

Stability and regularization for ill-posed Cauchy problem of a stochastic parabolic differential equation

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Abstract

In this paper, we investigate an ill-posed Cauchy problem involving a stochastic parabolic equation. We first establish a Carleman estimate for this equation. Leveraging this estimate, we are able to derive the conditional stability and convergence rate of the Tikhonov regularization method for the aforementioned ill-posed Cauchy problem. We also retain archived deterministic-kernel boundary-fitting experiments. These experiments do not discretize the stochastic Tikhonov functional proved here and do not validate its convergence theorem.

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1 Introduction

To beginning with, we introduce some notations concerning stochastic analysis. More details for these can be found in [24].

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined. Let H be a Fréchet space. We denote by $L^2_{\mathbb{F}}(0, T; H)$ the Fréchet space consisting of all H -valued $\mathbb{F}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty$; by $L^\infty_{\mathbb{F}}(0, T; H)$ the Fréchet space consisting of all H -valued $\mathbb{F}$ -adapted bounded processes; and by $L^2_{\mathbb{F}}(\Omega; C([0, T]; H))$ the Fréchet space consisting of all H -valued $\mathbb{F}$ -adapted continuous processes X such that $\mathbb{E}(|X|^2_{C([0, T]; H)}) < \infty$. All of the above spaces are equipped with the canonical quasi-norms. Furthermore, all of the above spaces are Banach spaces equipped with the canonical norms, if H is a Banach space.

For simplicity, we use the notation $y_i \equiv y_i(x) = \frac{\partial y(x)}{\partial x_i}$, where x_i is the i -th coordinate of a generic point $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$. In a similar manner, we use the notation z_i, v_i , etc. for the partial derivatives of z and v with respect to x_i . Also, we denote the scalar product in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$, and use C to denote a generic positive constant independent of the solution y , which may change from line to line.

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Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with the C^4 boundary ∂G , and Γ be a given nonempty open subset of ∂G . We denote $\nu(x) = (\nu^1(x), \dots, \nu^n(x))$ as the unit outward normal vector (of G) at $x \in \partial G$.

Let $a_1 \in L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(G; \mathbb{R}^n))$, $a_2 \in L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(G))$ and $a_3 \in L_{\mathbb{F}}^{\infty}(0, T; W^{1,\infty}(G))$. We assume $f \in L_{\mathbb{F}}^2(0, T; L^2(G))$, $g_1 \in L_{\mathbb{F}}^2(\Omega; H^1(0, T; L^2(\Gamma))) \cap L_{\mathbb{F}}^2(0, T; H^1(\Gamma))$, $g_2 \in L_{\mathbb{F}}^2(0, T; L^2(\Gamma))$, and $a^{ij} : \Omega \times [0, T] \times \overline{G} \rightarrow \mathbb{R}^{n \times n}$ ($i, j = 1, 2, \dots, n$) satisfies

(H1) $a^{ij} \in L_{\mathbb{F}}^{\infty}(\Omega; C^1([0, T]; W^{2,\infty}(G)))$ and $a^{ij} = a^{ji}$; in particular, the coefficients and their indicated derivatives have a common deterministic bound. All constants below may depend on this bound.

(H2) $\sum_{i,j=1}^n a^{ij}(\omega, t, x) \xi^i \xi^j \geq s_0 |\xi|^2$, $(\omega, t, x, \xi) \equiv (\omega, t, x, \xi^1, \dots, \xi^n) \in \Omega \times (0, T) \times G \times \mathbb{R}^n$ for some $s_0 > 0$.

