

EXACT CONTROLLABILITY FOR STOCHASTIC FIRST-ORDER MULTI-DIMENSIONAL HYPERBOLIC SYSTEMS*

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Abstract. This paper investigates the exact controllability problem for multi-dimensional stochastic first-order symmetric hyperbolic systems on smooth domains with a two-sided nested spectral boundary condition and incoming controls measured in the flux norm. The control inputs act in two distinct ways: an internal control applied to the diffusion term and a boundary control applied to the drift term. By means of a classical duality argument, the controllability problem is reduced to an observability estimate for the corresponding backward stochastic system. The main technical contribution is the establishment of a new global Carleman estimate for such backward systems, combined with a weighted energy identity. This enables us to prove the desired observability inequality under a geometric structural condition, which encodes uniform escape of the propagation modes toward the boundary. As a result, we obtain exact controllability provided the control time T exceeds a sufficient threshold T_0 given explicitly in terms of the system geometry. Furthermore, we complement the positive result with several negative controllability theorems, which establish restrictions on the control configuration under the stated additional hypotheses. Our analysis not only extends controllability theory from deterministic to stochastic multi-dimensional hyperbolic systems but also provides, as a byproduct, new results for deterministic systems under a structural hypothesis. Applications to stochastic traffic flow, epidemiological models, and shallow-water equations are discussed.

Key words. Stochastic symmetric hyperbolic systems, exact controllability, observability, Carleman estimate

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1. Introduction and main results. First-order hyperbolic systems govern the evolution of numerous physical quantities. Representative examples include the Saint–Venant equations for open channels, the Aw–Rascle equations for road traffic, gas dynamics, supply chains, and heat exchangers [3, 4, 19]. Due to their wide range of applications, the control theory of such systems has attracted considerable attention from both mathematical and engineering communities. Extensive research has been devoted to the boundary controllability of deterministic first-order hyperbolic systems (see, e.g., [9, 18, 20, 30] and the references therein). It is noteworthy, however, that existing controllability results are largely confined to one-dimensional (1-D) systems or multi-dimensional scalar equations.

In many realistic scenarios, uncertainties—due to environmental noise, parameter fluctuations, or unmodeled dynamics—cannot be ignored. Such randomness is naturally incorporated through stochastic forcing, leading to systems of stochastic hyperbolic equations (e.g., [6, 8, 10, 29]). The presence of noise not only affects the well-posedness and stability but also fundamentally alters the controllability properties, as the control must counteract both deterministic and stochastic dynamics. A key question arises: Can we steer a stochastic multi-dimensional hyperbolic system from any initial state to any desired target state in finite time, and if so, under what conditions?

In this paper, we address this question for a class of stochastic first-order sym-

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metric hyperbolic systems in a bounded spatial domain $G \subset \mathbb{R}^n$. We extend the study of controllability from the deterministic 1-D and scalar multi-dimensional settings to the stochastic multi-dimensional system framework. Notably, a by-product of our analysis is a new controllability result for deterministic first-order multi-dimensional symmetric hyperbolic systems under a suitable structural assumption.

Before describing the control problem studied in this paper, let us first introduce some notations.

Let $T > 0$ be a fixed time horizon, and let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional Brownian motion $W(\cdot)$ is defined. Here, $\mathbf{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ denotes the natural filtration generated by $W(\cdot)$ and augmented by all $\mathbb{P}$ -null sets. We write $\mathbb{F}$ for the progressive σ -algebra associated with $\mathbf{F}$.

Let $\mathcal{H}$ be a Banach space. For any $t \in [0, T]$, denote by $L^2_{\mathcal{F}_t}(\Omega; \mathcal{H})$ the Banach space of all $\mathcal{H}$ -valued $\mathcal{F}_t$ -measurable random variables f such that $\mathbb{E}\|f\|_{\mathcal{H}}^2 < \infty$. Denote by $L^2_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space of all $\mathcal{H}$ -valued, $\mathbb{F}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}\|X(\cdot)\|_{L^2(0, T; \mathcal{H})}^2 < \infty$, by $L^\infty_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space of all essentially bounded $\mathcal{H}$ -valued, $\mathbb{F}$ -adapted processes, by $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathcal{H}))$ the space of all $\mathcal{H}$ -valued, $\mathbb{F}$ -adapted continuous processes $X(\cdot)$ satisfying $\mathbb{E}\|X(\cdot)\|_{C([0, T]; \mathcal{H})}^2 < \infty$. Denote by $C_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ the space of $\mathcal{H}$ -valued, $\mathbb{F}$ -adapted processes whose paths are continuous as maps into $L^2(\Omega; \mathcal{H})$. Analogously, one defines the spaces $C^1_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ and $C_{\mathbb{F}}([0, T]; L^\infty(\Omega; \mathcal{H}))$. All these spaces are equipped with their standard norms.

For brevity set $H = L^2(G; \mathbb{R}^N)$ once the domain G is fixed, and write $S^2(H) = L^2_{\mathbb{F}}(\Omega; C([0, T]; H))$ and $\mathbf{L}^2(H) = L^2_{\mathbb{F}}(0, T; H)$; the same notation is used on shorter intervals.

Throughout the paper, for any vector $\mathbf{v}$ we set $|\mathbf{v}| \triangleq \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product.

Let $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded domain with a C^∞ -smooth boundary Γ . Given $\tau \in (0, T]$, we introduce the following space-time domains:

$$Q_\tau \triangleq (0, \tau) \times G, \quad \Sigma_\tau \triangleq (0, \tau) \times \Gamma.$$

For $i = 1, 2, \dots, n$, let $A_i(x)$ be $N \times N$ symmetric matrices whose entries belong to $C^\infty(\bar{G})$. Let $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ denote the unit outward normal vector of G at a boundary point $x \in \Gamma$. By symmetry, one can choose a measurable orthogonal matrix $\Pi = \Pi(x)$ such that

$$(1.1) \quad \Lambda(x) = \Pi^{-1}(x) \left(\sum_{i=1}^n \nu_i(x) A_i(x) \right) \Pi(x) = \begin{pmatrix} \Lambda_+(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_-(x) \end{pmatrix},$$

where $\Lambda_+(x) \in \mathbb{R}^{n_+ \times n_+}$ and $\Lambda_-(x) \in \mathbb{R}^{n_- \times n_-}$ are diagonal matrices containing the positive and negative eigenvalues of $\sum_{i=1}^n \nu_i(x) A_i(x)$, respectively. The numbers $n_+ = n_+(x)$ and $n_- = n_-(x)$ may depend on the boundary point $x \in \Gamma$.

The diagonalization is used only as a measurable choice of coordinates; no smooth global eigenvector frame is assumed. Write

$$\mathbf{N} = \sum_i \nu_i A_i, \quad \mathbf{P}_\pm = \mathbf{1}_{\mathbb{R}_\pm}(\mathbf{N}), \quad \mathbf{N}_\pm = \frac{1}{2}(|\mathbf{N}| \pm \mathbf{N}).$$

Here $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$. The incoming control and adjoint trace space

is the flux Hilbert space

$$(1.2) \quad U_- = \left\{ u(x) \in \mathbb{R}^{n-(x)} \text{ measurable} : \|u\|_{U_-}^2 = \int_{\Gamma} \langle -\Lambda_- u, u \rangle d\Gamma < \infty \right\}.$$

Functions are identified a.e. In dimension one, boundary measure is counting measure. Equivalently, set

$$(1.3) \quad \mathcal{U} = \{h \in L^2(\Gamma; \mathbb{R}^N) : P_- h = h\}, \quad h = \Pi \begin{pmatrix} 0 \\ 0 \\ (-\Lambda_-)^{1/2} u \end{pmatrix}.$$

This is an isometric bijection from U_- onto the closed subspace $\mathcal{U}$. The inverse is pointwise on the negative spectral subspace; a uniform lower bound on the nonzero boundary eigenvalues is not required.

ASSUMPTION 1.1 (Boundary geometry). *For $n \geq 2$, there is a finite disjoint union S of smooth closed embedded $(n-2)$ -dimensional submanifolds of Γ , possibly empty, such that the rank of $\mathbf{N}$ is constant on each component of $\Gamma \setminus S$. Near each point of S , each of the spaces $M = E_{\geq 0}(\mathbf{N}) = \ker P_-$ and $M = E_{\leq 0}(\mathbf{N}) = \ker P_+$ has smooth fixed-dimensional extensions $M_{\text{small}} \subseteq M_{\text{big}}$ on the entire neighborhood: on either side of S , M equals one of these two bundles, and $\ker \mathbf{N} \subseteq M_{\text{big}}$ throughout the neighborhood. For $n = 1$, take $S = \emptyset$ and assume that $A_1(x)$ is invertible on $\overline{G}$; this is automatic under Condition 1.1 below.*

This geometric hypothesis is used for the operator realization in Appendix A; it is imposed throughout. It does not require simultaneous diagonalization of the principal matrices. Away from S , constant rank gives smooth spectral subbundles, which need not admit global eigenvector frames. The regularity imposed here is sufficient; its optimality is not claimed.

Using $\Pi(x)$ from (1.1), we set

$$(1.4) \quad \zeta(t, x) = \begin{pmatrix} \zeta_+(t, x) \\ \zeta_0(t, x) \\ \zeta_-(t, x) \end{pmatrix} \triangleq \Pi^{-1}(x) y(t, x).$$

Consider the following controlled system:

$$(1.5) \quad \begin{cases} dy + \sum_{i=1}^n A_i y_{x_i} dt = (B_1 y + B_3 v) dt + (B_2 y + v) dW(t) & \text{in } Q_T, \\ \zeta_- = u & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } G, \end{cases}$$

with initial data $y_0 \in L^2(G; \mathbb{R}^N)$ and coefficient matrices

$$B_1, B_2, B_3 \in L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(G; \mathbb{R}^{N \times N})).$$

Here $y(\cdot)$ is the state, and $(u, v) \in L_{\mathbb{F}}^2(0, T; U_-) \times L_{\mathbb{F}}^2(0, T; L^2(G; \mathbb{R}^N))$ are the control inputs. Here the sample point ω will be omitted when no confusion arises. Throughout the paper, the subscript x_i denotes the partial derivative with respect to x_i .

The component ζ_+ (resp. ζ_-) describes the information passing through the boundary along (resp. opposite to) the outward normal direction. The functions $\zeta_+(t, x) \in \mathbb{R}^{n+}$ and $\zeta_-(t, x) \in \mathbb{R}^{n-}$ are called the outgoing and incoming variables of

system (1.5) at $x \in \Gamma$, respectively. The outgoing part carries information leaving the domain and can therefore be measured by sensors placed on the boundary, while the incoming part represents information entering the domain and can be modified by actuators through the boundary input. Hence, the outgoing variable is observed, whereas the incoming variable is controlled. The boundary condition of system (1.5) is imposed on the incoming variable $\zeta_-(t, x)$.

For nonsmooth solutions the boundary line in (1.5) is understood through Definition 1.1. It is not an assertion of an unweighted L^2 trace. The flux norm is essential near characteristic boundary points; a counterexample to the unweighted formulation is given at the end of Section 2.

The control system (1.5) is a first-order stochastic hyperbolic system with a non-homogeneous boundary condition. Solutions to (1.5) will be understood in the transposition sense. For this purpose, we introduce the corresponding backward stochastic hyperbolic system. For $\tau \in (0, T]$, consider

$$(1.6) \quad \begin{cases} dz + \sum_{i=1}^n A_i z_{x_i} dt = - \left[(B_1^\top - B_2^\top B_3^\top + \sum_{i=1}^n A_{i, x_i}) z + B_2^\top Z \right] dt \\ \quad + (Z - B_3^\top z) dW(t) & \text{in } Q_\tau, \\ \xi_+ = 0 & \text{on } \Sigma_\tau, \\ z(\tau) = z_\tau & \text{in } G, \end{cases}$$

where $z_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))$ and

$$\xi(t, x) = \begin{pmatrix} \xi_+(t, x) \\ \xi_0(t, x) \\ \xi_-(t, x) \end{pmatrix} \triangleq \Pi^{-1}(x)z(t, x),$$

with $\xi_+(t, x) \in \mathbb{R}^{n_+}$ and $\xi_-(t, x) \in \mathbb{R}^{n_-}$. For the backward system (1.6), the time direction is reversed, and consequently ξ_+ (resp. ξ_-) becomes the incoming (resp. outgoing) variable. For mild solutions, $\xi_+ = 0$ means the homogeneous outgoing graph boundary condition in Appendix A. The weighted outgoing trace ξ_- is constructed in Proposition 1.2; pointwise boundary values are not otherwise assumed. The coordinate-free trace is related by

$$\mathcal{C}_- z = \Pi \begin{pmatrix} 0 \\ 0 \\ (-\Lambda_-)^{1/2} \xi_- \end{pmatrix}, \quad \|\mathcal{C}_- z\|_{\mathcal{U}} = \|\xi_-\|_{\mathcal{U}_-}.$$

Throughout the paper, the superscript $^\top$ denotes the transpose of a vector or matrix.

