Inverse problems for stochastic partial differential equations

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Outline

1. Introduction

Formulation of inverse problems for SPDEs

Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a C^2 boundary Γ . Denote $Q = (0, T) \times G$ and $\Sigma = (0, T) \times \Gamma$.

Consider the following stochastic hyperbolic equation:

$$\begin{cases} dy_t - \Delta y dt = (ay + f)dt + (by + g)dW(t) & \text{in } Q, \\ y = \varphi & \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } G, \end{cases} \tag{1}$$

where a, b, f, g and φ are suitable functions/stochastic processes and $W(t)$ denotes the standard Brownian motion.

Direct problem: Determine the solution y of system (1) by specifying $(a, b, f, g, \varphi, y_0, y_1)$ along with Q. This can be achieved using classical well-posedness results of stochastic evolution equations.

Inverse problem: Involves certain unknown components in $(a, b, f, g, \varphi, y_0, y_1)$ and Q, which need to be determined through measuring some information about the solution y:

$$\mathcal{M}(y)(t,x) = h(t,x), \qquad (t,x) \in \mathcal{O} \subset [0,T] \times \overline{G}.$$

Here, $\mathcal{M}(\cdot)$ is a known operator, representing the method of obtaining measurement data, and $\mathcal{O}$ denotes the position of the measurement.

There are three distinct types of measurements: distributed, terminal, and boundary measurements.

For a given inverse problem, generally speaking, three main concerns are the following:

- To what extent can measurement data uniquely determine the unknown parameters, and what information can be derived from it?
- Is it feasible to reconstruct unknown components from measurement data?
- Do there exist efficient algorithms for constructing approximate solutions?

Deterministic Case: Uniqueness Fails

Consider the following inverse source problem:

$$\begin{cases} y_{tt} - \Delta y = f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_t(0) = 0 & \text{in } G. \end{cases} \tag{2}$$

Inverse Problem: Can we uniquely determine f from the measurements $\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma}$ and $y(T)$?

Answer: NO.

For any $y^* \in C_0^\infty(Q)$, let $f = y^*_{tt} - \Delta y^*$. Then y^* solves (2) with $\left.\frac{\partial y^*}{\partial \nu}\right|_{\Sigma} = 0$, $y^*(T) = 0$. But we can choose y^* such that $f \neq 0$ in Q.

Stochastic Case: Uniqueness Holds

Consider the analogous problem in the stochastic setting:

$$\begin{cases} dz_t - \Delta z\, dt = g\, dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_t(0) = 0 & \text{in } G. \end{cases}$$

Inverse Problem: Can we uniquely determine g from the measurements $\frac{\partial z}{\partial \nu}\Big|_{\Sigma}$ and $z(T)$?

Answer: YES. One can show that

$$\frac{\partial z}{\partial \nu}\Big|_{\Sigma} = 0 \quad \text{and} \quad z(T) = 0 \quad \Longrightarrow \quad g = 0 \ \text{ in } Q, \quad \mathbb{P}\text{-a.s.}$$

(Lü and Zhang, *Comm. Pure Appl. Math.*, 2015)

2. Inverse Source Problem for Stochastic Hyperbolic Equations

Semilinear Stochastic Hyperbolic Equation

Let $T > 0$, and let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete filtered probability space, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by a one-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$.

Consider the following semilinear stochastic hyperbolic equation:

$$\begin{cases} du_t - \sum_{j,k=1}^{n} (b^{jk} u_{x_j})_{x_k} dt = F(u, u_t, \nabla u) dt + a u dW(t) & \text{in } Q, \\ u_t(0) = 0 & \text{in } G, \\ u(0) = u_0 & \text{in } G, \end{cases} \tag{3}$$

Coefficients: Let $(b^{jk})_{1\le j,k\le n} \in C^1(G;\mathbb{R}^{n\times n})$ satisfy that $b^{jk} = b^{kj}$ for all $j,k = 1,\cdots,n$ and for some constant $s_0 > 0$,

$$\sum_{j,k=1}^{n} b^{jk}\xi_j\xi_k \ge s_0|\xi|^2, \text{ for all } (x,\xi) \in G\times\mathbb{R}^n.$$

Furthermore, $a \in L^\infty_{\mathbb{F}}(0,T;L^\infty(G))$.

