

Stability Estimates for the Inverse Recovery of Drift and Diffusion Point Sources in Stochastic Parabolic Equations

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Abstract

This paper studies the simultaneous recovery of point-source locations and temporal strengths in both the drift and diffusion terms of a stochastic parabolic equation with Neumann boundary conditions, using full-boundary observations over the observation period, supplemented by higher-regularity observations on a nonempty boundary portion after the sources have become inactive. By separating the stochastic equation into its expectation and centered parts, the drift channel reduces to a deterministic source problem with scalar Laplace moments, while the diffusion channel is controlled by Itô's isometry, which preserves the full L^2 -energy of the integrand. Under suitable nondegeneracy and separation assumptions, we establish Lipschitz stability for the location of a single source in either channel, logarithmic stability for the drift strength, and Lipschitz stability for the complete diffusion strength including its sign. For multiple sources, we obtain Lipschitz stability up to permutation for drift locations and weighted temporal moments in dimensions two and three, and for diffusion locations and strengths in dimensions one through three. The proof employs explicit adjoint probes, including complex-isotropic polynomial annihilators in dimensions two and three and spectral synthesis in one dimension, combined with boundary-to-source estimates derived from Green's identities. These results extend and improve upon existing deterministic stability theories by revealing the fundamentally different temporal information carried by stochastic integrals, a distinction that is further illustrated by numerical experiments.

Keywords: Inverse source problem, stochastic parabolic equation, point sources, Dirac measure, Lipschitz stability

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1 Introduction

Point-source models are used when a localized release, such as in heat generation or pollutant transport, is small compared with the observation length scale. Such models arise, for instance, in pollutant-source identification, where time-dependent point releases are described by advection-dispersion-reaction equations [5], and white-noise-modulated sources have been used in diffusion-convection models of turbulent transport [6]. In this context, the drift and diffusion sources represent mean localized releases and zero-mean fluctuations, respectively, about a background concentration. The corresponding inverse problem involves both finite-dimensional geometric unknowns, namely the source locations, and infinite-dimensional temporal unknowns, namely the release histories. In the deterministic setting, this mixed nature already leads to different stability regimes for these two types of unknowns. The goal of this paper is to investigate how this picture changes when both the drift and the Itô diffusion channels contain point sources in a stochastic parabolic equation.

Let $n \in \{1, 2, 3\}$, $G \subset \mathbb{R}^n$ be a bounded connected domain with boundary of class C^2 , $\Gamma \subset \partial G$ be nonempty and relatively open, and $0 < T_0 < T_1 < T$. Set $\Sigma = (0, T) \times \partial G$. On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ carrying a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ with augmented filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$, let $\mathbb{F}$ denote the progressive σ -field associated with $\mathbf{F}$. For a separable Hilbert space $\mathcal{H}$, $L^2_{\mathbb{F}}(0, T; \mathcal{H})$ denotes the space of progressively measurable $\mathcal{H}$ -valued processes X such that $\mathbb{E} \int_0^T \|X(t)\|_{\mathcal{H}}^2 dt < \infty$, and $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathcal{H}))$ denotes the space of adapted continuous processes satisfying

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$\mathbb{E} \sup_{0 \leq t \leq T} \|X(t)\|_{\mathcal{H}}^2 < \infty$. We use $L^2_{\mathbb{F}}(\Omega; H^1(T_1, T; \mathcal{H}))$ for the adapted processes in $L^2(\Omega; H^1(T_1, T; \mathcal{H}))$. Here and throughout, ν denotes the outward unit normal vector on ∂G .

Fix known integers $N_0, N_1 \geq 0$ with $N_0 + N_1 \geq 1$, and interpret an empty sum as zero. We consider

$$\begin{cases} du + \mathcal{L}u \, dt = \sum_{k=1}^{N_0} \lambda_k(t) \delta_{x_k} \, dt + \sum_{l=1}^{N_1} \gamma_l(t) \delta_{y_l} \, dW(t) & \text{in } (0, T) \times G, \\ \partial_\nu u = 0 & \text{on } (0, T) \times \partial G, \\ u(0) = u_0 & \text{in } G, \end{cases} \quad (1.1)$$

where

$$\mathcal{L} = -\Delta + A \cdot \nabla + \mu, \quad A \in \mathbb{R}^n, \quad \mu \in \mathbb{R}.$$

Here δ_ξ denotes the Dirac measure at $\xi \in G$. The initial datum $u_0 \in L^2(G)$ is deterministic and known. The source locations $x_k, y_l \in G$ and the deterministic strengths $\lambda_k, \gamma_l \in L^2(0, T)$ are unknown.

Problem 1 (Inverse point-source problem). *Determine the source locations and temporal strengths $\{(x_k, \lambda_k)\}_{k=1}^{N_0}$ and $\{(y_l, \gamma_l)\}_{l=1}^{N_1}$ from the boundary trace $u|_\Sigma$.*

Identification and reconstruction problems for point sources in deterministic PDEs have been extensively studied. Early contributions established uniqueness for stationary sources that vanish prior to the end of the experiment, based on partial lateral boundary data [4]. Subsequent work extended these results to time-dependent intensities in one dimension [5], moving sources [1], and multiple moving pollutants [2]. Ling and Takeuchi [15] recovered source counts, locations, and strengths from sparse fixed-point measurements. More recent developments include reconstruction methods for time-dependent emissions [22], Dirac sources from sparse data [10], and sources from single-boundary observations [24]; uniqueness from finite flux measurements has also been addressed [8]. While these results clarify identifiability issues, they do not provide the stability estimates required for error quantification in our context.

For regular sources, Lipschitz stability via Carleman estimates is by now classical [12], whereas Dirac singularities demand special treatment. Komornik and Yamamoto [13] derived sharp conditional estimates for point masses with prescribed temporal profiles. The most directly relevant work for our drift channel is [11], which establishes Lipschitz stability for the location and logarithmic stability for the strength of single sources, as well as planar Hölder estimates for multiple sources under moment/matching assumptions, using Carleman estimates, Laplace analysis, and adjoint elliptic solutions. An optimal-transport framework for positive point-source measures appears in [23] under finite-dimensional temporal normalization. Thus, while the deterministic theory reveals distinct stability scales for locations and histories, it does not capture the additional information carried by stochastic integrals in the diffusion channel.

Stochastic inverse source problems are more recent; see [17, 19]. Lü [16] proved uniqueness via Carleman estimates. [26] considered two additive source types. More recently, [25] recovered temporal-factor moduli from partial flux, and [14] proved uniqueness of drift/diffusion supports via shape optimization. Despite these advances, quantitative stability for the simultaneous recovery of interior point locations and full temporal strengths in both channels, including signed diffusion profiles and finite-source matching, has not yet been established.

A key ingredient in the analysis below is the assumption that all sources become inactive before the end of the observation period. This switch-off condition is standard in the point-source literature [4, 5, 11] and is essential for obtaining the terminal-state regularity needed in the stability proof.

Condition 1.1. The source amplitudes vanish after T_0 :

$$\lambda_k(t) = 0 \quad (k = 1, \dots, N_0), \quad \gamma_l(t) = 0 \quad (l = 1, \dots, N_1), \quad \text{for a.e. } t \in (T_0, T). \quad (1.2)$$

To state the main stability results, we compare two candidate configurations with the same source counts,

$$\{(x_k^j, \lambda_k^j)\}_{k=1}^{N_0}, \quad \{(y_l^j, \gamma_l^j)\}_{l=1}^{N_1}, \quad j = 1, 2.$$

Both configurations satisfy (1.2). Let u^j be the corresponding solutions with the same initial datum. For $n = 1$, boundary integrals use counting measure; in particular, $H^1(\Gamma) = L^2(\Gamma)$.

We first consider $N_0, N_1 \in \{0, 1\}$, with at least one channel present. For each source that is present, we suppress the source index by writing $x^j = x_1^j$, $y^j = y_1^j$, $\lambda^j = \lambda_1^j$, and $\gamma^j = \gamma_1^j$ for $j = 1, 2$.

Quantifying drift-strength stability requires minimal regularity and an energy lower bound. We assume the first reference strength has nonzero L^2 -norm and that the difference of two candidate strengths lies in a suitable Sobolev space for frequency-domain analysis.

Assumption 1.1. *There exist $s \in (0, \frac{1}{2})$ and $M, r > 0$ such that $\lambda^1 \in H^s(0, T)$, $\lambda^1 - \lambda^2 \in H_0^1(0, T)$, and*

$$\|\lambda^1\|_{H^s(0, T)} + \|\lambda^1 - \lambda^2\|_{H^1(0, T)} \leq M, \quad \|\lambda^1\|_{L^2(0, T)} \geq r. \quad (1.3)$$

For the diffusion channel, the Itô isometry permits a direct L^2 -based estimate of the strength difference, so the required assumptions are milder. It suffices to impose a uniform upper bound on both strengths and a nondegeneracy lower bound on the reference strength.

Assumption 1.2. *There exist $M > 0$ and $r > 0$ such that*

$$\|\gamma^1\|_{L^2(0, T)} \geq r, \quad \|\gamma^1\|_{L^2(0, T)} + \|\gamma^2\|_{L^2(0, T)} \leq M. \quad (1.4)$$

Define the logarithmic modulus

$$\Phi(\eta) = \begin{cases} [\log(3 + \eta^{-1})]^{-2}, & \eta > 0, \\ 0, & \eta = 0. \end{cases} \quad (1.5)$$

For a boundary trace f for which the following quantities are finite, set

$$\|f\|_{\text{obs}} = \|f\|_{L_{\mathbb{F}}^2(0, T; L^2(\partial G))} + \|f\|_{L_{\mathbb{F}}^2(T_1, T; H^1(\Gamma))} + \|f\|_{L_{\mathbb{F}}^2(\Omega; H^1(T_1, T; L^2(\Gamma)))}.$$

For a state u , we write $\|u\|_{\text{obs}} = \|u\|_{\Sigma}|_{\text{obs}}$. For deterministic traces, the expectation is omitted.

Throughout Sections 1–4, C denotes a positive constant that may change from line to line. Unless otherwise stated, its dependence on $G, \Gamma, T, T_0, T_1, A$, and μ is suppressed. Any additional dependence is specified in the relevant statement or locally in the proof.

Theorem 1.1. *Let $n \in \{1, 2, 3\}$. Under [Condition 1.1](#), the following statements hold.*

(a) Drift source. *Let $N_0 = 1$, $x^j \in G$, and suppose that [Assumption 1.1](#) holds. There exists $C = C(s, M, r) > 0$ such that*

$$|x^1 - x^2| \leq C\|u^1 - u^2\|_{\text{obs}}, \quad (1.6)$$

$$\|\lambda^1 - \lambda^2\|_{L^2(0, T)} \leq C\Phi(\|u^1 - u^2\|_{\text{obs}}). \quad (1.7)$$

(b) Diffusion source. *Let $N_1 = 1$, $y^j \in G$, and suppose that [Assumption 1.2](#) holds. There exists $C = C(M, r) > 0$ such that*

$$|y^1 - y^2| + \|\gamma^1 - \gamma^2\|_{L^2(0, T)} \leq C\|u^1 - u^2\|_{\text{obs}}. \quad (1.8)$$

Remark 1.1 (Comparison with deterministic single-source stability). In the absence of the diffusion channel, [Theorem 1.1\(a\)](#) recovers the Lipschitz location and logarithmic strength estimates of [\[11, Theorems 2.1 and 2.3\]](#) for $n = 3$ and $n = 2$, respectively; for $n = 1$, it removes the restriction $\mu = -|A|^2/4$ and replaces the nondegeneracy condition on $\int_0^T \lambda^1(t) dt$ in [Theorem 2.4](#) by the $L^2(0, T)$ lower bound in [Assumption 1.1](#).

For multiple source configurations, the drift estimate determines one weighted temporal integral for each source, whereas the diffusion estimate recovers the strengths in $L^2(0, T)$.

