# Controllability of stochastic fourth order partial differential equations

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第 15 届数学控制理论及应用学术会议

Nov. 9, 2024

# Outline

1. Introduction
2. Null controllability for fourth order stochastic parabolic equations
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4. Controllability for discrete fourth order stochastic parabolic equations

# 1. Introduction

Controllability of two types of systems:

- Null controllability for fourth order stochastic parabolic equations.
- Exact controllability for a refined stochastic plate equation.

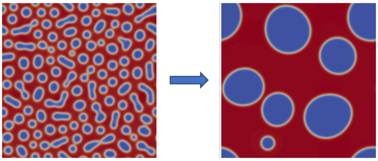

Figure: Cahn-Hilliard Equation

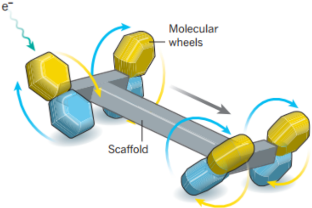

Figure: Molecular Motors

# One dimensional cases

- P. Gao, M. Chen and Y. Li, *A new global Carleman estimate for the one-dimensional Kuramoto-Sivashinsky equation and applications to exact controllability to the trajectories and an inverse problem*, SIAM J. Control Optim. 53 (2015), 475–500.
- Y. Yu and J.-F. Zhang, *Carleman estimates of refined stochastic beam equations and applications*, SIAM J. Control Optim. 60 (2022), 2947–2970.
- S. Zhang, H. Gao and G. Yuan, *New global Carleman estimates and null controllability for a stochastic Cahn-Hilliard type equation*, arXiv (2024).

## Notation

$(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is a complete filtered probability space. $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by a one-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$. $\mathbb{F}$ is the progressive $\sigma$-field with respect to $\mathbf{F}$.

Let $G$ be a bounded open set in $\mathbb{R}^n$ $(n \in \mathbb{N})$ with a $C^4$ boundary $\Gamma$. Denote

$$Q \triangleq (0,T) \times G, \qquad \Sigma \triangleq (0,T) \times \Gamma.$$

# 2. Null controllability for fourth order stochastic parabolic equations

# Controllability for fourth order stochastic parabolic equations

Consider the following fourth order stochastic parabolic equation:

$$\begin{cases} dy + \Delta^2 y dt = (b_1 y + \chi_{G_0} u + b_3 v)dt + (b_2 y + v)dW(t) & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{1}$$

Initial values: $y_0 \in L^2(G)$.

Coefficients: $b_1, b_2, b_3 \in L^\infty_{\mathbb{F}}(0,T;L^\infty(G))$.

Controls: $(u,v) \in L^2_{\mathbb{F}}(0,T;L^2(G_0)) \times L^2_{\mathbb{F}}(0,T;L^2(G))$.

System (1) admits a unique weak solution:

$$y \in L^2_{\mathbb{F}}(\Omega; C([0,T]; L^2(G))) \cap L^2_{\mathbb{F}}(0,T; H^2(G) \cap H^1_0(G)).$$

The system (1) is said to be null controllable at time $T$ if for any $y_0 \in L^2(G)$, there exist $(u, v)$ such that the corresponding solution to (1) fulfills that $y(T) = 0$, $\mathbb{P}$-a.s.

## Theorem 1 (Lü-Wang, SIAM J. Control Optim., 2022)

System (1) is null controllable at any time $T > 0$. Furthermore, there is a constant $C > 0$ such that for each given $y_0 \in L^2(G)$, one can choose controls $(u, v)$ which steer the state of (1) to 0 and satisfy

$$|(u,v)|_{L^2_{\mathbb{F}}(0,T;L^2(G_0))\times L^2_{\mathbb{F}}(0,T;L^2(G))} \leq C|y_0|_{L^2(G)} e^{C(1+r_1^2)(T^{-1}+1)},$$

where $r_1 = |a_1|_{L^\infty_{\mathbb{F}}(0,T;L^\infty(G))} + |a_2|_{L^\infty_{\mathbb{F}}(0,T;L^\infty(G))} + |a_3|_{L^\infty_{\mathbb{F}}(0,T;L^\infty(G))}$.