Nonlinearity: Let $F : [0,T]\times\Omega\times G\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be such that, for each $(\eta,\rho,\zeta) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$, $F(\cdot,\cdot,\cdot,\eta,\rho,\zeta) : [0,T]\times\Omega\times G$ is an $L^2(G)$-valued $\mathbf{F}$-adapted process, and there exists a constant L such that for a.e. $(t,\omega,x) \in [0,T]\times\Omega\times G$ and any $(\eta_i,\varrho_i,\zeta_i) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ $(i=1,2)$,

$$|F(t,\omega,x,\eta_1,\varrho_1,\zeta_1) - F(t,\omega,x,\eta_2,\varrho_2,\zeta_2)| \le L(|\eta_1-\eta_2| + |\varrho_1-\varrho_2| + |\zeta_1-\zeta_2|_{\mathbb{R}^n});$$

and $|F(\cdot,\cdot,\cdot,0,0,0)| \in L^2_{\mathbb{F}}(0,T;L^2(G))$.

Inverse Source Problem

Problem

Determine the source function u_0 from the lateral Cauchy data $u|_\Sigma$ and $\frac{\partial u}{\partial \nu}|_{\Sigma_0}$.

- $\Sigma = (0,T) \times \Gamma$, $\Sigma_0 = (0,T) \times \Gamma_0$ where $\Gamma_0 \subset \Gamma$ is a subset of the boundary.
- Only partial boundary data is available.

Conditions 1. There exists a positive function $\psi(\cdot) \in C^2(\overline{G})$ satisfying the following:

- For some constant $\mu_0 > 0$ and any $(x, \xi) \in \overline{G} \times \mathbb{R}^n$, it holds that

$$\sum_{j,k=1}^{n} \Big\{ \sum_{j',k'=1}^{n} \Big[2b^{jk'}(b^{j'k}\psi_{x_{j'}})_{x_{k'}} - b^{jk}_{x_{k'}} b^{j'k'}\psi_{x_{j'}} \Big] \Big\} \xi_j \xi_k \geq \mu_0 \sum_{j,k=1}^{n} b^{jk} \xi_j \xi_k.$$

- There is no critical point of $\psi(\cdot)$ in $\overline{G}$, i.e., $\min_{x \in \overline{G}} |\nabla \psi(x)| > 0$.

Put

$$\Gamma_0 \stackrel{\Delta}{=} \Big\{ x \in \Gamma \;\Big|\; \sum_{j,k=1}^{n} b^{jk}(x) \psi_{x_j}(x) \nu^k(x) > 0 \Big\}.$$

Theorem (Lü, Inverse Problems, 2013)

Under Conditions 1, there exists $T_0 > 0$ such that for $T > T_0$, there exists constant $C > 0$ so that for all $u_0, \hat{u}_0 \in H_0^1(G)$, the solutions satisfy

$$|u_0 - \hat{u}_0|_{H_0^1(G)} \leq C\left|\frac{\partial u}{\partial \nu} - \frac{\partial \hat{u}}{\partial \nu}\right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}.$$

- Identifiability: The solution u is uniquely determined by the boundary data.
- Stability: The solution depends continuously on the boundary data.
- Reconstruction: Is it possible to reconstruct the solution from the boundary data?

Iterative Regularization: Fixed-Point Scheme

Idea: Construct a sequence $\{u_n\}$ by minimizing the weighted Tikhonov functional at each iteration.