Writing $q = Z - B_3^\top z$, the backward equation is $dz = -(\mathcal{A}^* z + B_1^\top z + B_2^\top q)dt + q dW$. The change $(z, q) \leftrightarrow (z, Z)$ is bounded in the stated process spaces. By [26, Theorem 4.10] and the operator realization in Appendix A, we have the following well-posedness result for the backward stochastic system (1.6):

PROPOSITION 1.1. *For any $\tau \in (0, T]$ and any terminal datum*

$$z_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N)),$$

equation (1.6) admits a unique mild solution

$$(z, Z) \in L^2_{\mathbb{F}}(\Omega; C([0, \tau]; L^2(G; \mathbb{R}^N))) \times L^2_{\mathbb{F}}(0, \tau; L^2(G; \mathbb{R}^N))$$

satisfying

$$\|z\|_{L^2_{\mathbb{F}}(\Omega; C([0, \tau]; L^2(G; \mathbb{R}^N)))} + \|Z\|_{L^2_{\mathbb{F}}(0, \tau; L^2(G; \mathbb{R}^N))} \leq C \|z_\tau\|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))}.$$

Hereafter, unless explicitly stated otherwise, C denotes a generic positive constant that depends at most on T , the matrices A_i , the domain G , and the coefficients B_1, B_2, B_3 ; its value may vary from line to line.

Note that Proposition 1.1 does not directly imply $\xi_- \in L^2_{\mathbb{F}}(0, T; U_-)$. The corresponding flux regularity is provided by the following hidden regularity result.

PROPOSITION 1.2. *There exists a constant $C > 0$, independent of τ , such that for any $\tau \in (0, T]$ and any $z_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))$, the solution of the backward stochastic system (1.6) satisfies*

$$\|\xi_-\|_{L^2_{\mathbb{F}}(0, \tau; U_-)} \leq C \|z_\tau\|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))}.$$

Using the well-posedness and hidden regularity results for (1.6), we can now define transposition solutions of (1.5) as follows.

DEFINITION 1.1. A stochastic process $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G; \mathbb{R}^N)))$ is called a *transposition solution* of (1.5) if, for every $\tau \in (0, T]$ and every $z_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))$,

$$\mathbb{E} \int_G \langle y(\tau), z_\tau \rangle dx - \int_G \langle y_0, z(0) \rangle dx = \mathbb{E} \int_0^\tau \int_G \langle v, Z \rangle dx dt - \mathbb{E} \int_0^\tau \int_\Gamma \langle \xi_-, \Lambda_- u \rangle d\Gamma dt, \quad (1.7)$$

where (z, Z) is the solution of (1.6).

In Section 3 we shall establish the following well-posedness result for system (1.5).

PROPOSITION 1.3. *For any initial datum $y_0 \in L^2(G; \mathbb{R}^N)$, boundary control $u \in L^2_{\mathbb{F}}(0, T; U_-)$ and internal control $v \in L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))$, the system (1.5) admits a unique transposition solution $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G; \mathbb{R}^N)))$ satisfying*

$$\begin{aligned} & \|y\|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G; \mathbb{R}^N)))} \\ & \leq C \left(\|y_0\|_{L^2(G; \mathbb{R}^N)} + \|u\|_{L^2_{\mathbb{F}}(0, T; U_-)} + \|v\|_{L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))} \right). \end{aligned}$$

We now recall the notion of exact controllability for system (1.5).

DEFINITION 1.2. System (1.5) is called *exactly controllable* at time T if, for every initial state $y_0 \in L^2(G; \mathbb{R}^N)$ and every target state $y_1 \in L^2_{\mathcal{F}_T}(\Omega; L^2(G; \mathbb{R}^N))$, there exist a boundary control $u \in L^2_{\mathbb{F}}(0, T; U_-)$ and an internal control $v \in L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))$ such that the corresponding solution $y(\cdot)$ of (1.5) satisfies $y(T) = y_1$, $\mathbb{P}$ -a.s.

The main difficulty in studying the exact controllability problem of system (1.5) lies in obtaining a suitable observability estimate for the associated backward stochastic adjoint system, which through duality is equivalent to exact controllability.

Compared with the deterministic case, several additional analytical difficulties arise in the stochastic framework: (1) the corresponding observability estimate must necessarily capture not only the boundary trace of the adjoint state but also the stochastic unknown Z ; (2) the weighted energy argument must be carried out in the Itô sense, so that quadratic variation terms appear and must be handled carefully; (3) the transposition formulation requires suitable boundary regularity for the outgoing trace

of the adjoint solution, which is nontrivial in the stochastic setting; (4) adaptedness prevents one from treating the equation simply as a deterministic hyperbolic system with random coefficients. In particular, taking expectations does not reduce the stochastic estimates to deterministic ones: the martingale terms have zero expectation by localization and an H^1 martingale estimate, whereas the quadratic variation and covariation terms are finite-variation terms that remain in the estimates.

For deterministic hyperbolic systems, such estimates are often derived via multiplier methods or geometric optics. In the stochastic setting, however, these techniques do not directly apply due to the Itô correction and the non-anticipative nature of solutions. The primary technical novelty of this work is the development of a global Carleman estimate tailored for the adjoint system (1.6). Carleman estimates have proven powerful for control problems of stochastic PDEs (see, e.g., [12, 13, 21–26, 36, 37] and the references therein). In particular, [23] employed a Carleman estimate to prove exact controllability for a stochastic transport equation. However, Carleman estimates for systems in more than two variables remain challenging to derive and are still relatively scarce in the literature. Our global Carleman estimate is inspired by the one established for deterministic first-order symmetric hyperbolic systems in [11].

Although our Carleman estimate is inspired by [11], the present analysis is not a direct stochastic translation of that argument. In the stochastic setting, the derivation of the Carleman estimate must be carried out in the Itô framework and therefore contains quadratic variation terms with no deterministic analogue.

Remark 1.1. The boundary observation in [11] is the total boundary energy (see Theorem 1.2 therein), while our observation only uses the energy leaving the domain through the boundary (see Proposition 4.1). This difference comes from the treatment of the boundary condition: we use the pointwise incoming and outgoing decomposition of the boundary variables in (1.4), whereas [11] uses the decomposition of the boundary into the sets $\partial\Omega_+(t)$ and $\partial\Omega_-(t)$.

To establish the exact controllability of system (1.5), we impose the following structural assumption on the coefficient matrices $A_i(x)$ ($i = 1, 2, \dots, n$).

Condition 1.1. There exists a function $\eta(\cdot) \in C^1(\overline{G})$ and a constant $c_0 > 0$ such that

$$(1.8) \quad -\left\langle \left(\sum_{i=1}^n A_i(x) \eta_{x_i}(x) \right) \gamma, \gamma \right\rangle \geq c_0 |\gamma|^2, \quad \forall x \in \overline{G}, \gamma \in \mathbb{R}^N.$$

Condition 1.1 is essential for the derivation of our Carleman estimate. Its geometric interpretation as well as several physical examples that satisfy this condition will be discussed in Section 2. We are now ready to state our main exact controllability result.

THEOREM 1.1. *Assume that Assumption 1.1 and Condition 1.1 hold. If the terminal time T satisfies*

$$(1.9) \quad T > T_0 \triangleq \frac{1}{c_0} \left(\max_{x \in \overline{G}} \eta(x) - \min_{x \in \overline{G}} \eta(x) \right),$$

then system (1.5), with boundary inputs in $L^2_{\mathbb{F}}(0, T; U_-)$, is exactly controllable at time T .

Remark 1.2. Consider the scalar case $N = 1$. Let $\mathcal{O} = (A_1, \dots, A_n)^\top \in \mathbb{R}^n$ be constant, $\Pi = 1$, and define the inflow boundary by $\Gamma^- = \{x \in \Gamma \mid \mathcal{O} \cdot \nu(x) < 0\}$.

Assume $|\mathcal{O}| = 1$ and choose the origin in G . Then (1.5) reduces to the scalar stochastic transport equation

$$\begin{cases} dy + \mathcal{O} \cdot \nabla y \, dt = (B_1 y + B_3 v) \, dt + (B_2 y + v) \, dW(t), & \text{in } Q_T, \\ y = u, & \text{on } (0, T) \times \Gamma^-, \\ y(0) = y_0, & \text{in } G. \end{cases}$$

This system was studied in [26, Chapter 8], where exact controllability was proved under the condition $T > \tilde{T}_0 \triangleq 2 \max_{x \in \Gamma} |x|$.

Choosing the weight function $\eta(x) = -\mathcal{O} \cdot x$ in Condition 1.1 yields $c_0 = 1$. Consequently, Theorem 1.1 guarantees exact controllability whenever

$$T > T_0 = \max_{x \in \bar{G}} (\mathcal{O} \cdot x) - \min_{x \in \bar{G}} (\mathcal{O} \cdot x).$$

Observe that $T_0 \leq \tilde{T}_0$. This compares the sufficient-time bounds; the boundary space here is the incoming flux space. In general T_0 is not the minimal control time: the oscillation of an escape weight need not equal the longest characteristic crossing time. An example is given in Section 2.

In the study of exact controllability for scalar stochastic transport equations in [23], the author presented several results on the lack of exact controllability, which demonstrate the necessity of employing both boundary and internal controls. Similar controllability limitations hold for the more general stochastic first-order hyperbolic system (1.5).

We use two controls to achieve exact controllability of the control system (1.5). To underscore the necessity of the proposed control structure, we also establish several negative controllability results:

THEOREM 1.2. *Assume additionally that $B_1 \in L^\infty(0, T; L^\infty(G; \mathbb{R}^{N \times N}))$ and $B_3 = 0$. If $u \equiv 0$, then system (1.5) is not exactly controllable for any $T > 0$.*

THEOREM 1.3. *Assume additionally that $B_2 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega; L^\infty(G; \mathbb{R}^{N \times N})))$. Let G_0 be an open subset of G with $G \setminus \overline{G_0} \neq \emptyset$. If the control v is supported only in $(0, T) \times G_0$, then system (1.5) is not exactly controllable for any $T > 0$.*

Furthermore, control must also appear in the diffusion term. To see this, consider the following modified system:

$$(1.10) \quad \begin{cases} dy + \sum_{i=1}^n A_i(x) y_{x_i} \, dt = (B_1 y + \tilde{v}) \, dt + B_2 y \, dW(t) & \text{in } Q_T, \\ \zeta_- = u & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } G, \end{cases}$$

where B_1, B_2, u are as in (1.5) and $\tilde{v} \in L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))$ is an internal control acting only on the drift. For this system we obtain the following negative result.

THEOREM 1.4. *Assume additionally that $B_2 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega; L^\infty(G; \mathbb{R}^{N \times N})))$. Then system (1.10) is not exactly controllable for any $T > 0$.*

Remark 1.3. We emphasize that, because we are dealing with a system (coupled components), it may be possible under additional structural assumptions to achieve controllability with fewer controls; see e.g., [1, 16]. Determining the minimal number of required controls lies beyond the scope of the present work.

The rest of the paper is organized as follows. In Section 2, we give a geometric interpretation of Condition 1.1 and list several relevant examples. In Section 3, we prove the hidden regularity (Proposition 1.2) for the solution to (1.6) and establish the well-posedness (Proposition 1.3) for the system (1.5). Section 4 is devoted to proving Theorem 1.1. In Section 5, we prove Theorems 1.2–1.4. Finally, in Appendix A, we give the proofs of some technical results.

2. The geometric interpretation of Condition 1.1. In this section, we provide a geometric interpretation of Condition 1.1 and present several examples satisfying this condition. As an illustrative motivation, we consider the following deterministic 1-D scalar equation:

$$\begin{cases} y_t + y_x = 0, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = u(t), & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1). \end{cases}$$

Given a time $T > 0$ and a target function $y_1(x) \in L^2(0, 1)$, our goal is to choose boundary input $u(t)$ such that the solution y satisfies $y(T, x) = y_1(x)$. The characteristics are the lines $x - t = c$, with c constant, along which the boundary input propagates into the domain. We have two simple observations: characteristic rays indicate how the boundary input can influence the terminal state (see the left panel of Figure 1), and the controllability time T must be long enough (see the right panel). Indeed, if $T \geq 1$, one can prescribe $y(T, x) = y_1(x)$ by taking $u(T - x) = y_1(x)$; if $T < 1$, part of the terminal line is still determined by the initial datum y_0 , so exact controllability is impossible.

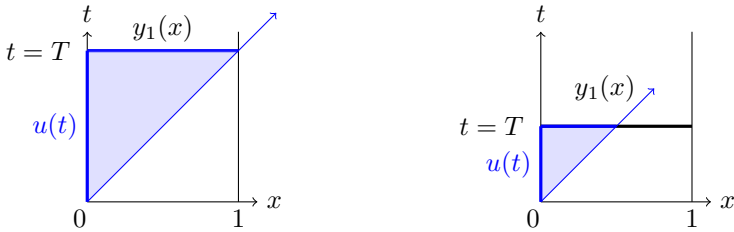


FIG. 1. A 1-D illustration of characteristic rays and controllability.