Set $p_{\sharp} = -\mu - \frac{|A|^2}{4}$. For multiple drift sources, localization via polynomial annihilation requires a minimal separation distance between sources and a lower bound on the weighted temporal moments to be recovered.

Assumption 1.3 (Multiple drift sources). *Suppose that, for some $\delta, r > 0$,*

$$|x_k^j - x_l^j| \geq \delta \quad (k \neq l, j = 1, 2), \quad (1.9)$$

$$\left| \int_0^T e^{-p_{\sharp} t} \lambda_k^1(t) dt \right| \geq r \quad (k = 1, \dots, N_0). \quad (1.10)$$

Multiple diffusion sources require the same separation condition, an L^2 -nondegeneracy lower bound on each reference strength, and a uniform total-energy upper bound on all candidates.

Assumption 1.4 (Multiple diffusion sources). *Suppose that, for some $\delta, r, M > 0$,*

$$|y_l^j - y_m^j| \geq \delta \quad (l \neq m, j = 1, 2), \quad (1.11)$$

$$\|\gamma_l^1\|_{L^2(0, T)} \geq r \quad (l = 1, \dots, N_1), \quad (1.12)$$

$$\sum_{l=1}^{N_1} (\|\gamma_l^1\|_{L^2(0, T)} + \|\gamma_l^2\|_{L^2(0, T)}) \leq M. \quad (1.13)$$

Theorem 1.2 (Finite-source stability). *Under [Condition 1.1](#), the following statements hold.*

- (a) Drift sources. Let $n \in \{2, 3\}$ and $N_0 \geq 2$, and suppose that Assumption 1.3 holds. There exist a constant $C = C(N_0, \delta, r) > 0$ and a permutation σ of $\{1, \dots, N_0\}$ such that

$$\max_{1 \leq k \leq N_0} |x_k^1 - x_{\sigma(k)}^2| \leq C \|u^1 - u^2\|_{\text{obs}}. \quad (1.14)$$

If, in addition,

$$\sum_{k=1}^{N_0} (\|\lambda_k^1\|_{L^2(0,T)} + \|\lambda_k^2\|_{L^2(0,T)}) \leq M \quad (1.15)$$

for some $M > 0$, then the same permutation satisfies, with $C = C(N_0, \delta, r, M)$,

$$\max_{1 \leq k \leq N_0} \left| \int_0^T e^{-p_{\sharp} t} (\lambda_k^1(t) - \lambda_{\sigma(k)}^2(t)) dt \right| \leq C \|u^1 - u^2\|_{\text{obs}}. \quad (1.16)$$

- (b) Diffusion sources. Let $n \in \{1, 2, 3\}$ and $N_1 \geq 2$, and suppose that Assumption 1.4 holds. There exist a constant $C = C(N_1, \delta, r, M) > 0$ and a permutation σ of $\{1, \dots, N_1\}$ such that

$$\max_{1 \leq l \leq N_1} (|y_l^1 - y_{\sigma(l)}^2| + \|\gamma_l^1 - \gamma_{\sigma(l)}^2\|_{L^2(0,T)}) \leq C \|u^1 - u^2\|_{\text{obs}}. \quad (1.17)$$

Remark 1.2 (Comparison with deterministic finite-source stability). In the planar case $\mu = -|A|^2/4$, we have $p_{\sharp} = 0$, so (1.10) reduces to the moment condition in [11, Theorem 2.2]. Under this specialization, Theorem 1.2(a) improves the Hölder location estimate with exponent $1/N_0$ to a Lipschitz estimate and removes the a priori nearest-neighbor matching condition imposed there; see also Remark 4.3.

Moreover, it removes the coefficient restriction $\mu = -|A|^2/4$ and extends the result to dimension three, with nondegeneracy imposed on the temporal moments weighted by $e^{-p_{\sharp} t}$, where $p_{\sharp} = -\mu - |A|^2/4$. Under the additional bound (1.15), the same permutation gives Lipschitz recovery of the weighted temporal moments.

Remark 1.3. The restriction $n \in \{2, 3\}$ in Theorem 1.2(a) is essential: in dimension one, two distinct configurations of two drift sources can produce identical boundary traces, even under the switch-off condition (1.2) and the moment condition (1.10). Such a counterexample follows by modifying the construction in [1, Section 4] so that the auxiliary heat solution returns to zero at T_0 .

The preceding counterexample exposes a fundamental limitation in one dimension while underscoring the importance of separation and nondegeneracy in higher dimensions. Our analysis refines the probe construction and localization of [11]. For a single drift source, a nonvanishing Laplace mode in a bounded frequency window, combined with frequency-dependent hyperbolic-sine probes, yields uniform one-dimensional estimates without the earlier coefficient and moment restrictions. For multiple sources, complex-isotropic polynomial probes at $p_{\sharp}$ extend the planar construction to three dimensions and accommodate general μ . Separation ensures at most one small distance factor in the product estimate, giving Lipschitz localization and a matching permutation; normalized probes then recover the coefficients.

After separating the stochastic equation into expectation and centered parts, a stochastic Green identity and Itô's isometry control source evaluation in $L^2(0, T)$, while deterministic testing controls scalar Laplace moments. This full-time integrand control yields Lipschitz recovery of signed diffusion strengths and permits combining finitely many spectral parameters into annihilating probes in one dimension, where finite-source diffusion recovery follows from the same localization and matching argument.

The remainder is organized as follows. Section 2 states the direct-problem regularity, separates channels, and derives boundary-to-source estimates. Sections 3 and 4 contain the single-source proof and the finite-source matching principle, probe constructions, and stability proof, respectively. Section 5 presents numerical experiments, and Appendix A establishes the direct problem and its boundary regularity.

2 Channel separation and boundary-to-source estimates

The drift and diffusion channels reduce to fundamentally different estimates: deterministic scalar moments for the drift, and stochastic $L^2(0, T)$ -valued estimates for the diffusion. Taking expectations removes the stochastic integral and leaves a deterministic equation for the mean, while the centered part is controlled by Itô's isometry, which preserves the full L^2 -energy of the integrand. The following subsections derive the boundary-to-source estimates for both channels.

Let $\mathcal{A} = -\mathcal{L}$ be the Neumann realization on $L^2(G)$, with

$$\mathcal{A}u = \Delta u - A \cdot \nabla u - \mu u, \quad D(\mathcal{A}) = \{u \in H^2(G) : \partial_{\nu} u = 0\}, \quad \mathcal{X}_{-2} = D(\mathcal{A}^*)', \quad (2.1)$$

where the dual is taken with pivot $L^2(G)$. We write $S(t)$ and $S_{-2}(t)$ for the corresponding semigroups; their construction is given in Appendix A. The formal adjoint expression is $\mathcal{L}^* = -\Delta - A \cdot \nabla + \mu$.

Proposition 2.1 (Well-posedness and boundary regularity). *For every fixed finite source configuration in (1.1), the direct problem admits a unique mild solution $u \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathcal{X}_{-2}))$. Moreover,*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u(t)\|_{\mathcal{X}_{-2}}^2 \leq C \left(\|u_0\|_{L^2(G)}^2 + \sum_{k=1}^{N_0} \|\lambda_k\|_{L^2(0,T)}^2 + \sum_{l=1}^{N_1} \|\gamma_l\|_{L^2(0,T)}^2 \right), \quad (2.2)$$

where C depends only on G, A, μ, T, N_0 , and N_1 . If Condition 1.1 holds, then $\|u\|_{\text{obs}} < \infty$ and $u \in L^2(\Omega; H^1(T_1, T; H^2(G)))$. In particular,

$$\|u\|_{\text{obs}}^2 + \mathbb{E} \|u\|_{H^1(T_1, T; H^2(G))}^2 + \mathbb{E} \|u(T)\|_{H^2(G)}^2 \leq C_{\text{conf}} \left(\|u_0\|_{L^2(G)}^2 + \sum_{k=1}^{N_0} \|\lambda_k\|_{L^2(0,T)}^2 + \sum_{l=1}^{N_1} \|\gamma_l\|_{L^2(0,T)}^2 \right). \quad (2.3)$$

Here C_{conf} may additionally depend on T_0, T_1 and the source locations, but not on the source strengths.

The proof of Proposition 2.1 is given in Appendix A.

For the two candidate solutions, set $v = u^1 - u^2$, $\bar{v} = \mathbb{E}v$ and $z = v - \bar{v}$. By the zero-mean property of the Itô integral, the mean $\bar{v}$ is the solution of

$$\begin{cases} \partial_t \bar{v} + \mathcal{L} \bar{v} = \sum_{k=1}^{N_0} \lambda_k^1(t) \delta_{x_k^1} - \sum_{k=1}^{N_0} \lambda_k^2(t) \delta_{x_k^2} & \text{in } (0, T) \times G, \\ \partial_\nu \bar{v} = 0 & \text{on } (0, T) \times \partial G, \\ \bar{v}(0) = 0 & \text{in } G, \end{cases} \quad (2.4)$$

whereas z is the mild solution of

$$\begin{cases} dz + \mathcal{L}z dt = \left(\sum_{l=1}^{N_1} \gamma_l^1(t) \delta_{y_l^1} - \sum_{l=1}^{N_1} \gamma_l^2(t) \delta_{y_l^2} \right) dW(t) & \text{in } (0, T) \times G, \\ \partial_\nu z = 0 & \text{on } (0, T) \times \partial G, \\ z(0) = 0 & \text{in } G. \end{cases} \quad (2.5)$$

The boundary regularity in Proposition 2.1 allows expectation to commute with the boundary traces. It follows that

$$\max\{\|\bar{v}\|_{\text{obs}}, \|z\|_{\text{obs}}\} \leq \|v\|_{\text{obs}}. \quad (2.6)$$

2.1 Boundary-to-source estimates

In this subsection, we derive boundary-to-source estimates for the two decoupled equations. A terminal-state estimate controls the terminal terms in Green's identities, yielding scalar Laplace-moment bounds for the drift and $L^2(0, T)$ bounds for the diffusion. These estimates are used in Sections 3 and 4.

Lemma 2.2 (Terminal-state estimate). *Let $w \in H^1(T_1, T; H^2(G))$ solve*

$$\partial_t w + \mathcal{L}w = 0 \quad \text{in } (T_1, T) \times G, \quad \partial_\nu w = 0 \quad \text{on } (T_1, T) \times \partial G.$$

Then

$$\|w(T)\|_{H^2(G)} \leq C (\|w\|_{L^2(T_1, T; H^1(G))} + \|w\|_{H^1(T_1, T; L^2(G))}). \quad (2.7)$$

Proof. The estimate follows from the partial-boundary Carleman estimate and parabolic smoothing, as in the proof of [11, Theorem 3.1, (3.2)]; see also [12, Lemma 2.4 and (2.12)]. $\square$

For $q \in L^2(0, T)$, define its finite-time Laplace transform by

$$\tilde{q}(p) = \int_0^T e^{-pt} q(t) dt, \quad p \in \mathbb{C}. \quad (2.8)$$

For $v \in H^2(G; \mathbb{C})$, set

$$\mathcal{B}^* v = \partial_\nu v + (A \cdot \nu) v \quad \text{on } \partial G. \quad (2.9)$$

The $L^2(G)$ pairing is extended complex bilinearly, and no boundary condition is imposed on v . For $r \in \mathbb{R}$, write $r_+ = \max\{r, 0\}$.