- Initialization: Start from any u_0 in the admissible set.
- Iteration: For $n \geq 0$, set

$$u_{n+1} = \mathfrak{G}(u_n) \overset{\Delta}{=} \arg\min_{\varphi \in \mathcal{U}} J(\varphi; u_n).$$

- No need for a good initial guess

Literature Review

Deterministic hyperbolic equations:

- **Carleman estimates.** Bukhgeim and Klibanov (1981), Klibanov (2013), Bellassoued and Yamamoto (2017), Klibanov and Li (2021) ...
- **Carleman estimates + fixed-point.** Nguyen and Klibanov (2022), Dang, Nguyen and Vu (2024)

Stochastic hyperbolic equations:

- **Explicit determination and reconstruction.** Lü (2013), Lü and Zhang (2015), Yuan (2015), Dou and Lü (2025) ...

3. Convergence result

Carleman Estimate

Carleman estimates are weighted energy inequalities which can be used for proving uniqueness, stability, and convergence in inverse problems.

Intuitive example: Let $\tau > 0$. It holds that

$$2\tau \int_{-1}^{1} |u|^2 e^{\tau t^2} dt \leq \int_{-1}^{1} \left| \frac{du}{dt} \right|^2 e^{\tau t^2} dt, \quad \forall u \in C_0^\infty(-1,1) \tag{4}$$

Proof. Set $v(t) = u(t)e^{\tau t^2/2}$. Then we have

$$\begin{aligned} |u'(t)|^2 e^{\tau t^2} &= |v'(t) - \tau t v(t)|^2 = v'(t)^2 - 2\tau t v(t) v'(t) + \tau^2 t^2 v(t)^2 \\ &= |v'(t) + \tau t v(t)|^2 - 4\tau t v(t) v'(t) \\ &= |v'(t) + \tau t v(t)|^2 + 2\tau |v(t)|^2 - 2\tau (t v(t)^2)' \end{aligned}$$

Integrate this over $[-1,1]$, noting that $v \in C_0^\infty(-1,1)$, we get (4).

Carleman Weight Function

- **Weight function:** For $c_0, \lambda > 0$,

$$\theta = e^{\ell}, \quad \ell(t,x) = \lambda(\psi(x) - c_0 t^2)$$

- **Boundary splitting:** Γ_0 defined by

$$\Gamma_0 \triangleq \Big\{ x \in \Gamma \;\Big|\; \sum_{j,k=1}^{n} b^{jk}(x)\psi_{x_j}(x)\nu^k(x) > 0 \Big\}.$$

- For $du_t - \Delta u dt$, function $\psi(x) = |x - x_0|^2$ with $x_0 \in \mathbb{R}^n \setminus \overline{G}$ satisfies the conditions and $\Gamma_0 = \{x \in \Gamma \mid (x - x_0) \cdot \nu > 0\}$.

Statement of the Carleman Estimate

Theorem (Lü and Wang, Inverse Problems, 2025)

Under Conditions 1, there exist positive constants C and λ_0 such that for all $\lambda \geq \lambda_0$, the solution z to the stochastic hyperbolic equation:

$$dz_t - \sum_{j,k=1}^{n} (b^{jk} z_{x_j})_{x_k} dt = \Upsilon dt + azdW(t)$$

with $z_t(0) = 0$, $z = 0$ on Σ, $\frac{\partial z}{\partial \nu} = 0$ on Σ_0 satisfies:

$$\mathbb{E}\int_Q \theta^2(\lambda z_t^2 + \lambda|\nabla z|^2 + \lambda^3 z^2)dxdt \leq C\mathbb{E}\int_G \theta^2(\lambda|\nabla z|^2 + \lambda^3|z|^2)_{t=T}dx + C\mathbb{E}\int_Q \theta^2|\Upsilon|^2 dxdt.$$