For the controllability of the multi-dimensional stochastic system, the same idea leads us to analyze the characteristics, which are determined by the principal first-order operator

$$\mathcal{L} \triangleq \partial_t + \sum_{i=1}^n A_i(x) \partial_{x_i}.$$

More precisely, the characteristic rays are determined by the deterministic principal symbol of $\mathcal{L}$:

$$P(t, x, \tau, \varpi) = \tau I_N + \sum_{i=1}^n \varpi_i A_i(x),$$

where $\tau \in \mathbb{R}$, $\varpi = (\varpi_1, \dots, \varpi_n) \in \mathbb{R}^n$ are the variables on frequency space, and I_N is the identity matrix in $\mathbb{R}^{N \times N}$. The characteristic set in the phase space (cotangent

bundle) is $\mathcal{T}^*(\mathbb{R} \times G)$:

$$(2.1) \quad \text{Char}(P) \triangleq \left\{ (t, x, \tau, \varpi) \in \mathcal{T}^*(\mathbb{R} \times G) \setminus \{0\} \mid \det \left(\tau I_N + \sum_{i=1}^n \varpi_i A_i(x) \right) = 0 \right\}.$$

Recalling that $A_i(x)$ ($i = 1, \dots, n$) are symmetric, hence we have

$$\det \left(\tau I_N + \sum_{i=1}^n \varpi_i A_i(x) \right) = \prod_{k=1}^N (\tau + \lambda_k(x, \varpi))$$

with $\lambda_k = \lambda_k(x, \varpi) \in \mathbb{R}$ being the k -th eigenvalue of the symmetric matrix $\sum_{i=1}^n \varpi_i A_i(x)$. On a region where a C^1 eigenpair is available, we define the corresponding Hamiltonian

$$H_k = H_k(t, x, \tau, \varpi) \triangleq \tau + \lambda_k(x, \varpi).$$

The corresponding k -th bicharacteristics $(t_k(s), \mathbf{x}_k(s), \tau_k(s), \varpi_k(s)) \subset \text{Char}(P)$ (where the bold letters are used to distinguish them from the space and frequency variables) are the integral curves of the k -th Hamiltonian vector field:

$$\begin{cases} \frac{dt_k}{ds} = \frac{\partial H_k}{\partial \tau} = 1, \\ \frac{d\mathbf{x}_k}{ds} = \nabla_{\varpi} H_k = \nabla_{\varpi} \lambda_k(\mathbf{x}_k, \varpi_k), \\ \frac{d\tau_k}{ds} = -\frac{\partial H_k}{\partial t} = 0, \\ \frac{d\varpi_k}{ds} = -\nabla_x H_k = -\nabla_x \lambda_k(\mathbf{x}_k, \varpi_k). \end{cases}$$

The curves $(t_k(s), \mathbf{x}_k(s))$ are called characteristic rays on regions where the eigenvalue branch and a normalized eigenvector can be chosen C^1 (in particular, on simple-eigenvalue regions). They represent the trajectory of energy propagation (wave front) there; no global differentiability through eigenvalue crossings is assumed. It is straightforward to check that, in 1-D, characteristic rays are independent of the frequency variable ϖ , whereas in the multi-dimensional case their velocities generally depend on ϖ , allowing infinitely many ray directions through the same spatial point. In view of this, Condition 1.1 is imposed precisely to constrain the directions of characteristic rays: there exists a potential function $\eta(\cdot)$ which decreases strictly and uniformly along all rays.

PROPOSITION 2.1. *Under Condition 1.1, on each such smooth eigenbranch, $\eta(x)$ strictly decreases along $\mathbf{x}_k(s)$ for all $k = 1, \dots, N$, with a uniform decay rate at least c_0 on its interval of definition (where c_0 is the constant in Condition 1.1):*

$$(2.2) \quad \frac{d}{ds} \eta(\mathbf{x}_k(s)) \leq -c_0, \quad \forall k \in \{1, \dots, N\}.$$

Proof. On the smooth eigenbranch under consideration, choose a C^1 normalized eigenvector $r_k = r_k(x, \varpi)$ with $|r_k| = 1$ so that

$$\left(\sum_{i=1}^n \varpi_i A_i(x) \right) r_k = \lambda_k(x, \varpi) r_k.$$

Taking the derivative with respect to ϖ_j on both sides of the above identity, and then taking the inner product with r_k , we obtain that

$$\langle r_k, A_j(x)r_k \rangle + \left\langle r_k, \left(\sum_{i=1}^n \varpi_i A_i(x) \right) \frac{\partial r_k}{\partial \varpi_j} \right\rangle = \left\langle r_k, \frac{\partial \lambda_k}{\partial \varpi_j} r_k \right\rangle + \lambda_k(x, \varpi) \left\langle r_k, \frac{\partial r_k}{\partial \varpi_j} \right\rangle.$$

By the symmetry and the fact that $|r_k| = 1$, we have

$$\langle r_k, A_j(x)r_k \rangle = \frac{\partial \lambda_k}{\partial \varpi_j}.$$

With the preparations above, we compute the evolution of η along the k -th rays:

$$\begin{aligned} \frac{d}{ds} \eta(\mathbf{x}_k(s)) &= \left\langle \nabla \eta, \frac{d\mathbf{x}_k}{ds} \right\rangle = \langle \nabla \eta, \nabla_{\varpi} \lambda_k \rangle \\ &= \sum_{i=1}^n \eta_{x_i} \langle r_k, A_i(x)r_k \rangle = \left\langle r_k, \left(\sum_{i=1}^n A_i(x) \eta_{x_i} \right) r_k \right\rangle. \end{aligned}$$

This, together with (1.8) and $|r_k| = 1$, implies (2.2) immediately. $\square$

Proposition 2.1 implies that, under Condition 1.1, the tangent direction at every point along each characteristic ray forms an acute angle with the direction of $-\nabla \eta$. Along every such ray segment the decrease is uniform; any continued trajectory obeying this inequality cannot remain in the domain longer than the oscillation bound below. This motivates the sufficient control time:

$$T_0 = \frac{1}{c_0} \left(\max_{x \in \overline{G}} \eta(x) - \min_{x \in \overline{G}} \eta(x) \right).$$

When $T > T_0$, the Carleman argument below proves controllability directly from the matrix condition. It does not require a choice of smooth eigenbranches at crossings.

Remark 2.1. Condition 1.1 is somewhat restrictive. In particular, several classical models may fail to satisfy it when rewritten as first-order hyperbolic systems, such as the wave equation and other wave-like equations (e.g., Maxwell or Klein–Gordon). In these settings, one typically needs to design an appropriate multiplier case-by-case; see [26, 35].

At the end of this section, we illustrate the applicability of Condition 1.1 with several representative models.

EXAMPLE 2.1 (Stochastic free traffic flow). In traffic flow modeling, the Aw–Rascle–Zhang system constitutes a first-order hyperbolic model defined on a one-dimensional road, with macroscopic state variables given by the traffic density ρ and the velocity v . Linearizing around a constant equilibrium (ρ_0, v_0) (see [4, Section 1.10]) yields the deterministic system

$$y_t + A_1 y_x = B_1 y, \quad y = (\rho, v)^\top,$$

where

$$A_1 = \begin{pmatrix} v_0 & \rho_0 \\ 0 & v_0 - \rho_0 P'(\rho_0) \end{pmatrix}, \quad B_1 = \sigma \begin{pmatrix} 0 & 0 \\ V'(\rho_0) & -1 \end{pmatrix}.$$

Here $P(\rho)$ (respectively $V(\rho)$) is a strictly increasing (resp. decreasing) function representing the traffic pressure (resp. average speed), and σ is proportional to the

drivers' reaction time. The eigenvalues of A_1 are v_0 and $v_0 - \rho_0 P'(\rho_0)$. Assuming $v_0 > \rho_0 P'(\rho_0) > 0$ corresponds to a free-flow regime, i.e. the road is relatively empty; consequently both eigenvalues are positive.

In reality, driver behaviour is uncertain, causing σ to fluctuate randomly, while additional internal stochastic perturbations may also affect the dynamics [34, Chapter 5]. Therefore we consider the stochastic version

$$dy + A_1 y_x dt = B_1(t, x)y dt + B_2(t, x)y dW(t).$$

Although A_1 itself is not symmetric, we introduce the symmetric positive-definite matrix

$$A_0 = \begin{pmatrix} P'(\rho_0) & 1 \\ 1 & 2/P'(\rho_0) \end{pmatrix},$$

for which $A_0 A_1$ becomes symmetric. Setting $\tilde{y} = A_0^{1/2} y$ gives

$$d\tilde{y} + \tilde{A}_1 \tilde{y}_x dt = \tilde{B}_1(t, x)\tilde{y} dt + \tilde{B}_2(t, x)\tilde{y} dW(t),$$

where $\tilde{A}_1 = A_0^{1/2} A_1 A_0^{-1/2}$ is symmetric. Because the eigenvalues of $\tilde{A}_1$ are positive, Condition 1.1 holds with $\eta(x) = -x$. On an interval $G = (0, L)$ both endpoint matrices are noncharacteristic, so Assumption 1.1 holds. After adding the two controls of (1.5) in the transformed coordinates, Theorem 1.1 applies for $T > L/(v_0 - \rho_0 P'(\rho_0))$. The constant invertible transformation also transforms the control variables.

EXAMPLE 2.2 (Linear stochastic age-structured SIR epidemiological equations). In mathematical epidemiology, populations subject to an infectious disease are commonly stratified into three compartments:

- **Susceptible** (S): individuals who are not infected but may contract the disease;
- **Infected** (I): individuals currently carrying the pathogen;
- **Recovered** (R): individuals who have recovered and acquired permanent immunity.

Let $S(t, x)$, $I(t, x)$, $R(t, x)$ denote the age-density of susceptible, infected, and recovered individuals at time t and age x . Under standard modelling assumptions (see, e.g., [4, Section 1.12]), the ageing part has coefficient matrix $A_1(x) = I_3$; the infection reaction is generally nonlinear. Linearization gives a linear hyperbolic system with this same principal matrix.

Deterministic models, however, presume fixed parameters. In practice, several sources of randomness cannot be ignored (cf. [2]):

1. Contact rates vary seasonally and with behavioural or policy changes;
2. Detection rates and compliance with isolation measures are uncertain;
3. Viral mutations introduce randomness in transmissibility;
4. Environmental factors (temperature, humidity) affect transmission stochastically.

To incorporate such inherent uncertainty, a motivating nonlinear stochastic age-structured model includes Brownian forcing:

$$(2.3) \quad \begin{cases} dS + S_x dt = -\left(\beta(x, t) \frac{SI}{N_{\text{pop}}} + \mu_S(x)S\right)dt + \sigma_S(t) dW(t), \\ dI + I_x dt = \left(\beta(x, t) \frac{SI}{N_{\text{pop}}} - (\gamma(x) + \mu_I(x))I\right)dt + \sigma_I(t) dW(t), \\ dR + R_x dt = (\gamma(x)I - \mu_R(x)R)dt + \sigma_R(t) dW(t). \end{cases}$$

Here $\partial/\partial x$ represents ageing; each $\sigma_j(t)dW(t)$ is a Brownian-driven stochastic perturbation; $\beta(x, t)$ denotes the (possibly time-varying) age-specific transmission rate; $\gamma(x)$ is the age-dependent recovery rate; and $\mu_\cdot(x)$ are age-dependent mortality rates.

Here $N_{\text{pop}} > 0$ is a fixed population normalization. Our linear controllability theorem concerns a linearization about a prescribed smooth bounded reference state, with bounded reaction coefficients and the two controls in (1.5); it does not assert controllability of the nonlinear system (2.3). Prescribed square-integrable affine forcing can be subtracted using a particular solution. On a finite age interval $(0, L)$, $A_1 = I_3$ gives noncharacteristic endpoints, $\eta = -x$ and $c_0 = 1$. Thus the controlled linearization satisfies both hypotheses and is exactly controllable for $T > L$.

EXAMPLE 2.3 (Stochastic shallow-water equations for torrential flow). The shallow-water equations are widely used in hydraulic engineering for the regulation of navigable rivers and irrigation networks [15], and in geophysics for the simulation of atmospheric and oceanic flows [17].

Consider the 2-D linearized shallow-water equations [33] around a constant equilibrium (H_0, U_0, V_0) . The corresponding coefficient matrices are

$$(2.4) \quad A_1 = \begin{pmatrix} U_0 & \sqrt{gH_0} & 0 \\ \sqrt{gH_0} & U_0 & 0 \\ 0 & 0 & U_0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} V_0 & 0 & \sqrt{gH_0} \\ 0 & V_0 & 0 \\ \sqrt{gH_0} & 0 & V_0 \end{pmatrix},$$

where g is the gravitational constant, and the constants H_0, U_0, V_0 represent the water height and the two horizontal velocity components of the steady flow, respectively.

Assume that

$$U_0^2 + V_0^2 > gH_0,$$

which means the flow velocity exceeds the gravitational wave speed. This regime is called torrential (or supercritical) flow. In this case, waves cannot propagate upstream against the current; all disturbances and energy are swept strictly downstream.

If a random perturbation, for instance due to stochastic wind stress (see [6, Chapter 1]), is present, we obtain the following stochastic shallow-water equation:

$$dy + (A_1 y_{x_1} + A_2 y_{x_2}) dt = B_1(t, x) y dt + B_2(t, x) y dW(t),$$

where $y = (h, u, v)^\top$ is the symmetrized state variable, and the matrices A_1, A_2 are given by (2.4).