Lemma 2.3 (Deterministic boundary-to-source estimate). *Let w be the solution of*

$$\begin{cases} \partial_t w + \mathcal{L}w = \sum_{k=1}^J q_k(t) \delta_{a_k} & \text{in } (0, T) \times G, \\ \partial_\nu w = 0 & \text{on } (0, T) \times \partial G, \\ w(0) = 0 & \text{in } G, \end{cases}$$

where $a_k \in G$ and $q_k \in L^2(0, T)$ vanish on (T_0, T) . If $p \in \mathbb{C}$, $v \in H^2(G; \mathbb{C})$, and $(\mathcal{L}^* + p)v = 0$, then

$$\sum_{k=1}^J \tilde{q}_k(p)v(a_k) = e^{-pT}(w(T), v)_{L^2(G)} + \int_0^T \int_{\partial G} w(t, \xi) e^{-pt} \mathcal{B}^* v(\xi) d\sigma dt. \quad (2.10)$$

Moreover,

$$\left| \sum_{k=1}^J \tilde{q}_k(p)v(a_k) \right| \leq C e^{(-\operatorname{Re} p) + T} \|v\|_{H^2(G)} \|w\|_{\text{obs}}. \quad (2.11)$$

The constant is independent of p , the source locations, and the strengths.

Proof. Set $\tilde{v}(t, \xi) = e^{-pt}v(\xi)$. Since $(\mathcal{L}^* + p)v = 0$, we have $-\partial_t \tilde{v} + \mathcal{L}^* \tilde{v} = 0$. Applying [11, Lemma 4.1] with this test function and using $w(0) = 0$ gives (2.10).

We next estimate the two terms on its right-hand side. Since the sources vanish after T_0 , Lemma 2.2 gives $\|w(T)\|_{H^2(G)} \leq C\|w\|_{\text{obs}}$. Hence

$$|e^{-pT}(w(T), v)_{L^2(G)}| \leq e^{(-\operatorname{Re} p) + T} \|w(T)\|_{L^2(G)} \|v\|_{L^2(G)} \leq C e^{(-\operatorname{Re} p) + T} \|v\|_{H^2(G)} \|w\|_{\text{obs}}. \quad (2.12)$$

For the boundary term, the Cauchy-Schwarz inequality and the trace bound $\|\mathcal{B}^* v\|_{L^2(\partial G)} \leq C\|v\|_{H^2(G)}$ yield

$$\left| \int_0^T \int_{\partial G} w(t, \xi) e^{-pt} \mathcal{B}^* v(\xi) d\sigma dt \right| \leq \sqrt{T} e^{(-\operatorname{Re} p) + T} \|w\|_{L^2(0, T; L^2(\partial G))} \|\mathcal{B}^* v\|_{L^2(\partial G)} \leq C e^{(-\operatorname{Re} p) + T} \|v\|_{H^2(G)} \|w\|_{\text{obs}}.$$

This, together with (2.12) and (2.10), proves (2.11). $\square$

For the diffusion channel, define the source evaluation

$$\mathfrak{D}[v](t) := \sum_{l=1}^{N_1} \gamma_l^1(t)v(y_l^1) - \sum_{l=1}^{N_1} \gamma_l^2(t)v(y_l^2). \quad (2.13)$$

Lemma 2.4 (Stochastic boundary-to-source estimate). *Let z be the solution of (2.5) under Condition 1.1. If $p \in \mathbb{R}$, $v \in H^2(G; \mathbb{C})$, and $(\mathcal{L}^* + p)v = 0$, then*

$$\int_0^T e^{-pt} \mathfrak{D}[v](t) dW(t) = e^{-pT}(z(T), v)_{L^2(G)} + \int_0^T \int_{\partial G} z e^{-pt} \mathcal{B}^* v d\sigma dt, \quad \mathbb{P}\text{-a.s.} \quad (2.14)$$

and

$$\|\mathfrak{D}[v]\|_{L^2(0, T)} \leq C_p \|v\|_{H^2(G)} \|z\|_{\text{obs}}. \quad (2.15)$$

Here $C_p = C(p)$ is independent of the source locations and strengths.

Proof. We regularize the point sources using

$$\rho(\xi) = \begin{cases} c_n \exp(-1/(1 - |\xi|^2)), & |\xi| < 1, \\ 0, & |\xi| \geq 1, \end{cases} \quad \rho_\epsilon^y(\xi) = \epsilon^{-n} \rho\left(\frac{\xi - y}{\epsilon}\right), \quad (2.16)$$

where $c_n > 0$ is chosen so that $\int_{\mathbb{R}^n} \rho(\xi) d\xi = 1$. For sufficiently small $\epsilon > 0$, all $\rho_\epsilon^{y_i^j}$ are supported in G . Let z_ϵ solve

$$\begin{cases} dz_\epsilon + \mathcal{L}z_\epsilon dt = \left(\sum_{l=1}^{N_1} \gamma_l^1(t) \rho_\epsilon^{y_l^1} - \sum_{l=1}^{N_1} \gamma_l^2(t) \rho_\epsilon^{y_l^2} \right) dW(t) & \text{in } (0, T) \times G, \\ \partial_\nu z_\epsilon = 0 & \text{on } (0, T) \times \partial G, \\ z_\epsilon(0) = 0 & \text{in } G. \end{cases} \quad (2.17)$$

Set

$$\mathfrak{D}_\epsilon[v](t) := \sum_{l=1}^{N_1} \gamma_l^1(t) (\rho_\epsilon^{y_l^1}, v)_{L^2(G)} - \sum_{l=1}^{N_1} \gamma_l^2(t) (\rho_\epsilon^{y_l^2}, v)_{L^2(G)}.$$

Applying Itô's formula and Green's formula to this equation yields

$$d(e^{-pt}(z_\epsilon(t), v)_{L^2(G)}) = e^{-pt} \mathfrak{D}_\epsilon[v](t) dW(t) - e^{-pt} \int_{\partial G} z_\epsilon(t, \xi) \mathcal{B}^* v(\xi) d\sigma dt.$$

Here the volume term vanishes because $(\mathcal{L}^* + p)v = 0$, and the deterministic test function contributes no quadratic covariation. Integrating in time and using $z_\epsilon(0) = 0$ gives

$$\int_0^T e^{-pt} \mathfrak{D}_\epsilon[v](t) dW(t) = e^{-pT}(z_\epsilon(T), v)_{L^2(G)} + \int_0^T \int_{\partial G} z_\epsilon(t, \xi) e^{-pt} \mathcal{B}^* v(\xi) d\sigma dt.$$

By Lemma A.1 in Appendix A, all terms converge in $L^2(\Omega; \mathbb{C})$ as $\epsilon \rightarrow 0$. Passing to the limit yields (2.14).

We now estimate the source evaluation. Since the sources vanish after T_0 , $z(\omega)$ satisfies the homogeneous parabolic equation on $(T_1, T) \times G$ for almost every ω . Applying Lemma 2.2 to each sample path, with a constant independent of ω , and taking $L^2(\Omega)$ norms gives

$$\|z(T)\|_{L^2(\Omega; L^2(G))} \leq C(\|z\|_{L^2_\frac{T}{2}(T_1, T; H^1(\Gamma))} + \|z\|_{L^2_\frac{T}{2}(\Omega; H^1(T_1, T; L^2(\Gamma)))}) \leq C\|z\|_{\text{obs}}.$$

Using Itô's isometry, the Cauchy–Schwarz inequality, and the trace bound $\|\mathcal{B}^*v\|_{L^2(\partial G)} \leq C\|v\|_{H^2(G)}$, we obtain

$$\begin{aligned} \|e^{-p \cdot} \mathfrak{D}[v]\|_{L^2(0, T)} &= \left(\mathbb{E} \left| \int_0^T e^{-pt} \mathfrak{D}[v](t) dW(t) \right|^2 \right)^{1/2} \\ &\leq e^{(-p)+T} \left(\|z(T)\|_{L^2(\Omega; L^2(G))} \|v\|_{L^2(G)} + \left\| \int_0^T \|z(t)\|_{L^2(\partial G)} dt \right\|_{L^2(\Omega)} \|\mathcal{B}^*v\|_{L^2(\partial G)} \right) \\ &\leq e^{(-p)+T} \left(\|z(T)\|_{L^2(\Omega; L^2(G))} \|v\|_{L^2(G)} + \sqrt{T} \|z\|_{L^2_\frac{T}{2}(0, T; L^2(\partial G))} \|\mathcal{B}^*v\|_{L^2(\partial G)} \right) \\ &\leq C_p \|v\|_{H^2(G)} \|z\|_{\text{obs}}. \end{aligned}$$

Finally, $e^{-pt} \geq e^{-p+T}$ on $[0, T]$, so removing the time weight gives (2.15). $\square$

3 Proof of the single-source theorem

Throughout this section, we take one source in the channel under consideration, assume Condition 1.1, and use the notation of Theorem 1.1. The same explicit spatial probes are used in both channels. The proofs differ only in the temporal coercivity supplied by the two boundary-to-source estimates.

3.1 Explicit adjoint probes

We construct a location probe that vanishes at one candidate source point and an amplitude probe normalized to one at that point. The location probe has different forms in dimensions $n = 2, 3$ and $n = 1$.

For $p \in \mathbb{C}$ and $v(\xi) = e^{-A \cdot \xi/2} h(\xi)$, direct differentiation gives

$$(\mathcal{L}^* + p)v = e^{-A \cdot \xi/2} \left[-\Delta h + \left(p + \mu + \frac{|A|^2}{4} \right) h \right]. \quad (3.1)$$

Choose a κ satisfying

$$\kappa^2 = p + \mu + \frac{|A|^2}{4}. \quad (3.2)$$

Dimensions $n = 2, 3$. For each unit vector $\varrho \in \mathbb{R}^n$, choose a unit vector $f_\varrho \perp \varrho$. For $a, b \in G$, define

$$v_{\varrho, a, b}^p(\xi) = e^{-\frac{1}{2}A \cdot (\xi - a)} (\varrho \cdot (\xi - b)) e^{\kappa f_\varrho \cdot (\xi - a)}. \quad (3.3)$$

By (3.1) and $f_\varrho \perp \varrho$,

$$(\mathcal{L}^* + p)v_{\varrho, a, b}^p = 0, \quad v_{\varrho, a, b}^p(b) = 0, \quad v_{\varrho, a, b}^p(a) = \varrho \cdot (a - b).$$

For $a \neq b$, choosing $\varrho = (a - b)/|a - b|$ gives $v_{\varrho, a, b}^p(a) = |a - b|$.

Dimension $n = 1$. After a translation, write $G = (0, \ell)$. For $a, b \in G$, define

$$\vartheta_{a, b}^p(\xi) = e^{-\frac{A}{2}(\xi - a)} \begin{cases} \frac{\sinh(\kappa(\xi - b))}{\kappa}, & \kappa \neq 0, \\ \xi - b, & \kappa = 0. \end{cases} \quad (3.4)$$

This probe satisfies

$$(\mathcal{L}^* + p)\vartheta_{a, b}^p = 0, \quad \vartheta_{a, b}^p(b) = 0.$$

For a uniform lower bound at a , let $\mathfrak{P}$ be a compact subset of

$$\left\{ p \in \mathbb{C} : \operatorname{Re} \left(p + \mu + \frac{A^2}{4} \right) > 0 \right\}, \quad (3.5)$$

and choose $\operatorname{Re} \kappa > 0$. The function $\sinh(\kappa r)/(\kappa r)$, continued by 1 at $r = 0$, is continuous and nonzero on the compact set of admissible roots and $r \in [-\operatorname{diam} G, \operatorname{diam} G]$. Hence

$$|\vartheta_{a, b}^p(a)| \geq c_{\mathfrak{P}} |a - b|, \quad a, b \in G, \quad p \in \mathfrak{P}, \quad (3.6)$$

where $c_{\mathfrak{P}} > 0$ depends only on $G, A, \mu, \mathfrak{P}$.