Lemma (Lü and Zhang, 2021). Let $\phi \in C^1((0,T)\times\mathbb{R}^n)$, $b^{jk}=b^{kj}\in C^2(\mathbb{R}^n)$ for $j,k=1,2,\cdots,n$, and $\ell,\Psi\in C^2((0,T)\times\mathbb{R}^n)$. Let φ be an $H^2(G)$-valued, $\mathbf{F}$-adapted process such that φ_t is an $L^2(G)$-valued Itô process. Set $\theta=e^{\ell}$ and $w=\theta\varphi$. Then, for a.e. $x\in G$ and $\mathbb{P}$-a.s. $\omega\in\Omega$,

$$
\begin{aligned}
&\theta\Big(-2\phi\ell_t w_t+2\sum_{j,k=1}^{n}b^{jk}\ell_{x_j}w_{x_k}+\Psi w\Big)\Big[\phi d\varphi_t-\sum_{j,k=1}^{n}(b^{jk}\varphi_{x_j})_{x_k}dt\Big]\\
&\quad+\sum_{j,k=1}^{n}\Big[\sum_{j',k'=1}^{n}\Big(2b^{jk}b^{j'k'}\ell_{x_{j'}}w_{x_j}w_{x_{k'}}-b^{jk}b^{j'k'}\ell_{x_j}w_{x_{j'}}w_{x_{k'}}\Big)-2\phi b^{jk}\ell_t w_{x_j}w_t+\phi b^{jk}\ell_{x_j}w_t^2\\
&\qquad+\Psi b^{jk}w_{x_j}w-\Big(A\ell_{x_j}+\frac{\Psi_{x_j}}{2}\Big)b^{jk}w^2\Big]_{x_k}dt\\
&\quad+d\Big\{\phi\sum_{j,k=1}^{n}b^{jk}\ell_t w_{x_j}w_{x_k}-2\phi\sum_{j,k=1}^{n}b^{jk}\ell_{x_j}w_{x_k}w_t+\phi^2\ell_t w_t^2-\phi\Psi w_t w+\Big[\phi A\ell_t+\frac{1}{2}(\phi\Psi)_t\Big]w^2\Big\}\\
&=\Big[(\phi^2\ell_t)_t+\sum_{j,k=1}^{n}(\phi b^{jk}\ell_{x_j})_{x_k}-\phi\Psi\Big]w_t^2dt-2\sum_{j,k=1}^{n}\big[(\phi b^{jk}\ell_{x_k})_t+b^{jk}(\phi\ell_t)_{x_k}\big]w_{x_j}w_t dt\\
&\quad+\sum_{j,k=1}^{n}c^{jk}w_{x_j}w_{x_k}dt+\mathcal{B}w^2dt+\Big(-2\phi\ell_t w_t+2\sum_{j,k=1}^{n}b^{jk}\ell_{x_j}w_{x_k}+\Psi w\Big)^2dt+\phi^2\theta^2\ell_t(d\varphi_t)^2.
\end{aligned}
$$

Step 1. Estimate the interior terms. Choose $\varphi = z, \quad \phi \equiv 1$, and $\Psi = \ell_{tt} + \sum_{j,k=1}^{n} (b^{jk}\ell_{x_j})_{x_k} - c_0\lambda$. Note that

$$\ell_t = -2c_1\lambda t, \quad \ell_{tt} = -2c_1\lambda, \quad \ell_{x_j} = \lambda\psi_{x_j}, \quad \ell_{tx_j} = 0, \quad j = 1, \cdots, n.$$

Then,

$$\Big[(\phi^2\ell_t)_t + \sum_{j,k=1}^{n} (\phi b^{jk}\ell_{x_j})_{x_k} - \phi\Psi\Big] w_t^2 = c_0\lambda w_t^2, \quad \sum_{j,k=1}^{n} c^{jk} w_{x_j} w_{x_k} \geq c\lambda|\nabla w|^2$$

$$-2\sum_{j,k=1}^{n} \big[(\phi b^{jk}\ell_{x_k})_t + b^{jk}(\phi\ell_t)_{x_k}\big] w_{x_j} w_t = 0,$$

$$\mathcal{B}w^2 \geq C\lambda^3 w^2, \quad \mathbb{E}\int_Q \theta^2 \ell_t |dz_t|^2 dx \geq -c\lambda\mathbb{E}\int_Q \theta^2 |z|^2 dxdt.$$

Step 2. Estimate the boundary terms.