Now, the Cauchy problem of the forward stochastic parabolic differential equation can be described as follows.

$$\begin{cases} dy - \sum_{i,j=1}^n (a^{ij} y_i)_j dt = [\langle a_1, \nabla y \rangle + a_2 y + f] dt + a_3 y dW(t) & \text{in } (0, T) \times G, \\ y = g_1 & \text{on } (0, T) \times \Gamma, \\ \frac{\partial y}{\partial \nu} = g_2 & \text{on } (0, T) \times \Gamma. \end{cases} \quad (1.1)$$

Let $\mathbf{G}_{\Gamma} \triangleq \{G' \subset G \mid \partial G' \cap \partial G \subset \Gamma\}$ and

$$H_{\Gamma}^2(G) \triangleq \{\eta \in H_{loc}^2(G) : \eta|_{G'} \in H^2(G'), \quad \forall G' \in \mathbf{G}_{\Gamma}, \quad \eta|_{\Gamma} = g_1, \quad \partial_{\nu} \eta|_{\Gamma} = g_2\}.$$

The aim of this paper is to study the Cauchy Problem with the Lateral Data for the stochastic parabolic differential equation:

Problem (CP) Find a function $y \in L_{\mathbb{F}}^2(\Omega; C([0, T]; L_{loc}^2(G))) \cap L_{\mathbb{F}}^2(0, T; H_{\Gamma}^2(G))$ that satisfies system (1.1).

Remark 1.1. We assume that $g_1 \in L_{\mathbb{F}}^2(\Omega; H^1(0, T; L^2(\Gamma))) \cap L_{\mathbb{F}}^2(0, T; H^1(\Gamma))$. This is because to handle the term $\mathbb{E} \int_0^T \int_{\Gamma} \sum_{i,j=1}^n a^{ij} v_i \nu^j dv d\Gamma$ in (2.28), we need y belongs to $L_{\mathbb{F}}^2(\Omega; H^1(0, T; L^2(\Gamma)))$. This difficulty also arises even in the one-dimensional case. It would be quite interesting to study the general case where $g_1 \in L_{\mathbb{F}}^2(0, T; H^1(\Gamma))$.

In certain applications involving diffusive, thermal, and heat transfer problems, measuring some boundary data can be challenging. For instance, in nuclear reactors and steel furnaces in the steel industry, the interior boundary can be difficult to measure. Similarly, the use of liquid crystal thermography in visualization can pose the same problem. To address this issue, engineers attempt to reconstruct the status of the boundary using measurements from the accessible boundary. This, in turn, creates the Cauchy problem for parabolic equations. Due to the requirements in real applications, numerous researchers have focused on solving the Cauchy problem of parabolic differential equations, particularly on the inverse problems for deterministic parabolic equations, as seen in [14–17, 19, 22, 31, 33], and their respective references. Among these works, some have studied the identification of coefficients in parabolic equations through lateral Cauchy observations, such as

uniqueness and stability [14, 19, 31, 33], and numerical reconstruction [15], assuming that the initial value is known. Meanwhile, the determination of the initial value has been considered in [16, 17, 22] under the assumption that all coefficients in the governing equation are known.

Stochastic parabolic equations have a wide range of applications for simulating various behaviors of stochastic models which are utilized in numerous fields such as random growth of bacteria populations, propagation of electric potential in neuron models, and physical systems that are subject to thermal fluctuations (e.g., [18, 24]). In addition, they can be considered as simplified models for complex phenomena such as turbulence, intermittence, and large-scale structure (e.g., [8]). Given the significant applications of these models, stochastic parabolic equations have been extensively studied in both physics and mathematics. Therefore, it is natural to study the Cauchy problem of stochastic parabolic equations in such situations. However, due to the complexity of stochastic parabolic equations, some tools become ineffective for solving these problems. Thus, the research on inverse problems for stochastic parabolic differential equations is relatively scarce. In [1], the author proved backward uniqueness of solutions to stochastic semilinear parabolic equations and tamed Navier-Stokes equations driven by linearly multiplicative Gaussian noises, via the logarithmic convexity property known to hold for solutions to linear evolution equations in Hilbert spaces with self-adjoint principal parts. In [11, 27], authors studied inverse random source problems for the stochastic time fractional diffusion equation driven by a spatial Gaussian random field, proving the uniqueness and representation for the inverse problems, as well as proposing a numerical method using Fourier methods and Tikhonov regularization. Carleman estimates play an important role in the study of inverse problems for stochastic parabolic differential equations, such as inverse source problems [23, 35], determination of the history for stochastic diffusion processes [4, 23, 30, 34], and unique continuation properties [6, 32]. We refer the reader to [25] for a survey on some recent advances in Carleman estimates and their applications to inverse problems for stochastic partial differential equations.