A common amplitude probe. For every $n \in \{1, 2, 3\}$, $p \in \mathbb{C}$, $b \in G$, and unit vector $\varpi \in \mathbb{R}^n$, define

$$\psi_{b,\varpi}^p(\xi) = e^{-\frac{1}{2}A \cdot (\xi-b)} e^{\kappa \varpi \cdot (\xi-b)}. \quad (3.7)$$

We take $\varpi = 1$ when $n = 1$. By (3.1), $(\mathcal{L}^* + p)\psi_{b,\varpi}^p = 0$ and $\psi_{b,\varpi}^p(b) = 1$. For each probe in its stated dimension, differentiating the explicit formula at most twice gives

$$\|\varphi\|_{H^2(G)} \leq C(1 + |\kappa|)^2 e^{C|\kappa|}, \quad \varphi \in \{v_{\varrho,a,b}^p, \vartheta_{a,b}^p, \psi_{b,\varpi}^p\}. \quad (3.8)$$

The mean-value formula for the exponential also yields

$$|\psi_{b,\varpi}^p(a) - 1| \leq C(1 + |\kappa|) e^{C|\kappa|} |a - b|. \quad (3.9)$$

Here C depends only on G and A , uniformly in the unit vectors.

3.2 Proof of the single-source stability estimates

Proof of Theorem 1.1. Set $\varepsilon = \|v\|_{\text{obs}}$, with $v, \bar{v}, z$ as in Section 2, and fix $p_0 = \max\{0, -\mu - |A|^2/4\} + 2$. By (2.6), both channel errors are at most ε .

Part I: drift channel. In this part, C may additionally depend on s, M , and r . Take $w = \bar{v}$ in Lemma 2.3.

Step 1: selection of a nondegenerate frequency. The zero-extension map $H^s(0, T) \rightarrow H^s(\mathbb{R})$ is bounded for $0 < s < 1/2$ [20, Lemma 3.32 and Theorem 3.33]. Thus, by Assumption 1.1, $f_1(t) = e^{-p_0 t} \lambda^1(t) \mathbf{1}_{(0,T)}(t)$ satisfies $\|f_1\|_{H^s(\mathbb{R})} \leq CM$ and $\|f_1\|_{L^2(\mathbb{R})} \geq e^{-p_0 T} r$.

With the Fourier convention $\widehat{f}_1(\tau) = \int_{\mathbb{R}} e^{-i\tau t} f_1(t) dt$, we have $\widehat{f}_1(\tau) = \widetilde{\lambda^1}(p_0 + i\tau)$. Plancherel's theorem and the H^s tail bound give

$$\int_{-R}^R |\widehat{f}_1(\tau)|^2 d\tau \geq 2\pi e^{-2p_0 T} r^2 - CM^2 R^{-2s}, \quad R \geq 1.$$

Choosing $R_0 \geq 1$ so that $CM^2 R_0^{-2s} \leq \pi e^{-2p_0 T} r^2$ yields

$$\sup_{|\tau| \leq R_0} |\widetilde{\lambda^1}(p_0 + i\tau)| \geq \left(\frac{\pi e^{-2p_0 T} r^2}{2R_0} \right)^{1/2} =: r_0 > 0.$$

Here R_0 and r_0 depend only on p_0, T, s, M, r .

Choose $\tau_0 \in [-R_0, R_0]$ such that

$$|\widetilde{\lambda^1}(p_0 + i\tau_0)| \geq r_0/2. \quad (3.10)$$

Set $p_{\dagger} = p_0 + i\tau_0$.

Step 2: location recovery. The location estimate is immediate when $x^1 = x^2$. Assume $x^1 \neq x^2$.

Case 1. $n = 2, 3$. Choose $\varrho = (x^1 - x^2)/|x^1 - x^2|$ and use $v_{\varrho, x^1, x^2}^{p_{\dagger}}$. By (2.11) and (3.8),

$$|\widetilde{\lambda^1}(p_{\dagger})| |x^1 - x^2| \leq C\varepsilon, \quad (3.11)$$

with C uniform for $\tau_0 \in [-R_0, R_0]$.

Case 2. $n = 1$. The compact frequency segment $\mathfrak{P}_0 = \{p_0 + i\tau : |\tau| \leq R_0\}$ lies in the set in (3.5) by the choice of p_0 . Using $\vartheta_{x^1, x^2}^{p_{\dagger}}$, which vanishes at x^2 , Lemma 2.3 together with (3.6) and (3.8) yields

$$c_{\mathfrak{P}_0} |\widetilde{\lambda^1}(p_{\dagger})| |x^1 - x^2| \leq C\varepsilon. \quad (3.12)$$

In both (3.11) and (3.12), (3.10) gives

$$|x^1 - x^2| \leq C\varepsilon. \quad (3.13)$$

Step 3: low-frequency strength estimate. Fix $R \geq 1$ and $|\tau| \leq R$, and set $p = p_0 + i\tau$. Use the common amplitude probe with ϖ fixed as in its construction, and set $\beta(p) = \psi_{x^2, \varpi}^p(x^1)$. Since $\psi_{x^2, \varpi}^p(x^2) = 1$, applying (2.11) yields

$$|\beta(p) \widetilde{\lambda^1}(p) - \widetilde{\lambda^2}(p)| \leq C \|\psi_{x^2, \varpi}^p\|_{H^2(G)} \varepsilon. \quad (3.14)$$

Furthermore, $|\widetilde{\lambda^1}(p)| \leq \sqrt{T} \|\lambda^1\|_{L^2(0,T)} \leq CM$. By (3.9) and (3.13),

$$|\beta(p) - 1| |\widetilde{\lambda^1}(p)| \leq C(1 + |\kappa|) e^{C|\kappa|} |x^1 - x^2| M \leq C(1 + |\kappa|) e^{C|\kappa|} \varepsilon.$$

Combining this estimate with (3.14) and noting that $|\kappa| \leq C(1 + R)^{1/2}$ for $|\tau| \leq R$, we obtain

$$\sup_{|\tau| \leq R} |(\widetilde{\lambda^1} - \widetilde{\lambda^2})(p_0 + i\tau)| \leq C(1 + R)^2 e^{C\sqrt{R}} \varepsilon. \quad (3.15)$$

Step 4: high-frequency tail and optimization. Extend $f(t) = e^{-p_0 t}(\lambda^1(t) - \lambda^2(t))$ by zero outside $(0, T)$. Then $\widehat{f}(\tau) = (\lambda^1 - \lambda^2)(p_0 + i\tau)$. Since $\lambda^1 - \lambda^2 \in H_0^1(0, T)$, boundedness of zero extension gives

$$\|f\|_{H^1(\mathbb{R})} \leq CM. \quad (3.16)$$

Plancherel's theorem, (3.15), and (3.16) imply that, for $R \geq 1$,

$$\begin{aligned} \|\lambda^1 - \lambda^2\|_{L^2(0, T)}^2 &\leq C \int_{\mathbb{R}} |\widehat{f}(\tau)|^2 d\tau \leq C \left(2R \sup_{|\tau| \leq R} |\widehat{f}(\tau)|^2 + \int_{|\tau| > R} |\widehat{f}(\tau)|^2 d\tau \right) \\ &\leq C(R(1+R)^4 e^{C\sqrt{R}} \varepsilon^2 + R^{-2} M^2) \\ &\leq C(e^{C_1 \sqrt{R}} \varepsilon^2 + R^{-2}). \end{aligned} \quad (3.17)$$

Here C and C_1 are independent of R and ε .

For $0 < \varepsilon < \varepsilon_*$, choose $\sqrt{R} = C_1^{-1} \log(\varepsilon^{-1})$, where the fixed threshold $0 < \varepsilon_* \leq 1/2$ is small enough that $R \geq 1$. Then (3.17) gives

$$\|\lambda^1 - \lambda^2\|_{L^2(0, T)} \leq C[\log(\varepsilon^{-1})]^{-2} \leq C\Phi(\varepsilon),$$

where we used (1.5) and $\log(3 + \varepsilon^{-1}) \leq 3 \log(\varepsilon^{-1})$ for $0 < \varepsilon \leq 1/2$. If $\varepsilon = 0$, letting $R \rightarrow \infty$ gives $\lambda^1 = \lambda^2$, consistent with $\Phi(0) = 0$. For $\varepsilon \geq \varepsilon_*$, we have $\Phi(\varepsilon) \geq \Phi(\varepsilon_*) > 0$, so Assumption 1.1 gives

$$\|\lambda^1 - \lambda^2\|_{L^2(0, T)} \leq M \leq \frac{M}{\Phi(\varepsilon_*)} \Phi(\varepsilon).$$

Increasing C if necessary proves (1.7) for all $\varepsilon \geq 0$. Together with (3.13), this proves part (a).

Part II: diffusion channel. In this part, C may additionally depend on M and r . At $p = p_0$, the square root $\kappa_0 > 0$ in (3.2) is real.

Step 1: location recovery. The location estimate is immediate when $y^1 = y^2$. Assume $y^1 \neq y^2$.

Case 1. $n = 2, 3$. Choose $\varrho = (y^1 - y^2)/|y^1 - y^2|$ and use $v_{\varrho, y^1, y^2}^{p_0}$. It vanishes at y^2 and equals $|y^1 - y^2|$ at y^1 , so Lemma 2.4 and (3.8) give

$$|y^1 - y^2| \|\gamma^1\|_{L^2(0, T)} \leq C\varepsilon. \quad (3.18)$$

Case 2. $n = 1$. Use the probe $\vartheta_{y^1, y^2}^{p_0}$ from (3.4). It vanishes at y^2 . Applying Lemma 2.4 with (3.8) and (3.6), where $\mathfrak{P} = \{p_0\}$, gives

$$c_{\{p_0\}} |y^1 - y^2| \|\gamma^1\|_{L^2(0, T)} \leq C\varepsilon. \quad (3.19)$$

In both (3.18) and (3.19), the lower bound in (1.4) yields

$$|y^1 - y^2| \leq C\varepsilon. \quad (3.20)$$

Step 2: strength recovery. Use $\psi_{y^2, \varpi}^{p_0}$ from (3.7), with ϖ fixed as in its construction, and set $\beta = \psi_{y^2, \varpi}^{p_0}(y^1)$. Lemma 2.4 gives

$$\|\beta\gamma^1 - \gamma^2\|_{L^2(0, T)} \leq C\varepsilon.$$

By (3.9), (3.20), and the upper bound in (1.4),

$$\|\gamma^1 - \gamma^2\|_{L^2(0, T)} \leq \|\beta\gamma^1 - \gamma^2\|_{L^2(0, T)} + |\beta - 1| \|\gamma^1\|_{L^2(0, T)} \leq C\varepsilon + C|y^1 - y^2|M \leq C\varepsilon.$$

Combining this estimate with (3.20) proves (1.8). $\square$

4 Finite-source stability

This section establishes that probes isolating a source or a matched pair provide location-coefficient stability, and then constructs such probes via polynomials in dimensions two and three, and via spectral synthesis for one-dimensional diffusion sources. Here N denotes the source count, with N_0 for drift and N_1 for diffusion.

4.1 From annihilating probes to stable matching

Consider two configurations $(a_k^j)_{k=1}^N$ in G , $j = 1, 2$. To locate a target source, we cancel the contributions from all other candidate points. After matching the locations, we instead retain the two sources in each matched pair to compare their coefficients. Write $c_k^j \in E$ for the coefficients, where E is a complex normed space. In the applications, $E = \mathbb{C}$ for drift moments and $E = L^2(0, T; \mathbb{C})$ for diffusion strengths.

For a scalar function $V : G \rightarrow \mathbb{C}$, set

$$\mathfrak{S}[V] := \sum_{k=1}^N c_k^1 V(a_k^1) - \sum_{k=1}^N c_k^2 V(a_k^2).$$

Let $\varepsilon \geq 0$ denote the observation error. The two types of probe are specified as follows.