To prove the controllability of (1), we introduce the dual equation

$$\begin{cases} dz-\Delta^2 z dt=-(b_1 z+b_2 Z)dt+Z dW(t) & \text{in } Q,\\ z=\Delta z=0 & \text{on } \Sigma,\\ z(T)=z_T & \text{in } G. \end{cases} \tag{2}$$

**Theorem 2 (Lü-Wang, SIAM J. Control Optim., 2022)**

There exists a constant $C>0$ such that the solution $(z,Z)$ of the system (2) satisfies

$$|z(0)|_{L^2(G)} \leq Ce^{C(1+r_1^2)(T^{-1}+1)}\big(|z|_{L^2_{\mathbb{F}}(0,T;L^2(G_0))}+|b_3 z+Z|_{L^2_{\mathbb{F}}(0,T;L^2(G))}\big),$$
$$\forall z_T\in L^2_{\mathcal{F}_T}(\Omega;L^2(G)).$$

**Theorem 3 (Lü-Wang, SIAM J. Control Optim., 2022)**

Let $\varphi$ be an $H^4(G)$-valued Itô process, $\theta = e^{\ell}$, $\ell = s\alpha$, $\psi = \theta\varphi$. $\kappa$ is a constant. Let $\eta \in C^2(\mathbb{R}^n)$. Then, for any $t \in [0,T]$ and a.e. $(x,\omega) \in G \times \Omega$,

$$\begin{aligned}
&-2\kappa \sum_{i,j=1}^{n} \left(\psi_{x_i x_i x_j} d\psi - \psi_{x_i x_j} d\psi_{x_i} + \Psi_2 \psi_{x_i} \delta_{ij} d\psi + \Psi_3^{ij} \psi_{x_i} d\psi\right)_{x_j} \\
&+ 2\theta K_1 (\kappa d\varphi + \Delta^2 \varphi dt) - 2\,\mathrm{div}(\widetilde{V}_1 + \widetilde{V}_2) dt \\
&= 2K_1^2 dt + 2K_1 K_3 + 2\widetilde{M}_1 dt + 2 \sum_{i,j,k,l=1}^{n} \Lambda_1^{ijkl} \psi_{x_i x_j} \psi_{x_k x_l} dt + 2 \sum_{i,j=1}^{n} \Lambda_2^{ij} \psi_{x_i} \psi_{x_j} dt \\
&\quad + 2\Lambda_3 \psi^2 dt + 2\kappa d\left(\frac{1}{2}|\nabla^2 \psi|^2 - \frac{1}{2}\Psi_2 |\nabla\psi|^2 - 2s^2\lambda^2\xi^2 |\nabla\eta\nabla\psi|^2 + \frac{1}{2}\Psi_6 \psi^2\right) \\
&\quad + 4\kappa s^2 \lambda^2 \xi^2 \theta^2 |\nabla\eta d\nabla\varphi + \nabla\eta\nabla\ell d\varphi|^2 + \kappa\theta^2 \Psi_2 |d\nabla\varphi + \nabla\ell d\varphi|^2 \\
&\quad - \kappa\theta^2 \sum_{i,j=1}^{n} \left[d\varphi_{x_i x_j} + 2\ell_{x_j} d\varphi_{x_i} + (\ell_{x_i x_j} + \ell_{x_i}\ell_{x_j}) d\varphi\right]^2 - \kappa\Psi_6 \theta^2 (d\varphi)^2.
\end{aligned}$$

The global Carleman estimate for the fourth order backward stochastic parabolic equation holds.

### Theorem 4 (Lü-Wang, SIAM J. Control Optim., 2022)

There exists a constant $C > 0$ such that for all $z_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$, the weak solution $(z(\cdot), Z(\cdot))$ to (2) satisfies that

$$\begin{aligned}
&\mathbb{E}\int_Q (s^6\lambda^8\xi^6\theta^2|z|^2 + s^4\lambda^6\xi^4\theta^2|\nabla z|^2 + s^3\lambda^4\xi^3\theta^2|\Delta z|^2 + s^2\lambda^4\xi^2\theta^2|\nabla^2 z|^2 \\
&\qquad + \lambda\theta^2|\nabla\Delta z|^2)dxdt \\
&\leq C\mathbb{E}\int_{Q_0} s^7\lambda^8\xi^7\theta^2|z|^2 dxdt + Ce^{C\lambda|\eta|_{C(\overline{G})}}\mathbb{E}\int_Q s^4\lambda^4\xi^4\theta^2|Z|^2 dxdt \\
&\quad + Cr_1^2\mathbb{E}\int_Q \theta^2(|z|^2 + |Z|^2)dxdt,
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^{1/2} + T)$.