$$
\begin{aligned}
&\mathbb{E}\int_G \Big(\sum_{j,k=1}^n b^{jk}\ell_t w_{x_j} w_{x_k} - 2\sum_{j,k=1}^n b^{jk}\ell_{x_j} w_{x_k} w_t + \ell_t w_t^2 - \Psi w_t w + \mathcal{A}\ell_t w^2 \Big)\Big|_{t=T} dx \\
&\le C\mathbb{E}\int_G (\lambda^3 |w(T)|^2 + \lambda |\nabla w(T)|^2) dx
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\int_\Sigma \sum_{j,k=1}^n \sum_{j',k'=1}^n \Big(2b^{jk}b^{j'k'}\psi_{x_{j'}} w_{x_j} w_{x_{k'}} - b^{jk}b^{j'k'}\psi_{x_j} w_{x_{j'}} w_{x_{k'}} \Big)\nu^k d\Gamma dt \\
&= \mathbb{E}\int_\Sigma \Big(\sum_{j,k=1}^n b^{jk}\nu^j\nu^k \Big)\Big(\sum_{j',k'=1}^n b^{j'k'}\psi_{x_{j'}}\nu^{k'} \Big)\theta^2 \Big|\frac{\partial z}{\partial \nu}\Big|^2 d\Gamma dt \\
&\le \mathbb{E}\int_{\Sigma_0} \Big(\sum_{j,k=1}^n b^{jk}\nu^j\nu^k \Big)\Big(\sum_{j',k'=1}^n b^{j'k'}\psi_{x_{j'}}\nu^{k'} \Big)\theta^2 \Big|\frac{\partial z}{\partial \nu}\Big|^2 d\Gamma dt = 0.
\end{aligned}
$$

Tikhonov Regularization Functional

- At each step, consider

$$J_n(\varphi) = |\mathcal{P}\varphi - \mathbf{I}\mathcal{F}(u_n)|^2_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + \kappa|\varphi|^2_{\mathcal{H}^2_{\lambda,0}}.$$

- $\mathcal{P}\varphi = \varphi_t(t,x) - \int_0^t \sum_{j,k=1}^n \left(b^{jk}(s,x)\varphi_{x_j}(s,x)\right)_{x_k} ds - \int_0^t a(s,x)\varphi(s,x)dW(s)$,
  $\mathbf{I}\mathcal{F}(u) = \int_0^t F(u,u_t,\nabla u)ds$.

- Admissible set:

$$\mathcal{U} \triangleq \left\{\varphi \in \mathcal{H}^2 \;\middle|\; \mathcal{P}\varphi \in L^2_{\mathbb{F}}(\Omega; H^1(0,T;L^2(G))), \quad \varphi_t(0) = 0, \quad \varphi|_{\Sigma} = f, \quad \frac{\partial\varphi}{\partial\nu}\Big|_{\Sigma_0} = g\right\},$$

$$L^{2,w}_{\mathbb{F}}(0,T;H^p(G)) = \{\varphi \in L^2_{\mathbb{F}}(0,T;H^p(G)) \mid \mathbb{E}\int_0^T\int_G \theta^2|D^\alpha\varphi|^2 dxdt < \infty,\ |\alpha| \le p\},$$

$$\mathcal{H}^p_{\lambda,c_0} = L^{2,w}_{\mathbb{F}}(0,T;H^p(G)) \cap L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;H^1(G))).$$

Global Convergence result

Theorem (Lü and Wang, Inverse Problems, 2025)

Under Conditions 1, fix $\gamma \in (0,1)$ and $\kappa > 0$. There exist positive constants C and $\tilde{\lambda}_0(\kappa)$ such that for all $\lambda \geq \tilde{\lambda}_0$, $\delta < e^{-C\lambda/\gamma}$, and $n \in \mathbb{N}$, the sequence $\{u_n\}$ satisfies

$$|u_{n+1} - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} \leq \left(\frac{C}{\lambda}\right)^{n+1} |u_0 - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} + C(\delta^{2-2\gamma} + \kappa |u^*|^2_{\mathcal{H}^2_{\lambda,0}}),$$

where u^* is the true solution.