Location probes. For a target with $\min_{1 \leq m \leq N} |a_l^1 - a_m^2| > 0$, we seek H_l satisfying

$$\begin{cases} H_l(a_k^1) = 0 & (k \neq l), & H_l(a_m^2) = 0 & (1 \leq m \leq N), \\ |H_l(a_l^1)| \geq c_0 \prod_{\substack{1 \leq k \leq N \\ k \neq l}} |a_l^1 - a_k^1| \prod_{m=1}^N |a_l^1 - a_m^2|, \\ \|\mathfrak{S}[H_l]\|_E \leq C_0 \varepsilon. \end{cases} \quad (4.1)$$

Here $c_0, C_0 > 0$ are independent of l and the configurations. The zeros leave only $\mathfrak{S}[H_l] = c_l^1 H_l(a_l^1)$, so the product lower bound connects the observation error to the distances from the target. The zero list contains the other $N - 1$ points of the first configuration and all N points of the second, giving $2N - 1$ entries.

Coefficient probes. After choosing a matching permutation σ , we seek probes V_l that retain only the matched pair and equal 1 at the reference point. With $\|V\|_{\text{Lip}(G)} = \|V\|_{L^\infty(G)} + \sup_{\{a, b \in G, a \neq b\}} |V(a) - V(b)|/|a - b|$, we require

$$\begin{cases} V_l(a_l^1) = 1, & V_l(a_k^1) = 0 \ (k \neq l), & V_l(a_m^2) = 0 \ (m \neq \sigma(l)), \\ \|\mathfrak{S}[V_l]\|_E \leq C_1 \varepsilon, & \|V_l\|_{\text{Lip}(G)} \leq C_1. \end{cases} \quad (4.2)$$

Here $C_1 > 0$ is uniform in l and the configurations. The Lipschitz bound makes $V_l(a_{\sigma(l)}^2)$ close to 1 when the matched locations are close. This zero list omits both points of the matched pair and has $2N - 2$ entries. Either list may contain repeated nodes when the configurations overlap.

The following lemma shows that these probe properties suffice for stable recovery.

Lemma 4.1 (Localization and matching). *Assume that $|a_k^j - a_l^j| \geq \delta > 0$ for $k \neq l$, $j = 1, 2$, and $\|c_k^1\|_E \geq r > 0$ for every k . If every target with $\min_{1 \leq m \leq N} |a_l^1 - a_m^2| > 0$ admits a probe satisfying (4.1), then there exist $C = C(G, N, \delta, r, c_0, C_0) > 0$ and a permutation σ such that*

$$\max_{1 \leq l \leq N} |a_l^1 - a_{\sigma(l)}^2| \leq C\varepsilon. \quad (4.3)$$

If, in addition, $\sum_{k=1}^N (\|c_k^1\|_E + \|c_k^2\|_E) \leq M$ and every permutation with $\max_l |a_l^1 - a_{\sigma(l)}^2| < \delta/4$ admits probes satisfying (4.2), then the same permutation satisfies

$$\max_{1 \leq l \leq N} \|c_l^1 - c_{\sigma(l)}^2\|_E \leq C\varepsilon, \quad (4.4)$$

where C may additionally depend on M and C_1 .

Proof. For each $l \in \{1, \dots, N\}$, choose a nearest point $a_{m_l}^2$. Separation of the second configuration gives

$$|a_l^1 - a_{m_l}^2| \geq \delta/2 \quad (m \neq m_l). \quad (4.5)$$

Indeed, this holds immediately if $|a_l^1 - a_{m_l}^2| \geq \delta/2$; otherwise, $|a_l^1 - a_{m_l}^2| \geq \delta - |a_l^1 - a_{m_l}^2| > \delta/2$. For $|a_l^1 - a_{m_l}^2| > 0$, the product-probe estimate and the reference lower bound yield

$$c_0 r \delta^{N-1} (\delta/2)^{N-1} |a_l^1 - a_{m_l}^2| \leq \|c_l^1\|_E |H_l(a_l^1)| = \|\mathfrak{S}[H_l]\|_E \leq C_0 \varepsilon.$$

Thus $|a_l^1 - a_{m_l}^2| \leq C\varepsilon$ for every l , including those with $|a_l^1 - a_{m_l}^2| = 0$. Choose $\varepsilon_* > 0$ so that $C\varepsilon_* < \delta/4$. For $\varepsilon \leq \varepsilon_*$, the nearest points are unique and form a permutation σ : two points in the first configuration assigned to the same point would be less than $\delta/2$ apart. This proves (4.3) for small errors. For $\varepsilon > \varepsilon_*$, any permutation gives the location estimate by boundedness of G .

Under the additional assumptions for coefficient recovery, consider first $\varepsilon \leq \varepsilon_*$. For the matching above, every nonmatched node is at distance at least $3\delta/4$ from a_l^1 , and the normalized probes give

$$\mathfrak{S}[V_l] = c_l^1 - c_{\sigma(l)}^2 V_l(a_{\sigma(l)}^2).$$

Their Lipschitz bounds and $V_l(a_l^1) = 1$ imply

$$\|c_l^1 - c_{\sigma(l)}^2\|_E \leq \|\mathfrak{S}[V_l]\|_E + \|c_{\sigma(l)}^2\|_E |V_l(a_{\sigma(l)}^2) - V_l(a_l^1)| \leq C\varepsilon + CM |a_l^1 - a_{\sigma(l)}^2| \leq C\varepsilon.$$

For $\varepsilon > \varepsilon_*$, the bound M gives the coefficient estimate for the same permutation after increasing the constant. $\square$

4.2 Annihilating probes in dimensions two and three

We now construct the probes required by Lemma 4.1 in dimensions two and three at the common spectral parameter $p_\sharp$. The $H^2(G)$ bounds below allow us to apply the boundary-to-source estimates in Lemmas 2.3 and 2.4. The dot product is extended complex bilinearly, whereas $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^n$.

Lemma 4.2 (Polynomial annihilators in dimensions two and three). *Let $n \in \{2, 3\}$, let $a \in G$, and let $b_1, \dots, b_J \in G \setminus \{a\}$, where $J \leq 2N - 1$. There exists $H_{a,\mathbf{b}} \in H^2(G; \mathbb{C})$ such that*

$$(\mathcal{L}^* + p_\sharp)H_{a,\mathbf{b}} = 0, \quad (4.6)$$

$$H_{a,\mathbf{b}}(b_j) = 0, \quad j = 1, \dots, J, \quad (4.7)$$

$$|H_{a,\mathbf{b}}(a)| \geq c \prod_{j=1}^J |a - b_j|, \quad \|H_{a,\mathbf{b}}\|_{H^2(G)} \leq C. \quad (4.8)$$

Here $c, C > 0$ depend only on G, A, N . If, in addition, $|a - b_j| \geq \rho > 0$ for every j , then the normalized probe $V_{a,\mathbf{b}} = H_{a,\mathbf{b}}/H_{a,\mathbf{b}}(a)$ satisfies

$$V_{a,\mathbf{b}}(a) = 1, \quad V_{a,\mathbf{b}}(b_j) = 0, \quad \|V_{a,\mathbf{b}}\|_{H^2(G)} + \|V_{a,\mathbf{b}}\|_{\text{Lip}(G)} \leq C_\rho, \quad (4.9)$$

where C_ρ depends only on G, A, N and ρ .

Remark 4.1. The planar construction below is the isotropic-vector formulation of the holomorphic polynomial probes used in [11]. The point of Lemma 4.2 is its quantitative extension to dimension three, obtained by choosing the complex-isotropic direction according to the finite source configuration.

Proof of Lemma 4.2. Step 1: choosing a uniformly transverse isotropic direction. Set $\alpha_N = 1 - (2N)^{-1}$ and $c_N = (1 - \alpha_N^2)^{1/2} \in (0, 1)$. We choose $\theta \in \mathbb{C}^n$ satisfying

$$\theta \cdot \theta = 0, \quad |\theta| = \sqrt{2}, \quad |\theta \cdot (a - b_j)| \geq c_N |a - b_j|, \quad j = 1, \dots, J.$$

The first condition makes polynomials in $\theta \cdot \xi$ harmonic. The last gives a uniform lower bound at the target point. For $n = 2$, take $\theta = \varrho_1 + i\varrho_2$, where ϱ_1, ϱ_2 are the standard coordinate vectors. Then $|\theta \cdot (a - b_j)| = |a - b_j|$.

For $n = 3$, set $u_j = (a - b_j)/|a - b_j|$ and let ς denote normalized surface measure on $\mathbb{S}^2$. The spherical cap area formula gives

$$\varsigma\{g \in \mathbb{S}^2 : |g \cdot u_j| > \alpha_N\} = 1 - \alpha_N = \frac{1}{2N}.$$

Since $J \leq 2N - 1$, the union of these exceptional sets has measure at most $J/(2N) < 1$. Thus there is $g \in \mathbb{S}^2$ with $|g \cdot u_j| \leq \alpha_N$ for every j . Take an orthonormal basis ϱ, f of $g^\perp$ and set $\theta = \varrho + if$. Then $\theta \cdot \theta = 0$, $|\theta| = \sqrt{2}$, and

$$|\theta \cdot u_j|^2 = |\varrho \cdot u_j|^2 + |f \cdot u_j|^2 = 1 - |g \cdot u_j|^2 \geq 1 - \alpha_N^2 = c_N^2.$$

For $J = 0$, choose any direction satisfying the first two conditions; empty products are interpreted as 1.

Step 2: constructing the annihilator. Define

$$P_{a,\mathbf{b}}(s) = \prod_{j=1}^J (s - \theta \cdot b_j), \quad s \in \mathbb{C}, \quad H_{a,\mathbf{b}}(\xi) = e^{-\frac{1}{2}A \cdot (\xi - a)} P_{a,\mathbf{b}}(\theta \cdot \xi).$$

The prescribed zeros follow from the factors in $P_{a,\mathbf{b}}$. Since $\theta \cdot \theta = 0$, we have $\Delta(P_{a,\mathbf{b}}(\theta \cdot \xi)) = P_{a,\mathbf{b}}''(\theta \cdot \xi) \theta \cdot \theta = 0$.

The gauge identity (3.1), with $p_\sharp = -\mu - \frac{|A|^2}{4}$, therefore gives (4.6).

Step 3: uniform bounds and normalization. Since $0 < c_N < 1$ and $J \leq 2N - 1$, the choice of θ gives

$$|H_{a,\mathbf{b}}(a)| = \prod_{j=1}^J |\theta \cdot (a - b_j)| \geq c_N^{2N-1} \prod_{j=1}^J |a - b_j|.$$

Fix a ball B containing $\overline{G}$, chosen in terms of G alone. Since $J \leq 2N - 1$, $|\theta| = \sqrt{2}$, and $a, b_j \in G$, differentiating the explicit formula gives

$$\|H_{a,\mathbf{b}}\|_{W^{2,\infty}(B)} \leq C,$$

where C depends only on G, A, N . Restriction to G yields the $H^2(G)$ bound in (4.8).

If $|a - b_j| \geq \rho > 0$ for every j , the product lower bound gives $|H_{a,\mathbf{b}}(a)| \geq c_{\rho,N} > 0$ uniformly for $0 \leq J \leq 2N - 1$. Consequently,

$$\|V_{a,\mathbf{b}}\|_{W^{2,\infty}(B)} \leq C_\rho.$$

Restriction to G gives the $H^2(G)$ bound, while the mean-value formula on the convex ball B gives the Lipschitz bound in (4.9). $\square$

4.3 Spectral synthesis for one-dimensional diffusion sources

In one dimension, since no nonzero complex-isotropic direction exists, the construction from Lemma 4.2 cannot be used. For diffusion sources, we therefore derive the probe estimates needed in Lemma 4.1 by combining finitely many spectral parameters.

Assume in this subsection that $G = (0, \ell)$ and $N = N_1$. Let z be the centered solution in (2.5), and set $\varepsilon = \|z\|_{\text{obs}}$. Set

$$p_j = j^2 + Aj - \mu, \quad \phi_j(\xi) = e^{j\xi}, \quad j = 0, \dots, 2N - 1. \quad (4.10)$$

Then each p_j is real and $(\mathcal{L}^* + p_j)\phi_j = 0$. Writing $q(\xi) = e^\xi$, we have $\phi_j = q^j$.