For constants α_1, α_2 satisfying $\alpha_1^2 + \alpha_2^2 = 1$, the eigenvalues of the symmetric matrix

$$\alpha_1 A_1 + \alpha_2 A_2 = \begin{pmatrix} \alpha_1 U_0 + \alpha_2 V_0 & \alpha_1 \sqrt{gH_0} & \alpha_2 \sqrt{gH_0} \\ \alpha_1 \sqrt{gH_0} & \alpha_1 U_0 + \alpha_2 V_0 & 0 \\ \alpha_2 \sqrt{gH_0} & 0 & \alpha_1 U_0 + \alpha_2 V_0 \end{pmatrix}$$

are

$$\lambda_1 = \alpha_1 U_0 + \alpha_2 V_0, \quad \lambda_2 = (\alpha_1 U_0 + \alpha_2 V_0) - \sqrt{gH_0}, \quad \lambda_3 = (\alpha_1 U_0 + \alpha_2 V_0) + \sqrt{gH_0}.$$

Choosing

$$(\alpha_1, \alpha_2) = \left(\frac{U_0}{\sqrt{U_0^2 + V_0^2}}, \frac{V_0}{\sqrt{U_0^2 + V_0^2}} \right)$$

makes the matrix $\alpha_1 A_1 + \alpha_2 A_2$ positive definite. Consequently, we may take the weight function

$$\eta(x) = -(\alpha_1 x_1 + \alpha_2 x_2).$$

To check the boundary hypothesis as well, take G to be a disk of radius R . Put $U = (U_0, V_0)$ and $a = \sqrt{gH_0}$. The normal eigenvalues are $U \cdot \nu - a$, $U \cdot \nu$, and $U \cdot \nu + a$. Since $|U| > a$, their zero sets on the circle are six distinct transverse points. Near each such point, only one smooth eigenline changes sign, giving the nested bundles in Assumption 1.1 (see Lemma A.1). Therefore $c_0 = |U| - a$ and $T_0 = 2R/(|U| - a)$ are admissible. The matrices genuinely do not commute: their commutator has entries $(2, 3) = a^2$ and $(3, 2) = -a^2$. On other smooth domains the same escape condition holds, while the boundary hypothesis must be checked separately.

EXAMPLE 2.4 (Stochastic supersonic gas equations). The evolution of density, velocities and entropy of a compressible, inviscid and non-heat-conducting gas is governed by the Euler equations, see [5, (13.2.26), p. 395]. For an n -dimensional ideal gas ($n = 2, 3$), we linearize the system around a constant equilibrium (ρ_0, u_0, s_0) , where the constants ρ_0 and s_0 are density and entropy, while $u_0 = (u_{01}, u_{02}, \dots, u_{0n})$ is the n -dimensional constant vector standing for the velocity. After symmetrization (see [5, p. 395]), we denote the symmetric coefficient matrices by A_i ($i = 1, \dots, n$), then

$$(2.5) \quad A_i = \begin{pmatrix} \frac{u_{0i}}{\rho_0 c_0^2} & e_i^\top & 0 \\ e_i & \rho_0 u_{0i} I_n & 0 \\ 0 & 0 & u_{0i} \end{pmatrix}, \quad i = 1, \dots, n,$$

where e_i is the n -dimensional unit vector with the i -th component being 1, and c_0 is the sound speed. The uncertain friction parameters naturally render the system stochastic (see [14] and references therein):

$$A_0 dy + \sum_{i=1}^n A_i y_{x_i} dt = B_1(t, x) y dt + B_2(t, x) y dW(t),$$

where A_i are given in (2.5) and the positive time matrix is

$$A_0 = \text{diag}((\rho_0 c_0^2)^{-1}, \rho_0 I_n, 1).$$

Indeed, for pressure, velocity and entropy perturbations $y = (p, v, s)^\top$, the principal equations are $(\partial_t + u_0 \cdot \nabla)p + \rho_0 c_0^2 \text{div } v = 0$, $\rho_0(\partial_t + u_0 \cdot \nabla)v + \nabla p = 0$, and $(\partial_t + u_0 \cdot \nabla)s = 0$. Dividing the first equation by $\rho_0 c_0^2$ gives these A_0, A_i . To use (1.5), set $w = Py$, $P = A_0^{1/2}$, and multiply the equation by P^{-1} . The normalized principal matrices are $\hat{A}_i = P^{-1} A_i P^{-1}$, and all lower-order matrices and controls are transformed accordingly.

Consider a constant n -dimensional unit vector $\alpha = (\alpha_1, \dots, \alpha_n)^\top$, then the eigenvalues of the $(n+2) \times (n+2)$ -matrix

$$\sum_{i=1}^n \alpha_i A_i = \begin{pmatrix} \frac{u_0 \cdot \alpha}{\rho_0 c_0^2} & \alpha^\top & 0 \\ \alpha & \rho_0 (u_0 \cdot \alpha) I_n & 0 \\ 0 & 0 & u_0 \cdot \alpha \end{pmatrix}$$

are

$$\lambda_1, \lambda_{n+2} = \frac{1}{2} \left(\rho_0 (u_0 \cdot \alpha) + \frac{u_0 \cdot \alpha}{\rho_0 c_0^2} \pm \sqrt{\left(\rho_0 (u_0 \cdot \alpha) - \frac{u_0 \cdot \alpha}{\rho_0 c_0^2} \right)^2 + 4} \right),$$

and

$$\lambda_2 = u_0 \cdot \alpha, \quad \lambda_3 = \cdots = \lambda_{n+1} = \rho_0(u_0 \cdot \alpha).$$

Assume the gas speed exceeds the sound speed (supersonic), i.e., $|u_0| > \mathbf{c}_0$, then we can choose $\alpha = u_0/|u_0|$ and $\eta(x) = -\sum_{i=1}^n \alpha_i x_i$.

In normalized coordinates,

$$\hat{A}_i = u_{0i} I_{n+2} + \mathbf{c}_0 \begin{pmatrix} 0 & e_i^\top & 0 \\ e_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently $\sum_i \alpha_i \hat{A}_i \geq (|u_0| - \mathbf{c}_0)I$ for $\alpha = u_0/|u_0|$. On the ball of radius R , the normal eigenvalues are $u_0 \cdot \nu \pm \mathbf{c}_0$ and $u_0 \cdot \nu$ (the latter has multiplicity n). Their three zero level sets are disjoint smooth submanifolds because $|u_0| > \mathbf{c}_0$. The acoustic projections are the outer products of $(1, \pm \nu, 0)/\sqrt{2}$; the complementary smooth projection is the repeated convective block. That whole block changes sign together. Local smooth frames therefore satisfy Lemma A.1, without a global tangent frame. The controlled linearized system, with controls prescribed after normalization, is covered for $T > 2R/(|u_0| - \mathbf{c}_0)$. This is a statement about the linearized system, not the nonlinear Euler equations.

Remark 2.2. The present operator proof is for smooth domains. For a polytope with constant principal matrices, the normal matrix is constant on each face, but this fact alone does not resolve the behavior at edges and corners. No general polytopal extension is asserted here.

Remark 2.3 (Necessity of the flux norm). On the unit disk in coordinates (x, r) take $D = \partial_x$, zero lower-order coefficients, $y_0 = 0$, and $T > 2$. Set $s(r) = \sqrt{1-r^2}$ and let $\Gamma^- = \{(-s(r), r) : -1 < r < 1\}$ be the inflow semicircle. At the left inflow point $(-s(r), r)$ one has $\mathbf{N}_- = s(r)$ and $d\Gamma = dr/s(r)$. The deterministic target $f(x, r) = (1-r)^{-1/2}$ satisfies

$$\|f\|_{L^2(G)}^2 = 2 \int_{-1}^1 \frac{\sqrt{1-r^2}}{1-r} dr = 2\pi.$$

Along the characteristic ending at (x, r) , the mean incoming control must obey $\mathbb{E}u(T - x - s(r), -s(r), r) = f(x, r)$. Jensen's inequality then forces

$$\mathbb{E} \int_0^T \int_{\Gamma^-} |u|^2 d\Gamma dt \geq 2 \int_{-1}^1 \frac{dr}{1-r} = \infty.$$

An internal diffusion control has zero mean and cannot remove this obstruction. The required input has finite flux norm instead: its norm squared on the used characteristics is $\int_G |f|^2 = 2\pi$. Thus merely repairing the old unweighted trace proof is impossible; the control norm really must change (or the target class must be reduced).

Remark 2.4 (Sufficient time versus minimal time). For scalar $D = \partial_x$ on $G = \{(x, r) : (x-r)^2 + r^2 < 1\}$, the weight $\eta = -x$ gives $d = 2\sqrt{2}$, $c_0 = 1$. The alternative weight $\tilde{\eta} = r - x$ has $d = 2$, $c_0 = 1$ on the same system and domain. Its scalar normal zeros are smooth simple glancing points, so the theorem already gives controllability for $T > 2$. Hence the threshold computed from an arbitrary admissible η is not generally a minimal time.

3. The well-posedness of the control system (1.5). In this section, we establish the well-posedness of the control system (1.5).

Put $D = \sum_i A_i \partial_{x_i}$ and $C = \sum_i A_{i,x_i}$. The homogeneous incoming realization is the graph closure

$$\mathcal{A} = \overline{(-D, \mathcal{D}_+)}, \quad \mathcal{D}_+ = \{y \in C^\infty(\overline{G}; \mathbb{R}^N) : P_- \gamma y = 0\}.$$

Thus $y \in D(\mathcal{A})$ means that smooth functions with the indicated boundary constraint converge to y together with their images under $-D$. Appendix A proves the exact adjoint realization with homogeneous outgoing boundary condition. This definition does not presume an unweighted trace for the unrestricted maximal graph space. Set $c = \|C\|_{L^\infty(G; \mathcal{L}(\mathbb{R}^N))}$.

LEMMA 3.1. *The operator $\mathcal{A}$ generates a C_0 -semigroup $S(\cdot)$ on $L^2(G; \mathbb{R}^N)$. Moreover, $S(\cdot)$ is generalized contractive, i.e., $\|S(t)\| \leq e^{Mt}$ for some $M > 0$ and all $t \geq 0$.*

The energy argument is standard (see [5, Chapter 9] and [7, Chapter 7]). For the variable-rank boundary geometry used here, Appendix A proves both resolvent ranges and the exact adjoint domain using the classical weak-strong theorem under Assumption 1.1.

We first prove Proposition 1.2.

Proof of Proposition 1.2. It suffices to take $\tau = T$. Put $q = Z - B_3^\top z$. Let $R_\mu = \mu(\mu - \mathcal{A}^*)^{-1}$ for real $\mu > c$, and put

$$z_\mu = R_\mu z, \quad q_\mu = R_\mu q, \quad f = -B_1^\top z - B_2^\top q, \quad f_\mu = R_\mu f.$$

These satisfy the strong equation

$$(3.1) \quad dz_\mu = -\mathcal{A}^* z_\mu dt + f_\mu dt + q_\mu dW_t, \quad z_\mu(T) = R_\mu z_T.$$

Indeed $R_\mu : H \rightarrow D(\mathcal{A}^*)$ and $\mathcal{A}^* R_\mu$ are bounded, so the assertion follows either from the mild equation and stochastic Fubini, or from the linear strong-solution criterion. In particular the noise is regularized too. Strong convergence and uniform boundedness of R_μ imply

$$z_\mu \rightarrow z \text{ in } S^2(H), \quad (q_\mu, f_\mu) \rightarrow (q, f) \text{ in } \mathcal{L}^2(H)^2.$$

For the first convergence, use compactness of each continuous sample path in H , followed by dominated convergence. We do *not* replace $R_\mu f$ by $-B_1^\top z_\mu - B_2^\top q_\mu$.

For $w = z_\mu - z_\rho$, $g = f_\mu - f_\rho$, and $p = q_\mu - q_\rho$, Itô's formula and (A.5) give

$$(3.2) \quad \begin{aligned} \mathbb{E} \int_0^T \|\mathcal{C}_- w\|^2 dt &= \mathbb{E} \|w(T)\|^2 - \mathbb{E} \|w(0)\|^2 + \mathbb{E} \int_0^T \langle \mathcal{C} w, w \rangle dt \\ &\quad - 2\mathbb{E} \int_0^T \langle w, g \rangle dt - \mathbb{E} \int_0^T \|p\|^2 dt \\ &\leq \mathbb{E} \|w(T)\|^2 + c\mathbb{E} \int_0^T \|w\|^2 dt + 2\|w\|_{\mathcal{L}^2(H)} \|g\|_{\mathcal{L}^2(H)} \longrightarrow 0. \end{aligned}$$

Thus the traces are Cauchy. Define $\mathcal{C}_- z$ to be their limit. A single-sequence version of this estimate, together with the backward solution estimate, gives the estimate in Proposition 1.2. It also shows independence of the particular sequence of resolvent

parameters. The limit is progressive and remains in $\mathcal{U}$. By the isometry in (1.3), it determines $\xi_- \in L^2_{\mathbb{F}}(0, T; U_-)$ uniquely.

If $r \in C^1([0, T] \times \overline{G})$ is deterministic and positive, then

$$\begin{aligned}
 & \mathbb{E} \int_G r(T) |z_T|^2 - \mathbb{E} \int_G r(0) |z(0)|^2 \\
 &= \mathbb{E} \int_0^T \int_{\Gamma} r |\mathcal{C}_- z|^2 + \mathbb{E} \int_0^T \int_G z^\top \left(r_t I + \sum_j A_j \partial_j r - r \mathbf{C} \right) z \\
 (3.3) \quad & - 2 \mathbb{E} \int_0^T \int_G r \langle z, B_1^\top z + B_2^\top q \rangle + \mathbb{E} \int_0^T \int_G r |q|^2.
 \end{aligned}$$

Space and time measures omitted from integral displays are $dx dt$ or $d\Gamma dt$ as appropriate. The same identity holds on every deterministic subinterval.