- Convergence is geometric in n.
- $\delta > 0$ quantifies the noise level in the measurements: for $\mathcal{E} \in \mathcal{H}^2$,

$$|\mathcal{P}\mathcal{E}|_{L^2_{\mathbb{F}}(0,T;H^1(0,T;L^2(G)))} + |\mathcal{E}|_{\mathcal{H}^2} \leq \delta, \quad f|_{\Sigma} = f^* + \mathcal{E}|_{\Sigma}, \quad g = g^* + \left.\frac{\partial \mathcal{E}}{\partial \nu}\right|_{\Sigma_0}.$$

Step 1. Error equation.

The Euler-Lagrange equation gives

$$\langle \mathcal{P}u_{n+1} - \mathbf{I}\mathcal{F}(u_n), \mathcal{P}\rho\rangle_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + \kappa\langle u_{n+1}, \rho\rangle_{\mathcal{H}^2_{\lambda,0}} = 0, \quad \forall\, \rho \in \mathcal{U}^w_0, \tag{5}$$

$$\mathcal{U}^w_0 \triangleq \Big\{\varphi \in \mathcal{H}^2 \;\Big|\; \mathcal{P}\varphi \in L^2_{\mathbb{F}}(\Omega; H^1(0,T; L^2(G))), \quad \varphi_t(0) = 0, \quad \varphi|_{\Sigma} = 0, \quad \frac{\partial\varphi}{\partial\nu}\Big|_{\Sigma_0} = 0\Big\}.$$

For the exact solution u^*,

$$\langle \mathcal{P}u^* - \mathbf{I}\mathcal{F}(u^*), \mathcal{P}\rho\rangle_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + \kappa\langle u^*, \rho\rangle_{\mathcal{H}^2_{\lambda,0}} = \kappa\langle u^*, \rho\rangle_{\mathcal{H}^2_{\lambda,0}}, \quad \forall\, \rho \in \mathcal{U}^w_0. \tag{6}$$

Subtracting (6) from (5) and letting $z = u_{n+1} - u^* - \mathcal{E}$, we obtain

$$\langle \mathcal{P}(z+\mathcal{E}) - (\mathbf{I}\mathcal{F}(u_n) - \mathbf{I}\mathcal{F}(u^*)), \mathcal{P}z\rangle_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + \kappa\langle z+\mathcal{E}, z\rangle_{\mathcal{H}^2_{\lambda,0}} = -\kappa\langle u^*, z\rangle_{\mathcal{H}^2_{\lambda,0}}.$$

Step 2. Apply Carleman estimate.

Thanks to the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&|\mathcal{P}z|^2_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + \kappa|z|^2_{\mathcal{H}^2_{\lambda,0}} \\
&\leq C(|u_n - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} + \kappa|u^*|^2_{\mathcal{H}^2_{\lambda,0}} + |\mathcal{P}\mathcal{E}|^2_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + |\mathcal{E}|^2_{\mathcal{H}^2_{\lambda,0}}).
\end{aligned}
$$

Apply Carleman estimate to z, for $\lambda \geq \lambda_0$, we have

$$
|\mathcal{P}z|^2_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} \geq \mathbb{E}\int_Q \theta^2(\lambda z_t^2 + \lambda|\nabla z|^2 + \lambda^3 z^2)dxdt - C\mathbb{E}\int_G \theta^2(\lambda|\nabla z|^2 + \lambda^3|z|^2)\Big|_{t=T} dx.
$$

Note that the last term can be estimated as follows:

$$
\mathbb{E}\int_G \theta^2(\lambda|\nabla z|^2 + \lambda^3|z|^2)\Big|_{t=T} dx \leq C\lambda^5 e^{-2c_0\lambda T^2}|z|^2_{\mathcal{H}^2_{\lambda,0}}.
$$

Step 3. Absorb terms.