By Lemma 2.4,

$$\|\mathfrak{D}[\phi_j]\|_{L^2(0,T)} \leq C\varepsilon, \quad j = 0, \dots, 2N - 1, \quad (4.11)$$

where C may additionally depend on N .

Lemma 4.3 (One-dimensional annihilators by spectral synthesis). *Let $a \in G$ and $b_1, \dots, b_J \in G \setminus \{a\}$, where $J \leq 2N - 1$, and define*

$$H_{a,\mathbf{b}}(\xi) = \prod_{j=1}^J (q(\xi) - q(b_j)). \quad (4.12)$$

Then $H_{a,\mathbf{b}}$ is a linear combination of $\phi_0, \dots, \phi_J$ with uniformly bounded coefficients, and

$$\|\mathfrak{D}[H_{a,\mathbf{b}}]\|_{L^2(0,T)} \leq C\varepsilon, \quad (4.13)$$

$$|H_{a,\mathbf{b}}(a)| \geq c \prod_{j=1}^J |a - b_j|. \quad (4.14)$$

If $|a - b_j| \geq \rho > 0$ for all j , the normalized function $V_{a,\mathbf{b}} = H_{a,\mathbf{b}}/H_{a,\mathbf{b}}(a)$ is a linear combination of $\phi_0, \dots, \phi_J$ with uniformly bounded coefficients and satisfies

$$\|\mathfrak{D}[V_{a,\mathbf{b}}]\|_{L^2(0,T)} \leq C_\rho \varepsilon, \quad \|V_{a,\mathbf{b}}\|_{\text{Lip}(G)} \leq C_\rho. \quad (4.15)$$

The constants may additionally depend on N ; the constants in the normalized estimates also depend on ρ .

Proof. We combine the $L^2(0, T)$ estimates (4.11) after testing at each individual p_j ; the synthesized function need not solve one common adjoint equation. Expanding the polynomial in (4.12) gives

$$H_{a,\mathbf{b}}(\xi) = \sum_{j=0}^J c_j \phi_j(\xi).$$

The roots $q(b_j)$ remain in a fixed compact interval, so the elementary symmetric-polynomial formula gives $\sum_{j=0}^J |c_j| \leq C$. Linearity and (4.11) prove (4.13).

The map q is bi-Lipschitz on $[0, \ell]$: there exist $0 < c_q \leq C_q$ such that

$$c_q |x - y| \leq |q(x) - q(y)| \leq C_q |x - y|, \quad x, y \in [0, \ell].$$

Consequently,

$$|H_{a,\mathbf{b}}(a)| = \prod_{j=1}^J |q(a) - q(b_j)| \geq c \prod_{j=1}^J |a - b_j|,$$

which proves (4.14). If all $|a - b_j| \geq \rho$, the denominator $H_{a,\mathbf{b}}(a)$ is bounded away from zero. The normalized polynomial coefficients and the first derivative of the explicit formula are then uniformly bounded, proving (4.15). $\square$

Remark 4.2. This synthesis uses the fact that $\mathfrak{D}[\phi_j]$ is estimated in the same space $L^2(0, T)$ for every p_j , with coefficients γ_l^j independent of p_j . For drift sources, changing the spectral parameter changes the scalar moments $\widetilde{\lambda}_k^j(p_j)$, so these estimates do not combine into the same annihilation identity.

4.4 Proof of the finite-source theorem

Proof of Theorem 1.2. Let $\bar{v}$ and z solve (2.4) and (2.5), respectively.

Drift sources ($n = 2, 3$). Set $N = N_0$, $\varepsilon = \|\bar{v}\|_{\text{obs}}$, $E = \mathbb{C}$, $a_k^j = x_k^j$, and $c_k^j = \widetilde{\lambda}_k^j(p_\sharp) = \int_0^T e^{-p_\sharp t} \lambda_k^j(t) dt$. Using the node lists in subsection 4.1, Lemmas 4.2 and 2.3 at $p = p_\sharp$ give the product-probe bound in (4.1).

Assumption 1.3 supplies separation and the coefficient lower bound. Thus Lemma 4.1 gives the drift location estimate, without an upper bound on the strengths. Under (1.15),

$$\sum_{k=1}^{N_0} (|c_k^1| + |c_k^2|) \leq \|e^{-p_\sharp \cdot}\|_{L^2(0,T)} M.$$

The same probe estimates with $\rho = 3\delta/4$ give (4.2), so the coefficient conclusion of Lemma 4.1 yields the weighted-moment estimate for the same permutation.

Diffusion sources. Set $N = N_1$, $\varepsilon = \|z\|_{\text{obs}}$, $E = L^2(0, T; \mathbb{C})$, $a_l^j = y_l^j$, and $c_l^j = \gamma_l^j$, viewing the real strengths in their complexification. For $n = 2, 3$, Lemmas 4.2 and 2.4 at $p = p_\sharp$ give the product and normalized probe bounds. For $n = 1$, Lemma 4.3 gives the same bounds. In both cases, use the prescribed node lists and $\rho = 3\delta/4$ for the normalized probes. Assumption 1.4 supplies separation and the coefficient bounds, so Lemma 4.1 gives the diffusion location and strength estimates with the same permutation. Finally, (2.6) bounds both channel errors by $\|u^1 - u^2\|_{\text{obs}}$ and proves all three conclusions. $\square$

Remark 4.3. In the planar deterministic setting of [11], with $\mu = -|A|^2/4$, the moments at $p_\sharp = 0$ are unweighted. The nearest-point separation bound (4.5) in the proof of Lemma 4.1 strengthens the intermediate product estimate in their Section 4.2, (4.11), directly to Lipschitz localization and constructs the matching permutation. This improvement uses the separation assumption already present in that argument.

5 Numerical experiments

We present three experiments illustrating the stability recovery mechanisms: simultaneous reconstruction of one source per channel, temporal conditioning at known locations, and recovery of two diffusion sources as their separation decreases.

The procedure combines channel separation, dictionary-based localization, and regularized least-squares recovery of temporal strengths. It is independent of the adjoint probe constructions used in the proofs. The experiments focus on recoverability and conditioning, not on estimating stability exponents.

5.1 Discretization and reconstruction

Experiments use the unit disk $G \subset \mathbb{R}^2$, $u_0 = 0$, with $T = 1$, $T_0 = 0.55$, $T_1 = 0.75$, $A = (0.35, -0.20)$, $\mu = 0.5$, and observation arc $\Gamma = \{(\cos \theta, \sin \theta) : -\pi/3 < \theta < \pi/2\}$.

Let $\mathcal{T}_h$ be a triangular mesh with spatial mesh size $h = \max_{E \in \mathcal{T}_h} \text{diam}(E)$. We use continuous P_1 finite elements on $\mathcal{T}_h$ with nodal basis $\{\varphi_i\}$ and implicit Euler time stepping. Let M_h and K_h denote the mass matrix and the finite element matrix associated with $\mathcal{L} = -\Delta + A \cdot \nabla + \mu$, respectively, and set $b_h(\xi)_i = \varphi_i(\xi)$. With U^n at $t_n = n\Delta t$, $\Delta t = T/N_t$, and $\Delta W_n = W(t_{n+1}) - W(t_n)$, the scheme for one source per channel is

$$(M_h + \Delta t K_h) U^{n+1} = M_h U^n + \Delta t \lambda(t_{n+1}) b_h(x) + \gamma(t_n) b_h(y) \Delta W_n.$$

Location data use a 29,958-triangle mesh with 480 time steps; inversion uses 8,068 triangles and 160 observation steps. Sources are off nodes.

The discrete norm approximates $\|\cdot\|_{\text{obs}}$. For boundary residuals r_n , let M_∂, M_Γ be the mass matrices on $\partial G, \Gamma$, respectively, and S_Γ the tangential stiffness matrix on Γ ; with $D_t r_n = (r_n - r_{n-1})/\Delta t$ for $n \geq 1$, we set

$$\|r\|_h^2 := \Delta t \sum_{n=0}^{N_t} r_n^\top M_\partial r_n + \Delta t \sum_{t_n \in [T_1, T]} [r_n^\top M_\Gamma r_n + (D_t r_n)^\top M_\Gamma D_t r_n + r_n^\top S_\Gamma r_n]. \quad (5.1)$$

For N_{MC} paths, the empirical norm is

$$\|r\|_{h, N_{\text{MC}}}^2 = \frac{1}{N_{\text{MC}}} \sum_{m=1}^{N_{\text{MC}}} \|r^{(m)}\|_h^2. \quad (5.2)$$

The same weights are used for calibration, reconstruction, and conditioning. Noise is smoothed and scaled to relative level η .

Each realization is paired with its Brownian path. Expectation/centering separates channels: reconstruct diffusion from centered observations, subtract its mean response, then recover drift from the corrected mean.

For fixed locations, response is linear in strengths. Each strength uses 18 cubic B-splines on $[0, T_0]$, extended by zero, with Tikhonov regularization. For candidates θ , coefficients a , response matrix $B(\theta)$, data $\mathbf{d}$, and weight matrix W_{obs} , set $\|r\|_{W_{\text{obs}}}^2 = r^* W_{\text{obs}} r$. The reconstruction is

$$a_\alpha(\theta) = \operatorname{argmin}_a \{ \|B(\theta)a - \mathbf{d}\|_{W_{\text{obs}}}^2 + \alpha \|Pa\|_2^2 + \beta_{\text{stab}}(\theta) \|a\|_2^2 \}, \quad (5.3)$$

with P the second-difference matrix and β_{stab} a diagonal stabilizer. Diffusion data and responses are centered. Regularization is chosen by residual matching.

Localization projects observations onto precomputed candidate responses: diffusion uses centered data correlated with Brownian increments and low-boundary Fourier modes; drift uses the corrected mean projected onto spline responses. Candidates with smallest residuals are refined locally; final drift location minimizes the full residual.

Single-source diffusion uses all 480 increments; localization and two-source recovery use pathwise-aggregated increments on coarser grids.

Each configuration uses 30 independent runs, separate from tuning. Location errors are normalized by $\text{diam}(G) = 2$, and profile errors are reported as relative $L^2(0, T_0)$ errors. Median errors are shown, with variability indicated in the figures and tables.

5.2 Simultaneous single-source recovery

In the first experiment, $N_0 = N_1 = 1$ (indices suppressed), with drift and diffusion sources at $x^* = (-0.38, 0.18)$ and $y^* = (0.32, -0.24)$. Set $\rho(t) = \sin^2(\pi t/T_0)$ on $[0, T_0]$ and zero on $(T_0, T]$. Before $L^2(0, T_0)$ normalization,

$$\lambda_{\text{raw}}(t) = \rho(t) [1 + 0.25 \cos(4\pi t/T_0) - 0.15 \sin(6\pi t/T_0)], \quad \gamma_{\text{raw}}(t) = \rho(t) [0.80 \sin(4\pi t/T_0) - 0.45 \sin(10\pi t/T_0)].$$

Both normalized profiles have unit $L^2(0, T_0)$ norm and are outside the spline space. With $N_{\text{MC}} = 512$ paths and $\eta \in \{0.25\%, 0.5\%, 1\%, 2\%\}$, Figure 1 shows recovered locations and profiles and their error dependence on noise; the representative run is the one closest to the sample median at 1% noise.