Apply Hilbert-space Itô's formula to $\int_G r |z_\mu|^2$, using Lemma A.4 to integrate the unbounded operator. Pass to the limit by the preceding convergences, including the boundary trace convergence. This gives (3.3). No differentiability of B_j is needed. To justify zero expectation of the local martingale, localization and BDG give

$$\mathbb{E} \left(\int_0^T |\langle r z_\mu, q_\mu \rangle_H|^2 dt \right)^{1/2} \leq \|r\|_\infty (\mathbb{E} \sup_t \|z_\mu(t)\|^2)^{1/2} (\mathbb{E} \int_0^T \|q_\mu\|^2 dt)^{1/2} < \infty.$$

Thus it is an H^1 martingale. No fourth-moment assumption is required. $\square$

Now we are in a position to prove the well-posedness of the control system (1.5).

Proof of Proposition 1.3. We divide the proof into three steps.

Step 1. Extension of the boundary control.

Let $u \in L^2_{\mathbb{F}}(0, T; U_-)$ and set $U = \Pi(0, 0, u)^\top$ as a measurable boundary vector. It need not belong to ordinary boundary L^2 . Truncate its magnitude: $U^{(m)} = \mathbf{1}_{\{|U| \leq m\}} U$. These truncations belong to ordinary $L^2_{\mathbb{F}}(0, T; L^2(\Gamma; \mathbb{R}^N))$, and

$$\mathbb{E} \int_0^T \int_{\Gamma} |\mathbf{N}_-^{1/2}(U - U^{(m)})|^2 d\Gamma dt \longrightarrow 0.$$

Using spatial smoothing and one-sided temporal mollification with a delay, choose adapted $U_m \in C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^{1/2}(\Gamma; \mathbb{R}^N)))$ with $U_m(0) = 0$ and ordinary L^2 distance from $U^{(m)}$ at most $1/m$. The time smoothing uses only past values and preserves adaptedness. Since $\mathbf{N}_-^{1/2}$ is bounded, the incoming components $u_m = (\Pi^{-1} U_m)_-$ satisfy

$$(3.4) \quad u_m \longrightarrow u \quad \text{in } L^2_{\mathbb{F}}(0, T; U_-).$$

A bounded right inverse of the boundary trace gives lifts

$$\tilde{U}_m \in C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^1(G; \mathbb{R}^N)))$$

with trace U_m and $\tilde{U}_m(0) = 0$.

Step 2. Reduction to a problem with homogeneous boundary data.

Consider the auxiliary stochastic hyperbolic system with zero boundary condition:

$$(3.5) \quad \begin{cases} d\tilde{y}_m + \sum_{i=1}^n A_i(x) \tilde{y}_{m,x_i} dt = (B_1 \tilde{y}_m + B_3 v + F_m) dt \\ \quad \quad \quad + [B_2(\tilde{y}_m + \tilde{U}_m) + v] dW(t) & \text{in } Q_T, \\ (\tilde{\zeta}_m)_- = 0 & \text{on } \Sigma_T, \\ \tilde{y}_m(0) = y_0 & \text{in } G, \end{cases}$$

where

$$F_m = B_1 \tilde{U}_m - \tilde{U}_{m,t} - \sum_{i=1}^n A_i \tilde{U}_{m,x_i}, \quad \tilde{\zeta}_m = ((\tilde{\zeta}_m)_+, (\tilde{\zeta}_m)_0, (\tilde{\zeta}_m)_-)^T \triangleq \Pi^{-1} \tilde{y}_m.$$

By Lemma 3.1 and [26, Theorem 3.21], system (3.5) admits a unique mild solution $\tilde{y}_m \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G; \mathbb{R}^N)))$.

Set $y_m \triangleq \tilde{y}_m + \tilde{U}_m$. On the boundary,

$$(\zeta_m)_- = [\Pi^{-1}(\tilde{y}_m + \tilde{U}_m)]_- = (\tilde{\zeta}_m)_- + u_m = u_m.$$

We next verify that y_m satisfies the transposition identity (1.7) with u replaced by u_m . To this end, fix $\tau \in (0, T]$ and $z_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))$, and let (z, Z) be the corresponding solution of (1.6). Set

$$\widehat{Z} = Z - B_3^\top z, \quad a_m = B_1 \tilde{y}_m + B_3 v + F_m, \quad c_m = B_2(\tilde{y}_m + \tilde{U}_m) + v.$$

Then

$$\begin{cases} d\tilde{y}_m = (\mathcal{A}\tilde{y}_m + a_m) dt + c_m dW(t), \\ dz = -[\mathcal{A}^* z + B_1^\top z + B_2^\top \widehat{Z}] dt + \widehat{Z} dW(t). \end{cases}$$

Both $\tilde{y}_m$ and z are mild solutions, so the classical Hilbert-space Itô formula is not directly applicable to $(\tilde{y}_m, z)_{L^2(G; \mathbb{R}^N)}$. We therefore use the standard resolvent regularization underlying the Yosida approximation; see, for instance, [26, Theorems 3.22 and 4.12].

For every sufficiently large λ , set $J_\lambda = \lambda(\lambda I - \mathcal{A})^{-1}$ and $J_\lambda^* = \lambda(\lambda I - \mathcal{A}^*)^{-1}$, and write $\tilde{y}_{m,\lambda} = J_\lambda \tilde{y}_m$ and $z_\lambda = J_\lambda^* z$. Since $\text{Ran}(J_\lambda) \subset D(\mathcal{A})$ and $\text{Ran}(J_\lambda^*) \subset D(\mathcal{A}^*)$, the regularized processes $\tilde{y}_{m,\lambda}$ and z_λ are strong solutions of the following equations:

$$\begin{cases} d\tilde{y}_{m,\lambda} = (\mathcal{A}\tilde{y}_{m,\lambda} + J_\lambda a_m) dt + J_\lambda c_m dW(t), \\ dz_\lambda = -[\mathcal{A}^* z_\lambda + J_\lambda^*(B_1^\top z + B_2^\top \widehat{Z})] dt + J_\lambda^* \widehat{Z} dW(t), \end{cases}$$

with $\tilde{y}_{m,\lambda}(0) = J_\lambda y_0$ and $z_\lambda(\tau) = J_\lambda^* z_\tau$.

Applying Itô's formula [26, Theorem 2.139] to $(\tilde{y}_{m,\lambda}, z_\lambda)_{L^2(G; \mathbb{R}^N)}$ gives

$$(3.6) \quad \begin{aligned} & \int_G \langle \tilde{y}_{m,\lambda}(\tau), z_\lambda(\tau) \rangle dx - \int_G \langle J_\lambda y_0, z_\lambda(0) \rangle dx \\ &= \int_0^\tau \int_G \left[\langle J_\lambda a_m, z_\lambda \rangle - \langle \tilde{y}_{m,\lambda}, J_\lambda^*(B_1^\top z + B_2^\top \widehat{Z}) \rangle + \langle J_\lambda c_m, J_\lambda^* \widehat{Z} \rangle \right] dx dt \end{aligned}$$

$$+ \int_0^\tau \left(\int_G \langle J_\lambda c_m, z_\lambda \rangle dx \right) dW(t) + \int_0^\tau \left(\int_G \langle \tilde{y}_{m,\lambda}, J_\lambda^* \widehat{Z} \rangle dx \right) dW(t).$$

The regularity of the solutions and the uniform boundedness of the resolvents imply that both stochastic integrands in (3.6) belong to $L^1_{\mathbb{F}}(\Omega; L^2(0, \tau))$. Hence, by [26, Theorem 2.130], the corresponding stochastic integrals have zero expectation. Taking expectations in (3.6) gives

$$\begin{aligned} & \mathbb{E} \int_G \langle \tilde{y}_{m,\lambda}(\tau), z_\lambda(\tau) \rangle dx - \int_G \langle J_\lambda y_0, z_\lambda(0) \rangle dx \\ &= \mathbb{E} \int_0^\tau \int_G \left[\langle J_\lambda a_m, z_\lambda \rangle - \langle \tilde{y}_{m,\lambda}, J_\lambda^* (B_1^\top z + B_2^\top \widehat{Z}) \rangle + \langle J_\lambda c_m, J_\lambda^* \widehat{Z} \rangle \right] dx dt. \end{aligned}$$

Since J_λ and J_λ^* converge strongly to the identity and are uniformly bounded, we may pass to the limit term by term as $\lambda \rightarrow \infty$, obtaining

$$(3.7) \quad \mathbb{E} \int_G \langle \tilde{y}_m(\tau), z_\tau \rangle dx - \int_G \langle y_0, z(0) \rangle dx = \mathbb{E} \int_0^\tau \int_G \left[\langle F_m, z \rangle + \langle B_2 \tilde{U}_m, \widehat{Z} \rangle + \langle v, Z \rangle \right] dx dt.$$

Similarly, Green's formula with the H^1 lift and a smooth-core approximation to z_λ gives the boundary pairing. Pass first to the adjoint graph limit and then to $\lambda \rightarrow \infty$ using the flux-trace convergence proved above. This yields

$$(3.8) \quad \begin{aligned} & \mathbb{E} \int_G \langle \tilde{U}_m(\tau), z_\tau \rangle dx + \mathbb{E} \int_0^\tau \int_G \left[\langle F_m, z \rangle + \langle B_2 \tilde{U}_m, \widehat{Z} \rangle \right] dx dt \\ &= -\mathbb{E} \int_0^\tau \int_\Gamma \langle \xi_-, \Lambda_- u_m \rangle d\Gamma dt. \end{aligned}$$

Combining (3.7) and (3.8) and recalling that $y_m = \tilde{y}_m + \tilde{U}_m$, we obtain

$$(3.9) \quad \begin{aligned} & \mathbb{E} \int_G \langle y_m(\tau), z_\tau \rangle dx - \int_G \langle y_0, z(0) \rangle dx \\ &= \mathbb{E} \int_0^\tau \int_G \langle v, Z \rangle dx dt - \mathbb{E} \int_0^\tau \int_\Gamma \langle \xi_-, \Lambda_- u_m \rangle d\Gamma dt. \end{aligned}$$

Step 3. Energy estimates and passage to the limit.

For any $m_1, m_2 \in \mathbb{N}$, (3.9) implies

$$\mathbb{E} \int_G \langle y_{m_1}(\tau) - y_{m_2}(\tau), z_\tau \rangle dx = -\mathbb{E} \int_0^\tau \int_\Gamma \langle \xi_-, \Lambda_- (u_{m_1} - u_{m_2}) \rangle d\Gamma dt.$$

Choose z_τ with $\|z_\tau\|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))} = 1$ such that

$$\mathbb{E} \int_G \langle y_{m_1}(\tau) - y_{m_2}(\tau), z_\tau \rangle dx \geq \frac{1}{2} \|y_{m_1}(\tau) - y_{m_2}(\tau)\|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))}.$$

Then, using Proposition 1.2, we have that

$$\begin{aligned} & \|y_{m_1}(\tau) - y_{m_2}(\tau)\|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G; \mathbb{R}^N))} \\ & \leq 2 \left| \mathbb{E} \int_0^\tau \int_\Gamma \langle \xi_-, \Lambda_- (u_{m_1} - u_{m_2}) \rangle d\Gamma dt \right| \end{aligned}$$

$$\begin{aligned}
&\leq 2\|u_{m_1} - u_{m_2}\|_{L^2_{\mathbb{F}}(0,T;U_-)} \|\xi_-\|_{L^2_{\mathbb{F}}(0,\tau;U_-)} \\
&\leq 2C\|u_{m_1} - u_{m_2}\|_{L^2_{\mathbb{F}}(0,T;U_-)} \|z_\tau\|_{L^2_{\mathcal{F}_\tau}(\Omega;L^2(G;\mathbb{R}^N))} \\
&= 2C\|u_{m_1} - u_{m_2}\|_{L^2_{\mathbb{F}}(0,T;U_-)},
\end{aligned}$$

where the constant C is independent of τ . Consequently,

$$\|y_{m_1} - y_{m_2}\|_{C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(G;\mathbb{R}^N)))} \leq C\|u_{m_1} - u_{m_2}\|_{L^2_{\mathbb{F}}(0,T;U_-)}.$$

Hence $\{y_m\}_{m=1}^\infty$ is a Cauchy sequence in $C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(G;\mathbb{R}^N)))$. Denote its limit by y . Letting $m \rightarrow \infty$ in (3.9) shows that y satisfies the transposition identity (1.7); therefore y is a transposition solution of (1.5).

Finally, choose z_τ with $\|z_\tau\|_{L^2_{\mathcal{F}_\tau}(\Omega;L^2(G;\mathbb{R}^N))} = 1$ such that

$$\mathbb{E} \int_G \langle y(\tau), z_\tau \rangle dx \geq \frac{1}{2} \|y(\tau)\|_{L^2_{\mathcal{F}_\tau}(\Omega;L^2(G;\mathbb{R}^N))}.$$

Combining (1.7) with Proposition 1.2 yields

$$\begin{aligned}
&\|y(\tau)\|_{L^2_{\mathcal{F}_\tau}(\Omega;L^2(G;\mathbb{R}^N))} \\
&\leq 2 \left(\left| \int_G \langle y_0, z(0) \rangle dx \right| + \left| \mathbb{E} \int_0^\tau \int_G \langle v, Z \rangle dx dt \right| + \left| \mathbb{E} \int_0^\tau \int_\Gamma \langle \xi_-, \Lambda_- u \rangle d\Gamma dt \right| \right) \\
&\leq C \left(\|y_0\|_{L^2(G;\mathbb{R}^N)} + \|v\|_{L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{R}^N))} + \|u\|_{L^2_{\mathbb{F}}(0,T;U_-)} \right),
\end{aligned}$$

with a constant $C > 0$ independent of τ . This completes the proof. $\square$

4. Proof of the observability estimate and the exact controllability. In this section, we prove our main exact controllability result Theorem 1.1. By the classical duality argument, we only need to prove the following observability estimate for the system (1.6).