For $\lambda \geq \lambda_0$, we arrive at

$$\mathbb{E}\int_Q \theta^2(\lambda z_t^2 + \lambda|\nabla z|^2 + \lambda^3 z^2)dxdt + \kappa|z|^2_{\mathcal{H}^2_{\lambda,0}} - C\lambda^5 e^{-2c_0\lambda T^2}|z|^2_{\mathcal{H}^2_{\lambda,0}}$$
$$\leq C\Big(|u_n - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} + \kappa|u^*|^2_{\mathcal{H}^2_{\lambda,0}} + |\mathcal{P}\mathcal{E}|^2_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + |\mathcal{E}|^2_{\mathcal{H}^2_{\lambda,0}}\Big).$$

Selecting λ_1 such that $C\lambda_1^5 e^{-2c_0\lambda_1 T^2} \leq \frac{\kappa}{2}$, for $\lambda \geq \max\{\lambda_0, \lambda_1\}$,

$$\lambda|u_{n+1} - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} \leq C(|u_n - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} + \kappa|u^*|^2_{\mathcal{H}^2_{\lambda,0}} + |\mathcal{P}\mathcal{E}|^2_{L^{2,w}_{\mathbb{F}}(\Omega;H^1(0,T;L^2(G)))} + |\mathcal{E}|^2_{\mathcal{H}^2_{\lambda,0}}).$$

Setting $\delta < e^{-C\lambda/\gamma}$, we obtain

$$|u_{n+1} - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} \leq \left(\frac{C}{\lambda}\right)|u_n - u^*|^2_{\mathcal{H}^1_{\lambda,c_0}} + C(\kappa|u^*|^2_{\mathcal{H}^2_{\lambda,0}} + \delta^{2-2\gamma}).$$

4. Numerical Algorithm and Experiments

Numerical Algorithm: Discretization and Optimization

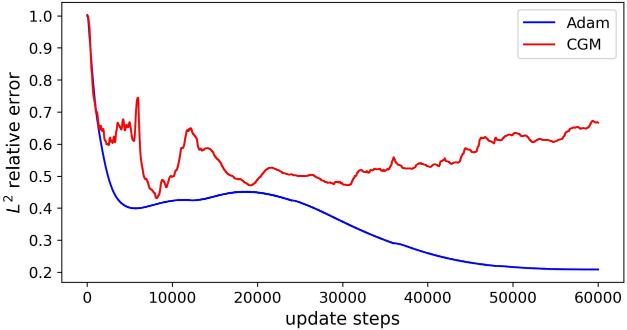
- **Discretization:** The spatial and temporal domains are discretized using finite difference schemes, with the Euler-Maruyama method applied for stochastic integration.
- **Functional:** The weighted Tikhonov functional is discretized in accordance with the chosen grid.
- **Optimization:** At each iteration, the Adam (Adaptive Moment Estimation) optimizer is employed to minimize the discretized functional.
- **Gradient computation:** Gradients are efficiently and accurately computed via automatic differentiation.

Algorithm Flowchart

Choose maximum iteration N_I and the number of sample paths N_S.
for $\ell = 1$ to N_S **do**
  Choose the initial guess $u_\ell^0 \in \mathcal{U}$.
  for $n = 1$ to N_I **do**
    Compute u_ℓ^n as the minimizer of the functional $J_{n,\ell}(u; u_\ell^{n-1})$.
  end for
end for
return compute solution $u_c = \frac{1}{N_S} \sum_{\ell=1}^{N_S} u_\ell^{N_I}$.

Why Adam and Autodiff?

- **Adam** is well-suited for high-dimensional, non-convex, and stochastic optimization tasks, making it effective for minimizing the regularized functional in our setting.
- **Automatic differentiation** enables efficient and accurate gradient computation, eliminating the need to explicitly solve backward stochastic partial differential equations.