In this paper, our objective is to solve **Problem (CP)**, i.e., we aim to retrieve the solution of equation (1.1) with observed data from the lateral boundary. To this end, we first prove the conditional stability based on a new Carleman estimate for the stochastic parabolic equation (1.1). Then we construct a Tikhonov functional for the Cauchy problem based on the Tikhonov regularization strategy and prove the uniqueness of the minimizer of the Tikhonov functional, as well as the convergence rate for the optimization problem using variational principles, Riesz theorems, and Carleman estimates established previously.

The numerical section records a deterministic-kernel boundary-fitting scheme and archived plots. For multiplicative Brownian noise, a finite sum of deterministic heat kernels with time-independent coefficients does not in general satisfy the stochastic equation. Consequently, this scheme must be distinguished from the stochastic Tikhonov functional analyzed in Section 3; no discrete convergence statement for the archived scheme is asserted here.

The strong solution of equation (1.1) is useful in the proof of convergence rate of regularization method. Thus, we recall it here.

Definition 1.1. We call $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_{loc}(G))) \cap L^2_{\mathbb{F}}(0, T; H^2_1(G))$ a solution to the equation (1.1) if for any $t \in [0, T]$ and a.e. $x \in G' \in \mathbf{G}_T$, it holds

$$\begin{aligned} & y(t, x) - y(0, x) \\ &= \int_0^t \left\{ \sum_{i,j=1}^n (a^{ij}(s, x) y_i(s, x))_j + [\langle a_1(s, x), \nabla y(s, x) \rangle + a_2(s, x) y(s, x) + f(s, x)] \right\} ds \\ &+ \int_0^t a_3(s, x) y(s, x) dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.2}$$

It should be noted that the assumption regarding the solution, namely that $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_{loc}(G))) \cap L^2_{\mathbb{F}}(0, T; H^2_{\Gamma}(G))$, implies a higher degree of smoothness than is strictly necessary for establishing Hölder stability of the Cauchy problem. However, this additional smoothness is required in order to facilitate the regularization process.

The remainder of this paper is organized as follows. Section 2 presents a proof of a Hölder-type conditional stability result, along with the establishment of the Carleman estimate as a particularly useful tool in this proof. The regularization method, using the Tikhonov regularization strategy, is introduced in Section 3 and we showcase both the uniqueness and convergence rate of the regularization solution. Section 4 records archived kernel-based boundary-fitting illustrations and states their limitations relative to the stochastic theory.

2 Conditional Stability

In this section, we prove a stability estimate for the Cauchy problem.

Theorem 2.1. (*Hölder stability estimate*). *For any given $G' \subset\subset G$ and $\varepsilon \in (0, \frac{T}{2})$, there exists $\delta_0 \in (0, 1)$, $\beta \in (0, 1)$ and a constant $C > 0$ such that if for $\delta \in (0, \delta_0)$,*

$$\max\{|f|_{L^2_{\mathbb{F}}(0, T; L^2(G))}, |g_1|_{L^2_{\mathbb{F}}(0, T; H^1(\Gamma))}, |g_{1,t}|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}, |g_2|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}\} \leq \delta, \quad (2.1)$$

then

$$\mathbb{E} \int_{\varepsilon}^{T-\varepsilon} \int_{G'} (y^2 + |\nabla y|^2) dx dt \leq C(1 + |y|_{L^2_{\mathbb{F}}(0, T; H^1(G))})^2 \delta^{2\beta}, \quad \forall \delta \in (0, \delta_0). \quad (2.2)$$

We first recall the following exponentially weighted energy identity, which will play a key role in the sequel.