PROPOSITION 4.1. *Assuming Assumption 1.1 and Condition 1.1 hold and $T > T_0$, then there exists a constant $C > 0$ such that for any $z_T \in L^2_{\mathcal{F}_T}(\Omega;L^2(G;\mathbb{R}^N))$, it holds that*

$$(4.1) \quad \|z_T\|_{L^2_{\mathcal{F}_T}(\Omega;L^2(G;\mathbb{R}^N))}^2 \leq C(\|\xi_-\|_{L^2_{\mathbb{F}}(0,T;U_-)}^2 + \|Z\|_{L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{R}^N))}^2),$$

where (z, Z) is the solution to (1.6).

In order to prove Proposition 4.1, we first establish the following fundamental weighted identity.

LEMMA 4.1. *Let h be a sufficiently regular $H^1(G;\mathbb{R}^N)$ -valued Itô process, so that its drift and diffusion are locally integrable and locally square-integrable, respectively, with values in H^1 . The identity below is interpreted in spatial distributions and after time integration; the integrated identity for the actual mild adjoint solution is justified by the approximation in Proposition 1.2. Set $\varphi \in C^1(\mathbb{R} \times \mathbb{R}^n)$, $\lambda > 0$, $\theta = e^{\lambda \varphi}$, and*

$w = \theta h$. Then the following identity holds in the sense just specified:

$$\begin{aligned}
 (4.2) \quad & 2\lambda \left\langle w, \theta \left(dh + \sum_{i=1}^n A_i h_{x_i} dt \right) \right\rangle \\
 &= \lambda \left[\sum_{i=1}^n \partial_{x_i} \langle w, A_i w \rangle dt + d\langle w, w \rangle - \langle dw, dw \rangle - \sum_{i=1}^n \langle w, A_{i,x_i} w \rangle dt \right] \\
 &\quad - 2\lambda^2 \left\langle w, \left(\sum_{i=1}^n A_i \varphi_{x_i} + \varphi_t \right) w \right\rangle dt.
 \end{aligned}$$

Proof. By the definition of w , we have

$$\theta \left(dh + \sum_{i=1}^n A_i h_{x_i} dt \right) = dw - \lambda \varphi_t w dt - \lambda \sum_{i=1}^n A_i \varphi_{x_i} w dt + \sum_{i=1}^n A_i w_{x_i} dt.$$

Thanks to Itô's formula, it holds that $d\langle w, w \rangle = 2\langle dw, w \rangle + \langle dw, dw \rangle$. For each $i = 1, \dots, n$, we obtain that

$$\left\langle w, \sum_{i=1}^n A_i w_{x_i} \right\rangle dt = \frac{1}{2} \sum_{i=1}^n \partial_{x_i} \langle w, A_i w \rangle dt - \frac{1}{2} \sum_{i=1}^n \langle w, A_{i,x_i} w \rangle dt.$$

Combining the above identities yields the desired result. $\square$

Proof of Proposition 4.1. We split the proof into three steps.

Step 1. Derivation of a Carleman estimate.

Apply the weighted identity first through the graph-resolvent approximation in the proof of Proposition 1.2, and pass to the limit using (3.3). In the resulting integrated identity we write $h = z$ and $w = \theta z$. Thus the calculation below applies to the mild solution; it does not require z itself to be H^1 -valued. Set $\varphi(t, x) = \beta t + \eta(x)$, where $\beta > 0$ will be fixed later and $\eta \in C^1(\overline{G})$ is the function from Condition 1.1. Integrating (4.2) over $Q_T = (0, T) \times G$, using integration by parts, taking expectation, we obtain

$$\begin{aligned}
 (4.3) \quad & 2\lambda \mathbb{E} \int_0^T \int_G \theta \left\langle w, \left(dz + \sum_{i=1}^n A_i z_{x_i} dt \right) \right\rangle dx \\
 &= \lambda \mathbb{E} \int_0^T \int_\Gamma \sum_{i=1}^n \langle w, A_i \nu_i w \rangle d\Gamma dt + \lambda \mathbb{E} \int_G |w(T)|^2 dx - \lambda \int_G |w(0)|^2 dx \\
 &\quad - \lambda \mathbb{E} \int_0^T \int_G \langle dw, dw \rangle dx - \lambda \mathbb{E} \int_0^T \int_G \sum_{i=1}^n \langle w, A_{i,x_i} w \rangle dx dt \\
 &\quad - 2\lambda^2 \mathbb{E} \int_0^T \int_G \left\langle w, \left(\sum_{i=1}^n A_i \varphi_{x_i} + \varphi_t \right) w \right\rangle dx dt.
 \end{aligned}$$

By the boundary condition $\xi_+ = 0$ and the definition $\xi = \Pi^{-1}z$,

$$\begin{aligned}
 (4.4) \quad & \lambda \mathbb{E} \int_0^T \int_\Gamma \sum_{i=1}^n \langle w, A_i \nu_i w \rangle d\Gamma dt = \lambda \mathbb{E} \int_0^T \int_\Gamma \theta^2 \langle \xi_-, \Lambda_- \xi_- \rangle d\Gamma dt \\
 &= -\lambda \mathbb{E} \int_0^T \int_\Gamma \theta^2 \langle -\Lambda_- \xi_-, \xi_- \rangle d\Gamma dt.
 \end{aligned}$$

Choose $\beta < c_0$ with c_0 given in Condition 1.1. Then, because $\varphi_{x_i} = \eta_{x_i}$ and $\varphi_t = \beta$, we have

$$\begin{aligned}
 & -2\lambda^2 \mathbb{E} \int_0^T \int_G \left\langle w, \left(\sum_{i=1}^n A_i \varphi_{x_i} + \varphi_t \right) w \right\rangle dx dt \\
 (4.5) \quad & = -2\lambda^2 \mathbb{E} \int_0^T \int_G \theta^2 \left\langle z, \left(\sum_{i=1}^n A_i \eta_{x_i} + \beta \right) z \right\rangle dx dt \\
 & \geq 2(c_0 - \beta) \lambda^2 \mathbb{E} \int_0^T \int_G \theta^2 |z|^2 dx dt.
 \end{aligned}$$

Moreover,

$$(4.6) \quad -\lambda \mathbb{E} \int_0^T \int_G \sum_{i=1}^n \langle w, A_{i,x_i} w \rangle dx dt \geq -C\lambda \mathbb{E} \int_0^T \int_G \theta^2 |z|^2 dx dt.$$

We next compute the remaining quadratic variation term. Since the martingale part of $w = \theta z$ is $\theta(Z - B_3^\top z)dW(t)$, using the Cauchy–Schwarz inequality and the boundedness of B_3 , we obtain

$$\begin{aligned}
 (4.7) \quad & -\lambda \mathbb{E} \int_0^T \int_G \langle dw, dw \rangle dx = -\lambda \mathbb{E} \int_0^T \int_G \theta^2 |Z - B_3^\top z|^2 dx dt \\
 & \geq -C\lambda \mathbb{E} \int_0^T \int_G \theta^2 |Z|^2 dx dt - C\lambda \mathbb{E} \int_0^T \int_G \theta^2 |z|^2 dx dt.
 \end{aligned}$$

We next estimate the left-hand side of (4.3). By (1.6), we have

$$\begin{aligned}
 & 2\lambda \mathbb{E} \int_0^T \int_G \theta \left\langle w, dz + \sum_{i=1}^n A_i z_{x_i} dt \right\rangle dx \\
 & = -2\lambda \mathbb{E} \int_0^T \int_G \theta \left\langle w, \left(B_1^\top - B_2^\top B_3^\top + \sum_{i=1}^n A_{i,x_i} \right) z + B_2^\top Z \right\rangle dx dt \\
 & \quad + 2\lambda \mathbb{E} \int_0^T \left(\int_G \theta \langle w, Z - B_3^\top z \rangle dx \right) dW(t).
 \end{aligned}$$

The last term has zero expectation. Indeed, with $q = Z - B_3^\top z$,

$$\mathbb{E} \left(\int_0^T \left| \int_G \theta^2 \langle z, q \rangle dx \right|^2 dt \right)^{1/2} \leq \|\theta^2\|_\infty (\mathbb{E} \sup_t \|z(t)\|_H^2)^{1/2} (\mathbb{E} \int_0^T \|q\|_H^2 dt)^{1/2} < \infty.$$

Localization and BDG therefore make it an H^1 martingale. Hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
 (4.8) \quad & 2\lambda \mathbb{E} \int_0^T \int_G \theta \left\langle w, \left(dz + \sum_{i=1}^n A_i z_{x_i} dt \right) \right\rangle dx \\
 & \leq C\lambda \mathbb{E} \int_0^T \int_G \theta^2 |z|^2 dx dt + C\lambda \mathbb{E} \int_0^T \int_G \theta^2 |Z|^2 dx dt.
 \end{aligned}$$

Collecting (4.3)–(4.8), and noting that the term (4.5) is quadratic in λ , we know that there exists a constant $\hat{\lambda}_1 > 0$ such that for all $\lambda \geq \hat{\lambda}_1$,

$$(4.9) \quad \begin{aligned} & C\lambda \mathbb{E} \int_0^T \int_G \theta^2 |Z|^2 dx dt + C\lambda \mathbb{E} \int_0^T \int_\Gamma \theta^2 \langle -\Lambda_- \xi_-, \xi_- \rangle d\Gamma dt + \lambda \int_G \theta(0)^2 |z(0)|^2 dx \\ & \geq \lambda \mathbb{E} \int_G \theta(T)^2 |z_T|^2 dx + (c_0 - \beta) \lambda^2 \mathbb{E} \int_0^T \int_G \theta^2 |z|^2 dx dt. \end{aligned}$$

Step 2. Estimating $\int_G \theta(0)^2 |z(0)|^2 dx$.

Applying Itô's formula to $|z|^2$ on $[t_1, t_2] \subset [0, T]$ and using (1.6), we first obtain,

$$\begin{aligned} & \int_G |z(t_2)|^2 dx - \int_G |z(t_1)|^2 dx \\ & = \int_{t_1}^{t_2} \int_G \left[2 \left\langle z, - \sum_{i=1}^n A_i z_{x_i} - \left(B_1^\top - B_2^\top B_3^\top + \sum_{i=1}^n A_{i,x_i} \right) z - B_2^\top Z \right\rangle \right. \\ & \quad \left. + |Z - B_3^\top z|^2 \right] dx dt \\ & \quad + 2 \int_{t_1}^{t_2} \int_G \langle z, Z - B_3^\top z \rangle dx dW(t). \end{aligned}$$

The last term has zero expectation by the same H^1 martingale estimate as in Step 1. Therefore,

$$\begin{aligned} & \mathbb{E} \int_G |z(t_2)|^2 dx - \mathbb{E} \int_G |z(t_1)|^2 dx \\ & = \mathbb{E} \int_{t_1}^{t_2} \int_G \left[2 \left\langle z, - \sum_{i=1}^n A_i z_{x_i} - \left(B_1^\top - B_2^\top B_3^\top + \sum_{i=1}^n A_{i,x_i} \right) z - B_2^\top Z \right\rangle \right. \\ & \quad \left. + |Z - B_3^\top z|^2 \right] dx dt. \end{aligned}$$

Using integration by parts and the boundary condition, we get

$$\begin{aligned} & -2\mathbb{E} \int_{t_1}^{t_2} \int_G \left\langle z, \sum_{i=1}^n A_i z_{x_i} \right\rangle dx dt \\ & = -\mathbb{E} \int_{t_1}^{t_2} \int_\Gamma \langle \xi_-, \Lambda_- \xi_- \rangle d\Gamma dt + \mathbb{E} \int_{t_1}^{t_2} \int_G \sum_{i=1}^n \langle z, A_{i,x_i} z \rangle dx dt \\ & \geq -C \mathbb{E} \int_{t_1}^{t_2} \int_G |z|^2 dx dt. \end{aligned}$$

By the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} & \mathbb{E} \int_{t_1}^{t_2} \int_G \left(-2 \langle z, B_2^\top Z \rangle + |Z - B_3^\top z|^2 \right) dx dt \\ & \geq \left(\frac{1}{2} - C\varepsilon \right) \mathbb{E} \int_{t_1}^{t_2} \int_G |Z|^2 dx dt - \left(\frac{C}{\varepsilon} + C_1 \right) \mathbb{E} \int_{t_1}^{t_2} \int_G |z|^2 dx dt. \end{aligned}$$

Combining the above inequalities and taking $\varepsilon < \frac{1}{2C}$ such that $\frac{1}{2} - C\varepsilon > 0$, we conclude that

$$\mathbb{E} \int_G |z(t_2)|^2 dx - \mathbb{E} \int_G |z(t_1)|^2 dx \geq -C \mathbb{E} \int_{t_1}^{t_2} \int_G |z|^2 dx dt.$$

Gronwall's inequality then gives

$$\int_G |z(0)|^2 dx \leq e^{CT} \mathbb{E} \int_G |z_T|^2 dx.$$

Recalling $\theta = e^{\lambda\varphi} = e^{\lambda(\beta t + \eta(x))}$, we obtain

$$\begin{aligned} \int_G \theta(0)^2 |z(0)|^2 dx &= \int_G e^{2\lambda\eta(x)} |z(0)|^2 dx \leq e^{2\lambda \max_{x \in \overline{G}} \eta(x)} \int_G |z(0)|^2 dx \\ &\leq e^{CT + 2\lambda \max_{x \in \overline{G}} \eta(x)} \mathbb{E} \int_G |z_T|^2 dx \\ &\leq e^{CT - 2\lambda\beta T + 2\lambda(\max_{x \in \overline{G}} \eta(x) - \min_{x \in \overline{G}} \eta(x))} \mathbb{E} \int_G \theta(T)^2 |z_T|^2 dx. \end{aligned}$$

Step 3. Completion of the proof.