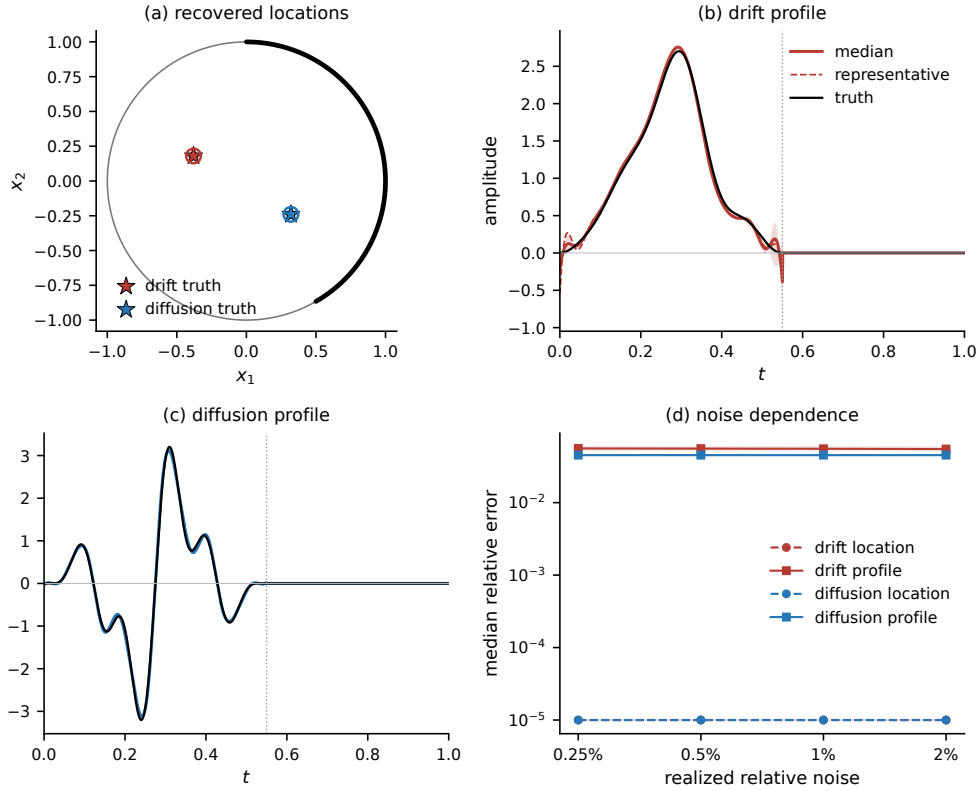


Figure 1: Simultaneous single-source recovery. Panel (a) shows the true and reconstructed locations at 1% noise. Panels (b)–(c) compare the true profiles with the pointwise medians, empirical 10%–90% bands, and a representative run (dashed). Panel (d) shows median errors against noise level, with 95% bootstrap intervals and a plotting floor of 10^{-5} .

At 1% noise, both normalized location errors are below 0.05 across all 30 runs. Median relative profile errors are 0.055 for the drift and 0.045 for the diffusion (see Table 1). The vanishing median location errors indicate that the true points are included in the refined search dictionary. The reconstructed profiles capture the drift variation and the sign changes of the diffusion strength, confirming that the two channels can be separated and recovered from the same noisy observations.

Channel	Location error		Profile error	
	Median	90th percentile	Median	90th percentile
drift	$< 10^{-12}$	2.5×10^{-3}	0.055	0.076
diffusion	$< 10^{-12}$	$< 10^{-12}$	0.045	0.046

Table 1: Single-source errors over 30 runs at $N_{MC} = 512$ and 1% relative noise. Location errors are normalized by the domain diameter, and profile errors are relative $L^2(0, T_0)$ errors; values below 10^{-12} are reported at that threshold.

The reconstruction is qualitatively unchanged on the three inversion meshes in Table 2. These have 3,017, 8,068, and 10,023 triangles, with 120, 160, and 240 observation time steps, respectively. Over ten runs at 0.25% noise, the median location errors remain below 10^{-12} . The drift-profile error decreases under refinement, while the diffusion-profile error remains near 0.045.

Mesh	Drift profile	Diffusion profile	Forward discrepancy
coarse	0.09198	0.04530	1.4063×10^{-3}
standard	0.05706	0.04531	6.6425×10^{-4}
fine	0.05302	0.04538	7.2141×10^{-4}

Table 2: Median relative profile errors over ten runs at 0.25% noise. The forward discrepancy compares data and inversion discretizations at the true sources using the noiseless 512-path ensemble of the first seed.

5.3 Temporal information in the two channels

Now fix locations at the true points and compare temporal oscillation transmission in the two forward maps, both using the standard mesh and 160 time steps. For $k = 1, \dots, 24$, define $q_k(t) = c_k \rho(t) \sin(k\pi t/T_0)$ with $\|q_k\|_{L^2(0, T_0)} = 1$. Generalized singular values measure conditioning relative to the input norm. With B_0 the drift response, $B_1^{(m)}$ the diffusion response for path m , and $(G_t)_{ij} = (q_i, q_j)_{L^2(0, T_0)}$ (trapezoidal quadrature), the response Gram matrices are

$$H_0 = B_0^* W_{\text{obs}} B_0, \quad H_1 = \frac{1}{2048} \sum_{m=1}^{2048} (B_1^{(m)})^* W_{\text{obs}} B_1^{(m)}.$$

Generalized singular values σ satisfy $H_j v = \sigma^2 G_t v$; their largest-to-smallest ratio gives the condition number on the span of the first K modes. Individual mode gains $\sqrt{(H_j)_{kk}}$ are normalized by the gain of q_1 .

To connect to reconstruction, use the same oscillatory profile in both channels:

$$q_H(t) = c_H \rho(t) [\sin(4\pi t/T_0) + 0.6 \sin(20\pi t/T_0)],$$

normalized to unit $L^2(0, T_0)$ norm. Reconstruction uses the span of $q_1, \dots, q_{24}$ (containing q_H) with an identity penalty in (5.3). Drift data are a single deterministic trace; diffusion data use 128 fixed paths. For each channel, we take 100 independent noise realizations at 0.1%, 0.3%, 1%, 3%, calibrated in that channel's norm.

Figure 2 shows a pronounced loss of sensitivity to temporal oscillations in the drift channel. At $k = 20$, its normalized gain is 0.00248, compared with 0.975 for the diffusion channel. At $K = 24$, the drift condition number is 833 times the diffusion condition number. This difference is also visible in reconstruction: at 1% noise, the median profile errors are 0.0181 and 1.18×10^{-4} , respectively. Temporal oscillations can cancel in the deterministic time integral, while Itô's isometry retains the energy of the diffusion integrand. The three comparisons illustrate this distinction underlying the strength estimates in Theorem 1.1.

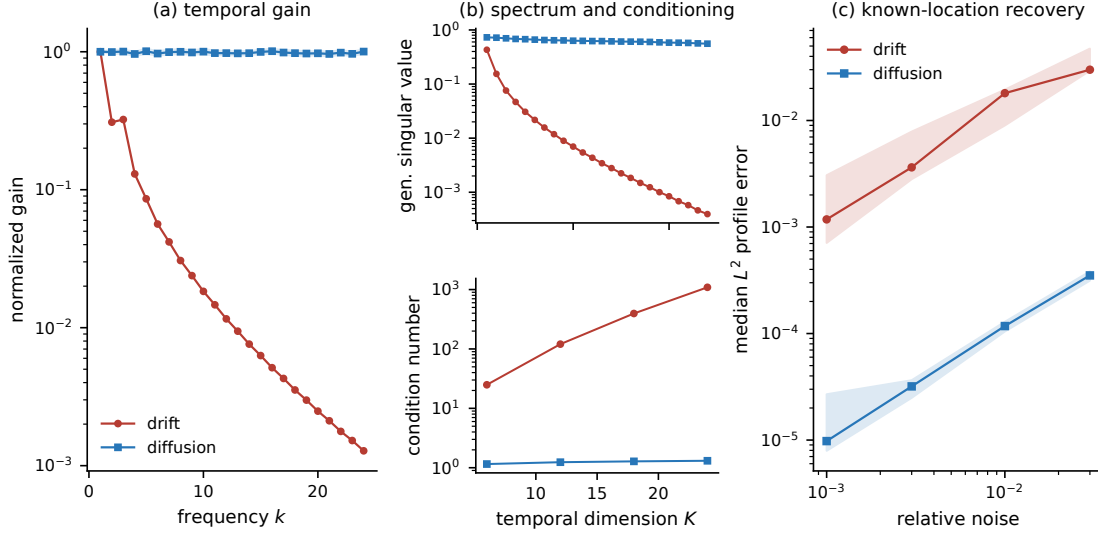


Figure 2: Temporal information in the two channels: (a) gains normalized by first-mode gain, with 95% bootstrap intervals for diffusion; (b) $K = 24$ singular values (nonincreasing) and condition numbers; (c) median relative L^2 errors and interquartile bands over 100 noise realizations for the common high-frequency profile at known locations.

5.4 Two-source recovery and separation

The final experiment examines separation effects on localization and profile reconstruction. We take $N_0 = 0$, $N_1 = 2$, $y_c = (0.08, -0.08)$, $\varrho = (\cos 35^\circ, \sin 35^\circ)$, $y_1^* = y_c - d\varrho/2$, $y_2^* = y_c + d\varrho/2$, and $d \in \{0.50, 0.35, 0.25, 0.18, 0.12\}$.

Define the unnormalized profiles by

$$\gamma_{1,\text{raw}}(t) = \rho(t)[0.85 \sin(4\pi t/T_0) + 0.30 \cos(8\pi t/T_0)], \quad \gamma_{2,\text{raw}}(t) = \rho(t)[0.75 \cos(6\pi t/T_0) - 0.40 \sin(12\pi t/T_0)].$$

We normalize these profiles to have $L^2(0, T_0)$ norms 1 and 0.8, respectively. Each separation uses $N_{\text{MC}} = 512$, 1% noise, and 30 runs.

Joint localization fixes the count at two, searches dictionary pairs with separation ≥ 0.08 , refines locally, and reconstructs profiles. Dimensionless separation d/h_{inv} , with $h_{\text{inv}} = \sqrt{2\pi/8000} \simeq 0.028025$, corresponds to the inversion mesh target. Evaluation matches locations via the Hungarian algorithm, using the same permutation σ for signed profile errors; success is $\max_{l=1,2} |\hat{y}_{\sigma(l)} - y_l^*| < d/4$.

The condition number uses generalized singular values of the two-source response and block temporal Gram matrices at true locations. Figure 3 shows matched reconstructions and their separation dependence.

Table 3 shows successful matching in all 90 runs with $d \geq 0.25$. As the separation decreases, the median location and profile errors increase, together with the condition number. The success rate decreases to 90% at $d = 0.18$ and to 40% at $d = 0.12$, where the median relative profile error is 0.344. Nearby sources produce less distinguishable boundary responses; the resulting deterioration illustrates the role of the separation assumption in Theorem 1.2.