# Controllability for coupled fourth order SPDEs

Consider the following coupled fourth order backward stochastic parabolic equations:

$$\begin{cases} dy_1 - \Delta^2 y_1 dt = (a_1 y_1 + a_2 y_2 + a_3 Y_1)dt + Y_1 dW(t) & \text{in } Q, \\ dy_2 - \Delta^2 y_2 dt = (b_1 y_1 + b_2 y_2 + b_3 Y_2 + \chi_{G_0} u)dt + Y_2 dW(t) & \text{in } Q, \\ y_1 = \dfrac{\partial y_1}{\partial \nu} = 0, \quad y_2 = \dfrac{\partial y_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ y_1(T) = y_1^T, \quad y_2(T) = y_2^T & \text{in } G, \end{cases} \tag{3}$$

Terminal state: $(y_1^T, y_2^T) \in \left[L^2_{\mathcal{F}_T}(\Omega; L^2(G))\right]^2$.

Coefficients: $\begin{cases} a_1, a_2 \in L^\infty_{\mathbb{F}}(0,T; L^\infty(G)), \quad a_3 \in L^\infty_{\mathbb{F}}(0,T; W^{2,\infty}(G)), \\ b_1, b_2 \in L^\infty_{\mathbb{F}}(0,T; L^\infty(G)), \quad b_3 \in L^\infty_{\mathbb{F}}(0,T; W^{2,\infty}(G)). \end{cases}$

Controls: $u \in L^2_{\mathbb{F}}(0,T; L^2(G_0))$.

The system (3) is called null controllable at a time $T > 0$ if for any given $(y_1^T, y_2^T) \in [L^2_{\mathcal{F}_T}(\Omega; L^2(G))]^2$, there exists a control $u$ such that the solution $(y_1, y_2; Y_1, Y_2)$ to (3) satisfies that $(y_1(0), y_2(0)) = (0, 0)$, $\mathbb{P}$-a.s.

Assumption 1. There exists a nonempty open set $G_1 \subset G_0$ and a constant $\sigma > 0$ such that $a_2(x,t) \geq \sigma$ or $a_2(x,t) \leq -\sigma$, a.e. $(x,t) \in G_1 \times (0,T)$, $\mathbb{P}$-a.s.

### Theorem 5 (Wang, JMAA, 2024)

Under Assumption 1, system (3) is null controllable at any time $T > 0$.

To prove the null controllability of (3), we introduce the adjoint equation:

$$\begin{cases} dz_1 + \Delta^2 z_1 dt = -(a_1 z_1 + b_1 z_2)dt - a_3 z_1 dW(t) & \text{in } Q, \\ dz_2 + \Delta^2 z_2 dt = -(a_2 z_1 + b_2 z_2)dt - b_3 z_2 dW(t) & \text{in } Q, \\ z_1 = \dfrac{\partial z_1}{\partial \nu} = 0, \quad z_2 = \dfrac{\partial z_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ z_1(0) = z_1^0, \quad z_2(0) = z_2^0 & \text{in } G. \end{cases} \tag{4}$$

## Theorem 6 (Wang, JMAA, 2024)

There exists a constant $C > 0$ such that the solution $(z_1, z_2)$ to system (4) satisfies

$$|(z_1(T), z_2(T))|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G)\times L^2(G))} \leq C|z_2|_{L^2_{\mathbb{F}}(0,T;L^2(G_0))}.$$

# 3. Exact controllability for the stochastic plate equation

# Exact controllability for a refined stochastic plate equation

Consider the stochastic plate equation:

$$\begin{cases} dy_t + \Delta^2 y dt = f dt + (a_1 y + g) dW(t) & \text{in } Q, \\ y = h_1, \ \dfrac{\partial y}{\partial \nu} = h_2 & \text{on } \Sigma, \\ (y(0), y_t(0)) = (y_0, y_1) & \text{in } G. \end{cases} \tag{5}$$

Initial values: $(y_0, y_1) \in L^2(G) \times H^{-2}(G)$.