For $T > T_0$ (where T_0 is given in (1.9)), choose $\beta = \frac{2c_0 T_0}{T + T_0} < c_0$, so that

$$T > \frac{1}{\beta} \left(\max_{x \in \overline{G}} \eta(x) - \min_{x \in \overline{G}} \eta(x) \right) = \frac{T + T_0}{2} > T_0.$$

Then

$$-\beta T + \left(\max_{x \in \overline{G}} \eta(x) - \min_{x \in \overline{G}} \eta(x) \right) = \beta \left[\frac{1}{\beta} \left(\max_{x \in \overline{G}} \eta(x) - \min_{x \in \overline{G}} \eta(x) \right) - T \right] = \frac{\beta(T_0 - T)}{2} < 0.$$

Consequently, there exists $\hat{\lambda}_2 > 0$ such that for all $\lambda \geq \hat{\lambda}_2$,

$$(4.10) \quad \int_G \theta(0)^2 |z(0)|^2 dx \leq \frac{1}{2} \mathbb{E} \int_G \theta(T)^2 |z_T|^2 dx.$$

Finally, set $\hat{\lambda}_3 = \max\{\hat{\lambda}_1, \hat{\lambda}_2\}$. For every $\lambda \geq \hat{\lambda}_3$ and $T > T_0$, combining (4.9) and (4.10) yields the observability estimate (4.1). This completes the proof. $\square$

Finally, we conclude this section by proving Theorem 1.1 via the observability estimate established in Proposition 4.1. The argument is a standard duality method; we include it here for completeness.

Proof of Theorem 1.1. Fix initial data $y_0 \in L^2(G; \mathbb{R}^N)$ and a target state $y_1 \in L^2_{\mathcal{F}_T}(\Omega; L^2(G; \mathbb{R}^N))$. Define a linear subspace $\mathcal{X}$ of $L^2_{\mathbb{F}}(0, T; U_-) \times L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))$ by

$$\mathcal{X} = \left\{ (\xi_-, Z) \mid (z, Z) \text{ solves (1.6) with some } z_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G; \mathbb{R}^N)) \right\},$$

and introduce a linear functional $\mathcal{L}$ on $\mathcal{X}$ by

$$\mathcal{L}(\xi_-, Z) = \mathbb{E} \int_G \langle y_1, z_T \rangle dx - \int_G \langle y_0, z(0) \rangle dx.$$

Proposition 4.1 guarantees that $\mathcal{L}$ is bounded on $\mathcal{X}$. By the Hahn–Banach theorem, $\mathcal{L}$ can be extended to a bounded linear functional on the whole space $L^2_{\mathbb{F}}(0, T; U_-) \times$

$L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))$; we still denote the extension by $\mathcal{L}$. Applying the Riesz representation theorem with the flux inner product $\langle u, \xi \rangle_{U_-} = \int_{\Gamma} \langle -\Lambda_- u, \xi \rangle d\Gamma$, we obtain a pair

$$(u, v) \in L^2_{\mathbb{F}}(0, T; U_-) \times L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))$$

such that

$$(4.11) \quad \mathbb{E} \int_G \langle y_1, z_T \rangle dx - \int_G \langle y_0, z(0) \rangle dx = -\mathbb{E} \int_0^T \int_{\Gamma} \langle u, \Lambda_- \xi_- \rangle d\Gamma dt + \mathbb{E} \int_0^T \int_G \langle v, Z \rangle dx dt.$$

Let y be the transposition solution of (1.5) corresponding to the controls (u, v) . According to Definition 1.1,

$$(4.12) \quad \mathbb{E} \int_G \langle y(T), z_T \rangle dx - \int_G \langle y_0, z(0) \rangle dx = -\mathbb{E} \int_0^T \int_{\Gamma} \langle u, \Lambda_- \xi_- \rangle d\Gamma dt + \mathbb{E} \int_0^T \int_G \langle v, Z \rangle dx dt.$$

Subtracting (4.11) from (4.12) gives

$$\mathbb{E} \int_G \langle y(T) - y_1, z_T \rangle dx = 0 \quad \text{for all } z_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G; \mathbb{R}^N)),$$

which implies $y(T) = y_1$, $\mathbb{P}$ -a.s. This completes the proof. $\square$

5. The lack of the exact controllability. In this section, we prove Theorems 1.2–1.4.

Proof of Theorem 1.2. In the case of $B_3 = 0$ and $u \equiv 0$, system (1.5) reads as

$$(5.1) \quad \begin{cases} dy + \sum_{i=1}^n A_i y_{x_i} dt = B_1 y dt + (B_2 y + v) dW(t) & \text{in } Q_T, \\ \zeta_- = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } G. \end{cases}$$

We only need to prove that the attainable set $\mathbb{A}_T$ of system (5.1) at time T with initial data $y_0 = 0$ is not $L^2_{\mathcal{F}_T}(\Omega; L^2(G; \mathbb{R}^N))$. For $y_0 = 0$ and any $t \in [0, T]$, the solution to (5.1) satisfies that

$$(5.2) \quad y(t) = \int_0^t S(t-s) B_1 y ds + \int_0^t S(t-s) (B_2 y + v) dW(s),$$

where $\{S(t)\}_{t \in [0, T]}$ is the C_0 -semigroup generated by $\mathcal{A}$. Taking mathematical expectation on both sides of (5.2), we obtain that

$$\mathbb{E}(y(t)) = \mathbb{E} \int_0^t S(t-s) B_1 y ds.$$

Taking $L^2(G; \mathbb{R}^N)$ norm on both sides and noting that B_1 is deterministic, we conclude that

$$(5.3) \quad \begin{aligned} \|\mathbb{E}(y(t))\|_{L^2(G; \mathbb{R}^N)} &= \left\| \int_0^t S(t-s) B_1 \mathbb{E}(y) ds \right\|_{L^2(G; \mathbb{R}^N)} \\ &\leq C_1 \int_0^t e^{C_2(t-s)} \|\mathbb{E}(y(s))\|_{L^2(G; \mathbb{R}^N)} ds, \end{aligned}$$

where $C_1, C_2 > 0$ are constants independent of t . Denote by

$$Y(t) \triangleq e^{-C_2 t} \|\mathbb{E}(y(t))\|_{L^2(G; \mathbb{R}^N)},$$

then (5.3) can be rewritten as

$$Y(t) \leq C_1 \int_0^t Y(s) ds.$$

Using Gronwall's inequality and noting that $Y(0) = 0$, we finally obtain that $Y(t) = 0$ for any $t \in [0, T]$, which implies that $\mathbb{E}(y(t)) = 0$ for any $t \in [0, T]$. Thus, for $y_0 = 0$, if we choose $y_1 \in L^2_{\mathcal{F}_T}(\Omega; L^2(G; \mathbb{R}^N))$ such that $\mathbb{E}(y_1) \neq 0$, then y_1 is not in the attainable set $\mathbb{A}_T$ of system (5.1) at time T , which completes the proof. $\square$

Remark 5.1. Naturally, we should consider whether the conclusion is valid after removing the assumptions that B_1 is deterministic and $B_3 = 0$. However, we still do not know how to do this, and we leave this for future work.

In order to prove Theorems 1.3 and 1.4, we first recall a useful result.

LEMMA 5.1 ([26, Proposition 6.2]). *There exists a random variable $\varrho \in L^2_{\mathcal{F}_T}(\Omega)$ such that it is impossible to find $(\varsigma_1, \varsigma_2) \in L^2_{\mathbb{F}}(0, T) \times C_{\mathbb{F}}([0, T]; L^2(\Omega))$ and $\varsigma_0 \in \mathbb{R}$ satisfying that*

$$\varrho = \varsigma_0 + \int_0^T \varsigma_1(t) dt + \int_0^T \varsigma_2(t) dW(t).$$

Proof of Theorem 1.3. Let $\chi_1 \in C_0^\infty(G \setminus \overline{G_0}; \mathbb{R}^N)$, where G_0 is the open subset introduced in Theorem 1.3, be a fixed function with $\|\chi_1\|_{L^2(G; \mathbb{R}^N)} = 1$. Assume, for contradiction, that system (1.5) is exactly controllable at time T . Then for any initial state $y_0 \in L^2(G; \mathbb{R}^N)$ there exist controls $(u, v) \in L^2_{\mathbb{F}}(0, T; U_-) \times L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{R}^N))$ with $\text{supp } v \subset G_0$ such that the corresponding solution of (1.5) satisfies $y(T) = \chi_1 \varrho$, where ϱ is the random variable given in Lemma 5.1.

Recall the approximating sequence $\{y_m\}_{m=1}^\infty$ constructed in the proof of Proposition 1.3, where $y_m = \tilde{y}_m + \tilde{U}_m$. Since $\tilde{y}_m$ is the mild solution of (3.5), it is also a weak solution. Testing the weak equation (3.5) against χ_1 and using the equation for $\tilde{U}_m$, we obtain the following identities. Spatial derivatives of the mild state are understood distributionally and transferred to the compactly supported test function:

$$\begin{aligned} & \int_G \langle \tilde{y}_m(T), \chi_1 \rangle dx - \int_G \langle y_0, \chi_1 \rangle dx \\ &= \int_0^T \int_G \left\langle -\sum_{i=1}^n A_i \tilde{y}_{m, x_i} + B_1 \tilde{y}_m - \sum_{i=1}^n A_i \tilde{U}_{m, x_i} + B_1 \tilde{U}_m - \tilde{U}_{m, t}, \chi_1 \right\rangle dx dt \\ & \quad + \int_0^T \int_G \langle B_2(\tilde{y}_m + \tilde{U}_m), \chi_1 \rangle dx dW(t) \\ &= \int_0^T \int_G \left\langle y_m, \sum_{i=1}^n (A_i \chi_1)_{x_i} + B_1^\top \chi_1 \right\rangle dx dt + \int_0^T \int_G \langle y_m, B_2^\top \chi_1 \rangle dx dW(t) \\ & \quad - \int_G \langle \tilde{U}_m(T), \chi_1 \rangle dx. \end{aligned}$$

Passing to the limit $m \rightarrow \infty$ yields

$$\begin{aligned} \int_G \langle y(T), \chi_1 \rangle dx &= \int_G \langle y_0, \chi_1 \rangle dx + \int_0^T \int_G \left\langle y, \sum_{i=1}^n (A_i \chi_1)_{x_i} + B_1^\top \chi_1 \right\rangle dx dt \\ &\quad + \int_0^T \int_G \langle y, B_2^\top \chi_1 \rangle dx dW(t). \end{aligned}$$

Because $y(T) = \chi_1 \varrho$ and $\|\chi_1\|_{L^2(G; \mathbb{R}^N)} = 1$, we deduce

$$\begin{aligned} \varrho &= \int_G \langle \chi_1 \varrho, \chi_1 \rangle dx = \int_G \langle y(T), \chi_1 \rangle dx \\ (5.4) \quad &= \int_G \langle y_0, \chi_1 \rangle dx + \int_0^T \int_G \left\langle y, \sum_{i=1}^n (A_i \chi_1)_{x_i} + B_1^\top \chi_1 \right\rangle dx dt \\ &\quad + \int_0^T \int_G \langle y, B_2^\top \chi_1 \rangle dx dW(t). \end{aligned}$$

From $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G; \mathbb{R}^N)))$ and the assumption

$$B_2 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega; L^\infty(G; \mathbb{R}^{N \times N}))),$$

it follows that

$$\int_G \left\langle y, \sum_{i=1}^n (A_i \chi_1)_{x_i} + B_1^\top \chi_1 \right\rangle dx \in L_{\mathbb{F}}^2(0, T),$$

and

$$\int_G \langle y, B_2^\top \chi_1 \rangle dx \in C_{\mathbb{F}}([0, T]; L^2(\Omega)).$$

These regularity properties show that (5.4) gives a representation of ϱ excluded by Lemma 5.1. This contradiction shows that system (1.5) cannot be exactly controllable when the distributed control v is supported only in G_0 . Hence the proof is complete. $\square$

Proof of Theorem 1.4. The argument parallels that of Theorem 1.3. Pick $\chi_2 \in C_0^\infty(G; \mathbb{R}^N)$ with $\|\chi_2\|_{L^2(G; \mathbb{R}^N)} = 1$. Assume, for a contradiction, that system (1.10) is exactly controllable at time T . Then for every initial state $y_0 \in L^2(G; \mathbb{R}^N)$ there exist controls $(u_1, v_1) \in L_{\mathbb{F}}^2(0, T; U_-) \times L_{\mathbb{F}}^2(0, T; L^2(G; \mathbb{R}^N))$ such that the corresponding solution of (1.10) satisfies $y(T) = \chi_2 \varrho$, where ϱ is the random variable given in Lemma 5.1.

