{E} \int_Q e^{-2s\phi} |y|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{M} \cap G_0} s^{-3} e^{-2s\phi} |u|^2 dt + \mathbb{E} \int_Q s^{-2} e^{-2s\phi} |v|^2 dt \\ & \leq C\mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0}, \end{aligned} \tag{5.2}$$

which implies that (1.4) holds true for $h \leq h_0$ and $\lambda h(\delta T)^{-1} \leq \varepsilon_0$.

From (4.4), (4.5) and (5.2), we obtain

$$\begin{aligned} \mathbb{E} \int_{\mathcal{M}} |y(T)|^2 & \leq C\mathcal{E}_{\lambda, \mu, h} \left(\mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0} + \mathbb{E} \int_Q s^{-3} e^{-2s\phi} |F|^2 dt \right) \\ & \leq C\mathcal{E}_{\lambda, \mu, h} \left(\mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0} + L^2 \mathbb{E} \int_Q e^{-2s\phi} |y|^2 dt \right) \\ & \leq C(1 + L^2) \mathcal{E}_{\lambda, \mu, h} \mathbb{E} \int_{\mathcal{M}} s^{-2} e^{-2s\phi} |y|^2 \Big|_{t=0} \\ & \leq C(1 + L^2) h^{-2} e^{C\lambda(2\theta(0) - \theta(T))} \mathbb{E} \int_{\mathcal{M}} |y_0|^2. \end{aligned}$$

Choose $\delta_0 > 0$ such that for all $\delta < \delta_0$, it holds that

$$2\theta(0) - \theta(T) = 4 - (\delta T)^{-1} \leq -\frac{1}{2}(\delta T)^{-1}.$$

Note that we need $h \leq h_0$ and $\lambda h(\delta T)^{-1} \leq \varepsilon_0$, where λ is fixed. Let $h_1 = \frac{\varepsilon_0 \delta_0 T}{\lambda}$. For any $h \leq \min\{h_0, h_1\}$, we choose $\delta = \frac{h}{h_1} \delta_0 \leq \delta_0$. Then, we obtain $\lambda h(\delta T)^{-1} = \varepsilon_0$. By the above argument, we get

$$\mathbb{E} \int_{\mathcal{M}} |y(T)|^2 \leq C e^{-\frac{C}{h}} \mathbb{E} \int_{\mathcal{M}} |y_0|^2.$$

□

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