Controls: $(f, g, h_1, h_2) \in L^2_{\mathbb{F}}(0,T; L^2(G)) \times L^2_{\mathbb{F}}(0,T; H^{-2}(G)) \times L^2_{\mathbb{F}}(0,T; L^2(\Gamma)) \times L^2_{\mathbb{F}}(0,T; L^2(\Gamma))$.

System (5) is exactly controllable at time $T$: for any

$$(y_0, y_1) \in L^2(G) \times H^{-2}(G), \quad (\tilde{y}_0, \tilde{y}_1) \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(G)),$$

there exist controls $(f, g, h_1, h_2)$ such that the solution $y$ of system (5) satisfies $(y(T), y_t(T)) = (\tilde{y}_0, \tilde{y}_1)$, $\mathbb{P}$-a.s.

The controls are the strongest possible ones that can be introduced into (5). Is the exact controllability trivial?

**Theorem 7 (Lü-Wang, arXiv, 2022)**

System (5) is not exactly controllable at any time $T > 0$.

Consider the refined stochastic plate equation:

$$\begin{cases} dy = \hat{y}dt + (a_3 y + f)dW(t) & \text{in } Q, \\ d\hat{y} + \Delta^2 y dt = (a_1 y + a_2 \cdot \nabla y + a_5 g)dt + (a_4 y + g)dW(t) & \text{in } Q, \\ y = h_1, \dfrac{\partial y}{\partial \nu} = h_2 & \text{on } \Sigma, \\ (y(0), \hat{y}(0)) = (y_0, \hat{y}_0) & \text{in } G. \end{cases} \tag{6}$$

Initial values: $(y_0, \hat{y}_0) \in H^{-1}(G) \times (H^3(G) \cap H_0^2(G))^*$.

Coefficients: $a_1, a_3, a_4 \in L^\infty_{\mathbb{F}}(0,T; W^{1,\infty}(G))$, $a_2 \in L^\infty_{\mathbb{F}}(0,T; W^{2,\infty}(G;\mathbb{R}^n))$, $a_5 \in L^\infty_{\mathbb{F}}(0,T; W^{3,\infty}(G))$.

Controls: $(f, g, h_1, h_2) \in L^2_{\mathbb{F}}(0,T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0,T; (H^3(G) \cap H_0^2(G))^*) \times L^2_{\mathbb{F}}(0,T; L^2(\Gamma)) \times L^2_{\mathbb{F}}(0,T; H^{-1}(\Gamma))$.

# Transposition solution

System (6) is a nonhomogeneous boundary value problem. Its solution is understood in the sense of transposition.

Introduce the following reference equation:

$$\begin{cases} dz = \hat{z}dt + (Z - a_5 z)dW(t) & \text{in } Q_\tau, \\ d\hat{z} + \Delta^2 z dt = [(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2 \cdot \nabla z - a_3 \hat{Z} + a_4 Z]dt & \text{in } Q_\tau, \\ \qquad\qquad + \hat{Z}dW(t) & \\ z = \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma_\tau, \\ (z(\tau), \hat{z}(\tau)) = (z^\tau, \hat{z}^\tau) & \text{in } G, \end{cases} \tag{7}$$

where $\tau \in (0,T]$, $Q_\tau = (0,\tau) \times G$, $\Sigma_\tau = (0,\tau) \times \Gamma$, $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; H^1_0(G))$.

System (7) admits a unique weak solution

$$(z, Z, \hat{z}, \hat{Z}) \in L^2_{\mathbb{F}}(\Omega; C([0,\tau]; (H^3(G) \cap H^2_0(G)))) \times L^2_{\mathbb{F}}(0,\tau; (H^3(G) \cap H^2_0(G))) \\ \times L^2_{\mathbb{F}}(\Omega; C([0,\tau]; H^1_0(G))) \times L^2_{\mathbb{F}}(0,\tau; H^1_0(G)).$$

### Proposition 1 (Hidden regularity)

The solution $(z, Z, \hat{z}, \hat{Z})$ of (7) satisfies $|\nabla \Delta z||_\Gamma \in L^2_{\mathbb{F}}(0,\tau; L^2(\Gamma))$.