Proceeding as in the proof of Theorem 1.3, we obtain

$$\begin{aligned} \varrho &= \int_G \langle \chi_2 \varrho, \chi_2 \rangle dx = \int_G \langle y(T), \chi_2 \rangle dx \\ &= \int_G \langle y_0(x), \chi_2(x) \rangle dx + \int_0^T \int_G \left[\left\langle y, \sum_{i=1}^n (A_i \chi_2)_{x_i} + B_1^\top \chi_2 \right\rangle + \langle v_1, \chi_2 \rangle \right] dx dt \\ (5.5) \quad &+ \int_0^T \int_G \langle y, B_2^\top \chi_2 \rangle dx dW(t). \end{aligned}$$

Because $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G; \mathbb{R}^N)))$ and $B_2 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega; L^\infty(G; \mathbb{R}^{N \times N})))$, we have

$$\int_G \left[\left\langle y, \sum_{i=1}^n (A_i \chi_2)_{x_i} + B_1^\top \chi_2 \right\rangle + \langle v_1, \chi_2 \rangle \right] dx \in L_{\mathbb{F}}^2(0, T),$$

and

$$\int_G \langle y, B_2^\top \chi_2 \rangle dx \in C_{\mathbb{F}}([0, T]; L^2(\Omega)).$$

Consequently, (5.5) gives a representation of ϱ excluded by Lemma 5.1. This contradiction shows that system (1.10) cannot be exactly controllable. The proof is complete. $\square$

Appendix A. The proof of Lemma 3.1.

A.1. The boundary criterion and the classical weak–strong principle.

LEMMA A.1 (A directly verifiable coherent-crossing class). *Suppose $\text{Ran } \mathbf{N}$ has constant dimension away from a finite disjoint union S as above. Near every point of S , suppose a smooth local orthogonal frame gives*

$$(A.1) \quad Q^\top \mathbf{N} Q = \text{diag}(D_p, \kappa D_c, 0, -D_m),$$

where the nonempty D_p, D_c, D_m are smooth positive definite blocks. There is a smooth local defining function h_0 of S with $d_\Gamma h_0 \neq 0$, such that $\kappa = 0$ on S and $\kappa h_0 > 0$ off S . Then Assumption 1.1 holds. The zero block and either of D_p, D_m may be absent.

Proof. Let the smooth bundles associated with the four blocks be E_p, E_c, E_0, E_m . For $\mathbf{N}$, the small and big allowed bundles are $E_p \oplus E_0$ and $E_p \oplus E_c \oplus E_0$. For $-\mathbf{N}$ they are $E_m \oplus E_0$ and $E_m \oplus E_c \oplus E_0$, with the sides interchanged. At S , $\ker \mathbf{N} = E_c \oplus E_0$, which belongs to both corresponding big bundles. All required inclusions and side identifications follow. $\square$

Only the normal matrix is put into a *local boundary* block form. No simultaneous diagonalization of $A_1, \dots, A_n$ in the domain occurs. The common special case is $\kappa = h_0$. Even $\kappa = h_0^{2k+1}$ is allowed: the proof needs the bundle geometry, not a nonzero derivative of each eigenvalue. Oppositely directed zero crossings at the same interface need not satisfy the nested condition and are not automatically included.

LEMMA A.2 (Imported weak–strong principle). *For a smooth symmetric first-order operator $L = \sum_j a_j \partial_j + b$ on a smooth bounded domain, equip the boundary with a maximal nonnegative space M for a_ν . Assume the rank is constant off smooth embedded boundary interfaces and M admits the local nested small/big extensions containing $\ker a_\nu$ as in Assumption 1.1. Every L^2 weak solution of $(\lambda + L)w = f$, with this homogeneous boundary condition, is strong in the graph sense: there exist smooth w_k satisfying that condition off the interfaces with*

$$w_k \longrightarrow w, \quad (\lambda + L)w_k \longrightarrow f \quad \text{in } L^2.$$

Here *weak* means testing against smooth functions in the adjoint boundary space $M^* = (a_\nu M)^\perp$.

Source and meaning. We use the classical weak–strong theorem of Rauch [28] in the form explicitly stated in Nishitani–Takayama [27, Section 1, pp. 168–169] and Secchi [32, Section 1]. The weak formulation is also explicitly given in [27, p. 170]. The graph approximation with exact homogeneous boundary constraints is illustrated in [31, Proposition 1.3 and its proof], which cites Rauch’s Corollary 2.5. That proposition’s *special* sign-changing class is not being mistaken for the general mixed-sign theorem. No H^1 -regularity conclusion, flow-sign condition for higher regularity, or small-coupling condition is imported. This classical result is the one non-elementary deterministic boundary input. We invoke the nested-bundle formulations stated in

these references; we do not assert weak–strong equivalence for arbitrary variable-rank boundary geometry.

For $n = 1$ under Assumption 1.1, the weak–strong statement also follows directly: invertibility of A_1 and the weak equation give $w' = A_1^{-1}(f - (\lambda + b)w) \in L^2$, hence $w \in H^1(G)$. Integration by parts identifies the endpoint boundary values from the adjoint tests. Smooth H^1 approximation with those endpoint values yields graph approximation. This applies to both orientations below.

For our $\mathbf{N}$, $E_{\geq 0}(\mathbf{N})$ is maximal nonnegative and

$$(A.2) \quad (\mathbf{N}E_{\geq 0}(\mathbf{N}))^\perp = E_{\leq 0}(\mathbf{N}).$$

Indeed, $\mathbf{N}E_{\geq 0}(\mathbf{N}) = E_{> 0}(\mathbf{N})$. The reversed identity holds for $-\mathbf{N}$. Thus Lemma A.2 applies in both orientations under Assumption 1.1, including the noncommuting examples of Section 2.

A.2. Both resolvent ranges and the exact adjoint domain. Define the smooth cores

$$\mathcal{D}_+ = \{y \in C^\infty(\overline{G}; \mathbb{R}^N) : \mathbf{P}_- \gamma y = 0\}, \quad \mathcal{D}_- = \{z \in C^\infty(\overline{G}; \mathbb{R}^N) : \mathbf{P}_+ \gamma z = 0\},$$

where constraints can equivalently be imposed off S . Define the graph closures

$$(A.3) \quad \mathcal{A} = \overline{(-D, \mathcal{D}_+)}, \quad \mathcal{B} = \overline{(D + \mathbf{C}, \mathcal{D}_-)}.$$

These differential operators are closable, by distributional convergence, and densely defined because their cores contain $C_c^\infty(G; \mathbb{R}^N)$. Let $c = \|\mathbf{C}\|_{L^\infty(G; \mathcal{L}(\mathbb{R}^N))}$.

LEMMA A.3 (One-sided flux traces). *There are continuous graph-norm maps*

$$\mathcal{C}_+ : D(\mathcal{A}) \rightarrow L^2(\Gamma; \mathbb{R}^N), \quad \mathcal{C}_- : D(\mathcal{B}) \rightarrow \mathcal{U}$$

which on the respective smooth cores are $\mathbf{N}_+^{1/2} \gamma y$ and $\mathbf{N}_-^{1/2} \gamma z$. They satisfy

$$(A.4) \quad 2 \langle \mathcal{A}y, y \rangle_H = -\|\mathcal{C}_+ y\|_{L^2(\Gamma)}^2 + \langle \mathbf{C}y, y \rangle_H,$$

$$(A.5) \quad 2 \langle \mathcal{B}z, z \rangle_H = -\|\mathcal{C}_- z\|_{\mathcal{U}}^2 + \langle \mathbf{C}z, z \rangle_H.$$

Furthermore $\mathcal{B} \subseteq \mathcal{A}^*$ and $\mathcal{A} \subseteq \mathcal{B}^*$.

Proof. Integration by parts on the cores gives the two energy identities. For example,

$$\left\| \mathbf{N}_-^{1/2} \gamma z \right\|_{\mathcal{U}}^2 \leq c \|z\|_H^2 + 2 \|(D + \mathbf{C})z\|_H \|z\|_H.$$

Apply this to differences of a graph-approximating sequence. The flux traces are Cauchy in the indicated Hilbert spaces, independent of the sequence, and the identities pass to the limit. On the smooth cores the cross-boundary form $\int_\Gamma y^\top \mathbf{N}z$ vanishes, since y is in the nonnegative space and z in the nonpositive space. The cross Green identity therefore passes to the graph closures and gives the two inclusions. This argument asserts no trace theorem on the unrestricted maximal graph space. $\square$

PROPOSITION A.1 (Range and adjoint theorem). *Under Assumption 1.1, for every real $\lambda > c/2$,*

$$(A.6) \quad \text{Ran}(\lambda - \mathcal{A}) = H, \quad \text{Ran}(\lambda - \mathcal{B}) = H, \quad \left\| (\lambda - \mathcal{A})^{-1} \right\|, \left\| (\lambda - \mathcal{B})^{-1} \right\| \leq (\lambda - c/2)^{-1}.$$

Consequently

$$(A.7) \quad \mathcal{A}^* = \mathcal{B}, \quad \mathcal{B}^* = \mathcal{A},$$

and both generate strongly continuous semigroups of norm at most $e^{ct/2}$.

Proof. We prove the first range statement without assuming any identity of adjoint domains. Given $f \in H$, for $v \in \mathcal{D}_-$, (A.5) implies

$$\|(\lambda - D - C)v\|_H \geq (\lambda - c/2) \|v\|_H.$$

The functional on $(\lambda - D - C)\mathcal{D}_-$ defined by

$$(\lambda - D - C)v \longmapsto \langle f, v \rangle_H$$

is well-defined and has norm at most $(\lambda - c/2)^{-1} \|f\|_H$. Hahn–Banach and Riesz give $u \in H$ such that

$$(A.8) \quad \langle u, (\lambda - D - C)v \rangle_H = \langle f, v \rangle_H \quad (v \in \mathcal{D}_-).$$

By (A.2), this is exactly the weak formulation of $(\lambda + D)u = f$ with boundary space $E_{\geq 0}(\mathbf{N})$. The shifted operator is symmetric positive because $2\lambda I - C \geq (2\lambda - c)I$. The hypotheses of Lemma A.2 hold by Assumption 1.1. Hence there are $u_k \in \mathcal{D}_+$ with $u_k \rightarrow u$ and $(\lambda + D)u_k \rightarrow f$ in H . Thus $u \in D(\mathcal{A})$ and $(\lambda - \mathcal{A})u = f$. The energy inequality gives injectivity and the stated inverse bound.

For the second range statement apply the identical construction to $L = -D - C$, whose normal matrix is $-\mathbf{N}$, with maximal nonnegative space $E_{\leq 0}(\mathbf{N})$. Its formal adjoint is D and its shifted symmetric part is again $2\lambda I - C$. The second orientation in Assumption 1.1 is exactly what is required here. This proves both parts of (A.6).

The energy identities show that $\mathcal{A} - cI/2$ and $\mathcal{B} - cI/2$ are dissipative; (A.6) and the Lumer–Phillips theorem give the semigroups. For completeness, let $z \in D(\mathcal{A}^*)$, and choose $w \in D(\mathcal{B})$ with $(\lambda - \mathcal{B})w = (\lambda - \mathcal{A}^*)z$. The inclusion in Lemma A.3 yields $(\lambda - \mathcal{A}^*)(z - w) = 0$. Since $\text{Ran}(\lambda - \mathcal{A}) = H$, the kernel of $\lambda - \mathcal{A}^*$ is zero, so $z = w$. Thus $D(\mathcal{A}^*) \subseteq D(\mathcal{B})$, proving the first equality in (A.7); taking adjoints proves the second. No energy estimate was applied to an unrecognized weak adjoint domain. $\square$

LEMMA A.4 (Scalar multiplier invariance). *For $a \in C^1(\overline{G})$, multiplication by a preserves $D(\mathcal{A})$ and $D(\mathcal{A}^*)$. In particular,*

$$\mathcal{A}^*(az) = a\mathcal{A}^*z + \sum_j A_j(\partial_j a)z, \quad \mathcal{C}_-(az) = a|_\Gamma \mathcal{C}_-z.$$

For real a ,

$$(A.9) \quad 2 \langle az, \mathcal{A}^*z \rangle_H = - \int_\Gamma a |\mathcal{C}_-z|^2 d\Gamma + \int_G z^\top \left(aC - \sum_j A_j \partial_j a \right) z dx.$$

Proof. For smooth scalar a , multiplication preserves the boundary cores, and the commutator is a bounded zeroth-order operator. Graph approximation proves the claim for domain elements. Approximate a C^1 scalar by smooth scalars in the C^1 norm to obtain the stated regularity. Weighted integration by parts on the core followed by graph convergence proves (A.9). Scalar multiplication is essential: no corresponding assertion for arbitrary smooth matrix multipliers on these graph spaces is used. $\square$

A.3. No universal graph-space trace and no global frame. On a flat boundary patch with inward coordinate $r > 0$ and tangential coordinates s, q , take

$$A_s = I_2, \quad A_r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The functions $e^{-kr}(\cos kq, \sin kq)^\top$ satisfy $(A_r \partial_r + A_q \partial_q)u = 0$. Multiply by fixed smooth cutoffs, one equal to one near $r = 0$, and by $\sqrt{k}$. Their L^2 graph norms remain bounded, whereas their boundary L^2 norms grow like $\sqrt{k}$. This invalidates an unrestricted graph-norm-to- L^2 trace assertion, not Lemma A.3, whose domains impose one-sided homogeneous boundary conditions. The proof also never assumes a global smooth eigenvector frame of $\mathbf{N}$.

Remark A.1. The range proof establishes both the forward and adjoint generators with the norm bound $e^{ct/2}$, which is sufficient for the stochastic evolution results used in Section 3.

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