Numerical example

- Consider the following semilinear stochastic hyperbolic equation:

$$\begin{cases} du_t - \Delta u dt = F(u, u_t, \nabla u)dt + audW(t) & \text{in } Q, \\ u_t(0) = 0 & \text{in } G, \\ u(0) = u_0 & \text{in } G, \end{cases}$$

where $G = (0,1) \times (0,1.5)$ and $T = 1$.

- The observation boundary

$$\Gamma_0 = \partial G \backslash \{(x,0) \mid x \in [0,1]\}$$

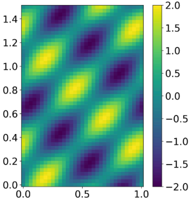
- Multiplicative noise

$$f = f^*(1 + \delta\xi), \qquad g = g^*(1 + \delta\xi),$$

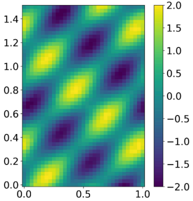
where ξ is a random variable uniformly distributed in $[-1,1]$.

Example 1

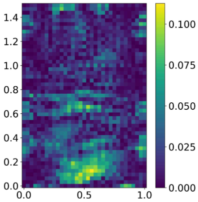
$$F(u, u_t, \nabla u) = \min\{e^u + |\nabla u|, 10\},$$
$$a = x^2 + y^2 + t^2$$



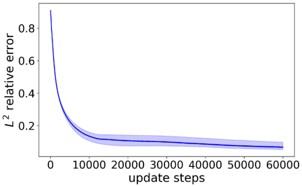
(a) The true source function u_0^*.



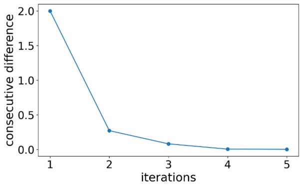
(b) The computed source function u_0.



(c) The relative difference $\frac{|u_0 - u_0^*|}{|u_0^*|_{L^\infty(G)}}$.



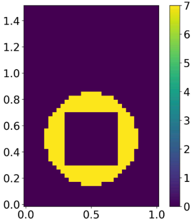
(d) The L^2 relative error $\frac{\mathbb{E}|u_0 - u_0^*|_{L^2(G)}}{\mathbb{E}|u_0^*|_{L^2(G)}}$ over update steps



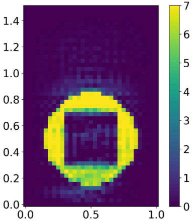
(e) The consecutive difference $\mathbb{E}|u_\ell^{n+1}(0) - u_\ell^n(0)|_{L^\infty(G)}$

Example 2

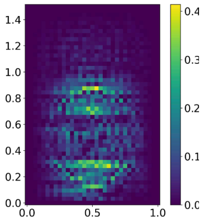
$$F(u, u_t, \nabla u) = \sqrt{1+u^2} + |\nabla u|,$$
$$a = 10xyt^2$$



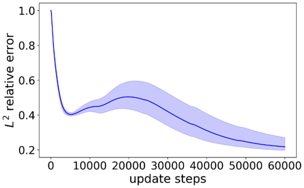
(a) The true source function u_0^*.



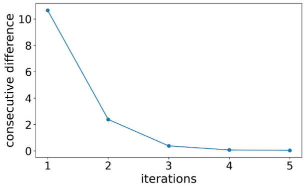
(b) The computed source function u_0.



(c) The relative difference $\frac{|u_0 - u_0^*|}{|u_0^*|_{L^\infty(G)}}$.



(d) The L^2 relative error $\frac{\mathbb{E}|u_0 - u_0^*|_{L^2(G)}}{\mathbb{E}|u_0^*|_{L^2(G)}}$ over update steps



(e) The consecutive difference $\mathbb{E}|u_\ell^{n+1}(0) - u_\ell^n(0)|_{L^\infty(G)}$

Summary

- **Accurate and robust:** The method reconstructs sources reliably under noise, nonlinearity, and partial data.
- **Modern optimization (Adam, autodiff)** enables efficient and robust computation.
- **Carleman-based regularization** is a powerful tool for the inverse problems of SPDEs.

Thank You!