A pair of stochastic processes $(y,\hat{y}) \in C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))\times C_{\mathbb{F}}([0,T];L^2(\Omega;(H^3(G)\cap H_0^2(G))^*))$ is a transposition solution to (6) if for any $\tau\in(0,T]$ and $(z^\tau,\hat{z}^\tau)$, we have

$$\begin{aligned}
&\mathbb{E}\langle \hat{y}(\tau),z^\tau\rangle_{(H^3(G)\cap H_0^2(G))^*,H^3(G)\cap H_0^2(G)}-\langle \hat{y}_0,z(0)\rangle_{(H^3(G)\cap H_0^2(G))^*,H^3(G)\cap H_0^2(G)}\\
&-\mathbb{E}\langle y(\tau),\hat{z}^\tau\rangle_{H^{-1}(G),H_0^1(G)}+\langle y_0,\hat{z}(0)\rangle_{H^{-1}(G),H_0^1(G)}\\
&=-\mathbb{E}\int_0^\tau\langle f,\widehat{Z}\rangle_{H^{-1}(G),H_0^1(G)}dt+\mathbb{E}\int_0^\tau\langle g,Z\rangle_{(H^3(G)\cap H_0^2(G))^*,H^3(G)\cap H_0^2(G)}dt\\
&\quad+\mathbb{E}\int_0^\tau\int_\Gamma\frac{\partial\Delta z}{\partial\nu}h_1\,d\Gamma dt-\mathbb{E}\int_0^\tau\langle h_2,\Delta z\rangle_{H^{-1}(\Gamma),H^1(\Gamma)}dt.
\end{aligned}$$

Here, $(z,Z,\hat{z},\hat{Z})$ solves (7).

## Proposition 2

System (6) admits a unique transposition solution.

System (6) is exactly controllable at time $T$: for any

$$(y_0, \hat{y}_0) \in H^{-1}(G) \times (H^3(G) \cap H_0^2(G))^*,$$
$$(y_1, \hat{y}_1) \in L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G)) \times L^2_{\mathcal{F}_T}(\Omega; (H^3(G) \cap H_0^2(G))^*),$$

there exist controls $(f, g, h_1, h_2)$ such that the solution $(y, \hat{y})$ to (6) satisfies that $(y(T,\cdot), \hat{y}(T,\cdot)) = (y_1, \hat{y}_1)$, $\mathbb{P}$-a.s.

### Theorem 8 (Lü-Wang, arXiv, 2022)

System (6) is exactly controllable at any time $T > 0$.

Recall the following reference equation :

$$
\begin{cases}
dz = \hat{z}dt + (Z - a_5 z)dW(t) & \text{in } Q,\\
d\hat{z} + \Delta^2 z dt = [(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2\cdot\nabla z - a_3\hat{Z} + a_4 Z]dt \\
\qquad\qquad + \hat{Z}dW(t) & \text{in } Q,\\
z = \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma,\\
(z(T), \hat{z}(T)) = (z^T, \hat{z}^T) & \text{in } G.
\end{cases}
$$

**Theorem 9 (Lü-Wang, arXiv, 2022)**

There exists a constant $C > 0$ such that for every $(z^T, \hat{z}^T)$, it holds that

$$
\begin{aligned}
&|(z^T, \hat{z}^T)|^2_{L^2_{\mathcal{F}_T}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_T}(\Omega;H_0^1(G))}\\
&\le C\mathbb{E}\int_\Sigma (|\nabla\Delta z|^2 + |\Delta z|^2)d\Gamma dt + C|(Z,\hat{Z})|^2_{L^2_{\mathbb{F}}(0,T;H^3(G)\cap H_0^2(G))\times L^2_{\mathbb{F}}(0,T;H_0^1(G))}.
\end{aligned}
$$

The boundary controls $h_1$ and $h_2$ in system (6) cannot be dropped simultaneously, and the internal controls $f$ and $g$ must be acted on the whole domain $G$.