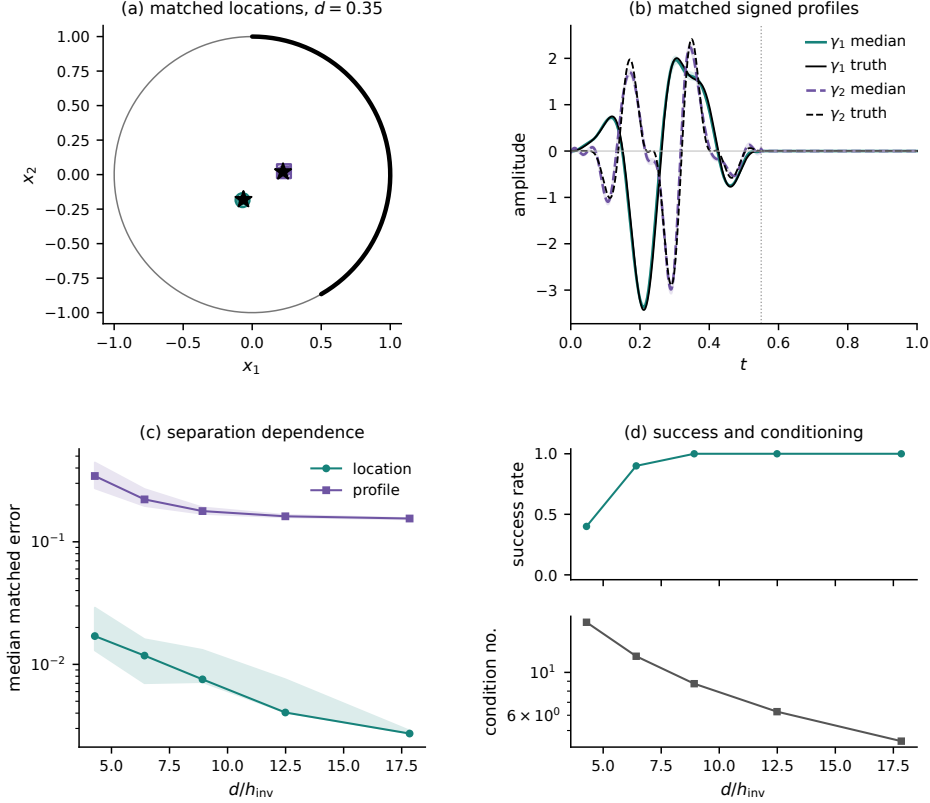


Figure 3: Two-source diffusion recovery: (a)–(b) matched locations and signed profiles at $d = 0.35$ with 10%–90% bands; (c) median diameter-normalized location and relative profile errors vs. d/h_{inv} , with interquartile bands; (d) matching success rate (threshold $d/4$) and condition number.

d/h_{inv}	Success	Location error		Profile error		Condition number
		Median	90th percentile	Median	90th percentile	
17.8	100%	0.003	0.006	0.155	0.157	4.4
12.5	100%	0.004	0.010	0.161	0.169	6.3
8.9	100%	0.008	0.017	0.178	0.221	8.7
6.4	90%	0.012	0.022	0.222	0.308	12.1
4.3	40%	0.017	0.040	0.344	0.475	18.2

Table 3: Two-source results: 30 runs per separation, $N_{\text{MC}} = 512$, 1% noise. Errors are maxima over matched sources; locations normalized by domain diameter, profiles relative $L^2(0, T_0)$. Success threshold $d/4$; condition number at true locations reported as median.

A The direct problem and boundary regularity

We prove [Proposition 2.1](#) using standard semigroup estimates, with particular attention to the full-time boundary trace: the spatial separation of the sources and the boundary makes the localized stochastic convolution square integrable. Smoothing after the sources become inactive gives the late-time regularity. We then justify the point-source approximation used in [Lemma 2.4](#). Constants may depend on G, A, μ, T, N_0, N_1 ; other dependencies are stated explicitly.

Proof of Proposition 2.1. The adjoint of the Neumann realization (2.1) is

$$\mathcal{A}^*v = \Delta v + A \cdot \nabla v - \mu v, \quad D(\mathcal{A}^*) = \{v \in H^2(G) : \partial_\nu v + (A \cdot \nu)v = 0\}.$$

Elliptic regularity [9, Theorem 2.3.3.2], after absorbing lower-order terms, gives equivalence between the graph norm of $D(\mathcal{A}^*)$ and the $H^2(G)$ norm. Since $n \leq 3$, Sobolev embedding $H^2(G) \hookrightarrow C(\overline{G})$ yields

$$\sup_{y \in G} \|\delta_y\|_{\mathcal{X}_{-2}} \leq C. \quad (\text{A.1})$$

Standard semigroup and extrapolation theory [21, Chapter 3], [7, Section II.5] provide the analytic semigroups $S(t)$ and $S_{-2}(t)$. Multiplication by $e^{A \cdot x/2}$ makes $\mathcal{A}$ similar to a self-adjoint Robin realization, so $S_{-2}(t)$ satisfies $\|S_{-2}(t)\| \leq e^{\rho t}$ for some $\rho \geq 0$ in an equivalent Hilbert norm. Applying [18, Theorems 3.18 and 3.20] on $\mathcal{X}_{-2}$ with $p = 2$ and (A.1) yields the unique mild solution and (2.2), where

$$u(t) = S(t)u_0 + \sum_{k=1}^{N_0} \int_0^t S_{-2}(t-s) \lambda_k(s) \delta_{x_k} ds + \sum_{l=1}^{N_1} \int_0^t S_{-2}(t-s) \gamma_l(s) \delta_{y_l} dW(s) \quad \text{in } \mathcal{X}_{-2}. \quad (\text{A.2})$$

Analytic smoothing on the extrapolation scale [21, Theorem 2.6.13], after a spectral shift, gives the estimates used below:

$$\begin{aligned} \|S(t)\|_{\mathcal{L}(L^2(G), H^2(G))} &\leq Ct^{-1}, \quad 0 < t \leq 2T, \\ \sup_{\tau \leq t \leq T} \sum_{j=0}^1 \|\partial_t^j S_{-2}(t)\|_{\mathcal{L}(\mathcal{X}_{-2}, H^2(G))} &\leq C_\tau, \quad 0 < \tau \leq T. \end{aligned} \quad (\text{A.3})$$

Write $K_y(t) = k(t, \cdot, y) = S_{-2}(t) \delta_y$ for the continuous interior representative of the heat kernel. The Gaussian bound in [3, Theorems 4.4 and 6.1] and spatial integration give, for $0 < t \leq 2T$,

$$|k(t, \xi, y)| \leq Ct^{-n/2} \exp\left(-\frac{|\xi - y|^2}{ct}\right), \quad (\text{A.4})$$

$$\sup_{y \in G} \|K_y(t)\|_{L^2(G)} \leq Ct^{-n/4}. \quad (\text{A.5})$$

Let u_d denote the deterministic source term in (A.2). Since $n \leq 3$, (A.5) and Young's inequality give $u_d \in L^2(0, T; L^2(G))$. By semigroup duality, it satisfies the Neumann transposition identity used in [11, Section 3]; uniqueness then identifies the two solutions. The same identification applies to (2.4).

Assume Condition 1.1 holds. The deterministic off-source regularity follows from the proof of [11, Theorem 3.1]. It remains to establish the full-time boundary bound for the stochastic term. Enclose the sources in a compact set $\mathcal{C}_{\text{src}} \Subset G$. Choose $\chi \in C^\infty(\overline{G})$ with $0 \leq \chi \leq 1$, equal to one near ∂G and zero near $\mathcal{C}_{\text{src}}$, and set $\mathbf{d}_\chi = \text{dist}(\text{supp } \chi, \mathcal{C}_{\text{src}}) > 0$. From (A.4) and (A.5), together with the semigroup property and analytic smoothing, we have for $y \in \mathcal{C}_{\text{src}}$ and $0 < t \leq T$,

$$\|\chi K_y(t)\|_{L^2(G)} \leq Ct^{-n/4} e^{-c\mathbf{d}_\chi^2/t}, \quad \|\chi K_y(t)\|_{H^2(G)} \leq C_\chi t^{-1-n/4}.$$

Standard interpolation then yields

$$\sup_{y \in \mathcal{C}_{\text{src}}} \|\chi K_y(t)\|_{H^1(G)} \leq C_\chi t^{-(n+2)/4} e^{-c\mathbf{d}_\chi^2/t}, \quad 0 < t \leq T. \quad (\text{A.6})$$

The right-hand side of (A.6) lies in $L^2(0, T)$. For each diffusion source, Itô's isometry and Tonelli's theorem give

$$\mathbb{E} \int_0^T \left\| \int_0^t \chi K_y(t-s) \gamma(s) dW(s) \right\|_{H^1(G)}^2 dt \leq \|\gamma\|_{L^2(0, T)}^2 \int_0^T \|\chi K_y(r)\|_{H^1(G)}^2 dr.$$

Combining this with Young's inequality for u_d , and writing $v = u - S(\cdot)u_0$, yields

$$\mathbb{E} \|\chi v\|_{L^2(0, T; H^1(G))}^2 \leq C_\chi \left(\sum_{k=1}^{N_0} \|\lambda_k\|_{L^2(0, T)}^2 + \sum_{l=1}^{N_1} \|\gamma_l\|_{L^2(0, T)}^2 \right). \quad (\text{A.7})$$

Since $\chi = 1$ near ∂G , taking traces gives the full-time boundary bound, with a constant depending on the source-to-boundary distances. For the initial component, analytic smoothing, interpolation, and the multiplicative trace inequality give the integrable bound

$$\|S(t)u_0\|_{L^2(\partial G)}^2 \leq C \|S(t)u_0\|_{L^2(G)} \|S(t)u_0\|_{H^1(G)} \leq Ct^{-1/2} \|u_0\|_{L^2(G)}^2.$$

Finally, since the sources vanish after T_0 , we have $u(t) = S_{-2}(t - T_0)u(T_0)$ for $t \geq T_0$. Thus (A.3) with $\tau = T_1 - T_0$, together with (2.2), gives $u \in L^2(\Omega; H^1(T_1, T; H^2(G)))$ and the terminal bound, uniformly in the source locations. Taking traces and combining the preceding bounds proves (2.3). $\square$

Lemma A.1 (Point-source approximation). *Under Condition 1.1, let z and z_ϵ solve (2.5) and (2.17), respectively. Then*

$$\lim_{\epsilon \rightarrow 0} (\|z_\epsilon - z\|_{\text{obs}} + \|z_\epsilon(T) - z(T)\|_{L^2_{\mathcal{F}_T}(\Omega; H^2(G))}) = 0. \quad (\text{A.8})$$

Moreover, for every $v \in H^2(G; \mathbb{C})$ and fixed $p \in \mathbb{R}$,

$$\lim_{\epsilon \rightarrow 0} \int_0^T e^{-pt} \mathfrak{D}_\epsilon[v](t) dW(t) = \int_0^T e^{-pt} \mathfrak{D}[v](t) dW(t) \quad \text{in } L^2(\Omega; \mathbb{C}).$$

Proof. For the mollifiers in (2.16), Sobolev embedding $H^2(G) \hookrightarrow C^{0,\alpha}(\overline{G})$ for some $\alpha > 0$ yields

$$\|\rho_\epsilon^y - \delta_y\|_{\mathcal{X}_{-2}} \leq C\epsilon^\alpha \rightarrow 0.$$

Since the sources vanish after T_0 , the smoothing estimate (A.3) with $\tau = T_1 - T_0$, together with Itô's isometry, gives $z_\epsilon \rightarrow z$ in $L^2(\Omega; H^1(T_1, T; H^2(G)))$ and $z_\epsilon(T) \rightarrow z(T)$ in $L^2(\Omega; H^2(G))$.

For the full-time boundary trace, fix a compact set $\mathcal{C}_{\text{src}} \Subset G$ containing all mollifier supports for sufficiently small ϵ , and choose a cutoff χ as above. The kernel representation gives

$$S(t)\rho_\epsilon^y = \int_{\mathcal{C}_{\text{src}}} K_\eta(t)\rho_\epsilon^y(\eta) d\eta.$$

Since the mollifiers are nonnegative with unit mass, Minkowski's inequality and (A.6) bound $\|\chi S(t)\rho_\epsilon^y\|_{H^1(G)}$ uniformly in ϵ by the same $L^2(0, T)$ majorant. At each fixed $t > 0$, (A.3) gives $S(t)\rho_\epsilon^y \rightarrow K_y(t)$ in $H^2(G)$. Dominated convergence therefore yields

$$\lim_{\epsilon \rightarrow 0} \int_0^T \|\chi(S(t)\rho_\epsilon^y - K_y(t))\|_{H^1(G)}^2 dt = 0.$$

Applying Itô's isometry gives $\chi(z_\epsilon - z) \rightarrow 0$ in $L^2(\Omega \times (0, T); H^1(G))$. The trace theorem, together with the late-time and terminal convergence, proves (A.8).

Finally, the same Sobolev embedding gives $(\rho_\epsilon^{y_i^j}, v)_{L^2(G)} \rightarrow v(y_i^j)$, and hence $\mathfrak{D}_\epsilon[v] \rightarrow \mathfrak{D}[v]$ in $L^2(0, T)$. Itô's isometry, combined with the bounded weight e^{-pt} , gives the remaining convergence. $\square$

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