**Theorem 10 (Lü-Wang, arXiv, 2022)**

The system (6) is not exactly controllable at any time $T > 0$ provided that one of the following three conditions is satisfied:

- $a_3 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega))$, $G \backslash \overline{G_0} \neq \emptyset$ and $\operatorname{supp} f \subset G_0$;
- $a_4 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega))$, $G \backslash \overline{G_0} \neq \emptyset$ and $\operatorname{supp} g \subset G_0$;
- $h_1 = h_2 = 0$.

# 4. Controllability for discrete fourth order stochastic parabolic equations

For $N \in \mathbb{N}$ and $h = 1/(N+1)$, consider the following semi-discretization of the heat equation:

$$\begin{cases} \dfrac{dy^i}{dt} - \dfrac{1}{h^2}(y^{i+1} - 2y^i + y^{i-1}) = 0 & t \in (0,T), \quad i = 1, \cdots, N, \\ y^0(t) = 0, \quad y^{N+1}(t) = u(t) & t \in (0,T), \\ y^i(0) = y_0^i & i = 1, \cdots, N. \end{cases} \tag{8}$$

Does the uniformly null controllability hold?

For any $\{y_0^i\}_{i=1}^N$ and $h > 0$, does there exist $u_h \in L^2(0,T)$, such that the solution satisfies $y^i(T) = 0$ for $i = 1, \cdots, N$ and

$$|u_h|^2_{L^2(0,T)} \leq Ch \sum_{i=1}^{N} |y_0^i|^2?$$

**Theorem 11 (López-Zuazua, 1998)**

The system (8) is uniformly null controllable.

A counterexample provided by E. Zuazua (2005) shows that the uniformly null controllability does NOT hold in the 2D case.

Null controllability for the lower part of the spectrum:

- F. Boyer, F. Hubert, J. Le Rousseau, J. Math. Pures Appl., 2010.
- F. Boyer, F. Hubert, J. Le Rousseau, SIAM J. Control Optim., 2010.

$\phi-$null controllability, i.e., $h\sum_{i=1}^{N}|y^i(T)|^2 \leq \phi(h)$ and $\lim_{h\to 0}\phi(h)=0$.

- F. Boyer, J. Le Rousseau, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 2014.
- E. Cerpa, R. Lecaros, T. N. T. Nguyen, A. Pérez, J. Math. Pures Appl., 2022.

Consider the following stochastic semi-discrete system:

$$\begin{cases} dy^i + \dfrac{1}{h^4}(y^{i+2} - 4y^{i+1} + 6y^i - 4y^{i-1} + y^{i-2})dt & \\ \qquad = (a_1^i y^i + \chi_{G_0}^i u^i)dt + (a_2^i y^i + v^i)dW(t) & t \in (0,T), \quad i = 1, \cdots, N, \\ y^0(t) = y^{N+1}(t) = 0 & t \in (0,T), \\ y^{-1}(t) = y^{N+2}(t) = 0 & t \in (0,T), \\ y^i(0) = y_0^i & i = 1, \cdots, N. \end{cases}$$

Initial values: $y_0 \in L_h^2(\mathcal{M})$.

Coefficients: $a_1, a_2 \in L_{\mathbb{F}}^\infty(0,T; L_h^\infty(\mathcal{M}))$.

Controls: $(u,v) \in L_{\mathbb{F}}^2(0,T; L_h^2(\mathcal{M} \cap G_0)) \times L_{\mathbb{F}}^2(0,T; L_h^2(\mathcal{M}))$.

## Theorem 12 (Wang-Zhao, arXiv, 2024)

There exist positive constants $h_0$ and $C$ such that for all $h \leq h_0$, there exist $(u, v)$ such that the solution $y$ satisfies

$$\mathbb{E}\int_Q |v|^2 dt + \mathbb{E}\int_0^T \int_{G_0 \cap \mathcal{M}} |u|^2 dt \leq C \int_{\mathcal{M}} |y_0|^2,$$

and

$$\mathbb{E}\int_{\mathcal{M}} |y(T)|^2 \leq C e^{-\frac{C}{h}} \int_{\mathcal{M}} |y_0|^2.$$

# Thank You!