Lemma 2.1. [29, Theorem 3.1] *Let n be a positive integer,*

$$b^{ij} = b^{ji} \in L^2_{\mathbb{F}}(\Omega; C^1([0, T]; W^{2,\infty}(\mathbb{R}^n))), \quad i, j = 1, 2, \dots, n, \quad (2.3)$$

and $\ell \in C^{1,4}((0, T) \times \mathbb{R}^n)$, $\Psi \in C^{1,2}((0, T) \times \mathbb{R}^n)$. Assume u is an $H^2(\mathbb{R}^n)$ -valued continuous semi-martingale. Set $\theta = e^{\ell}$ and $v = \theta u$. Then for a.e. $x \in \mathbb{R}^n$ and $\mathbb{P}$ -a.s. $\omega \in \Omega$,

$$\begin{aligned} & 2 \int_0^T \theta \left[- \sum_{i,j=1}^n (b^{ij} v_i)_j + \mathcal{A}v \right] \left[du - \sum_{i,j=1}^n (b^{ij} u_i)_j dt \right] + 2 \int_0^T \sum_{i,j=1}^n (b^{ij} v_i dv)_j \\ & + 2 \int_0^T \sum_{i,j=1}^n \left[\sum_{i',j'=1}^n \left(2b^{ij} b^{i'j'} \ell_{i'} v_i v_{j'} - b^{ij} b^{i'j'} \ell_i v_{i'} v_{j'} \right) + \Psi b^{ij} v_i v - b^{ij} \left(\mathcal{A} \ell_i + \frac{\Psi_i}{2} \right) v^2 \right]_j dt \\ & = 2 \int_0^T \sum_{i,j=1}^n \left\{ \sum_{i',j'=1}^n \left[2b^{ij'} \left(b^{i'j} \ell_{i'} \right)_{j'} - \left(b^{ij} b^{i'j'} \ell_{i'} \right)_{j'} \right] - \frac{b_t^{ij}}{2} + \Psi b^{ij} \right\} v_i v_j dt \\ & + \int_0^T \mathcal{B} v^2 dt + 2 \int_0^T \left[- \sum_{i,j=1}^n (b^{ij} v_i)_j + \mathcal{A}v \right] \left[- \sum_{i,j=1}^n (b^{ij} v_i)_j + (\mathcal{A} - \ell_t) v \right] dt \\ & + \left(\sum_{i,j=1}^n b^{ij} v_i v_j + \mathcal{A} v^2 \right) \Big|_0^T - \int_0^T \theta^2 \sum_{i,j=1}^n b^{ij} [(du_i + \ell_i du)(du_j + \ell_j du)] - \int_0^T \theta^2 \mathcal{A}(du)^2, \quad (2.4) \end{aligned}$$

where

$$\begin{cases} \mathcal{A} \triangleq - \sum_{i,j=1}^n (b^{ij} \ell_i \ell_j - b_j^{ij} \ell_i - b^{ij} \ell_{ij}) - \Psi, \\ \mathcal{B} \triangleq 2 \left[\mathcal{A} \Psi - \sum_{i,j=1}^n (\mathcal{A} b^{ij} \ell_i)_j \right] - \mathcal{A}_t - \sum_{i,j=1}^n (b^{ij} \Psi_j)_i. \end{cases} \quad (2.5)$$

In the sequel, for a positive integer p , denote by $O(\mu^p)$ a function of order μ^p for large μ (which is independent of λ); by $O_\mu(\lambda^p)$ a function of order λ^p for fixed μ and for large λ .

Proof of Theorem 2.1. We borrow some idea from [21].

Choose a nonempty smooth boundary patch Γ_0 with $\overline{\Gamma_0} \subset \Gamma$. By a smooth outward deformation supported in Γ_0 , choose a bounded C^4 domain $\tilde{G}$ containing G , with a nonempty exterior open part, such that $\partial G \cap \tilde{G} \subset \Gamma$. Then we have

$$G \subset \tilde{G}, \quad \overline{\partial G \cap \tilde{G}} \subset \Gamma, \quad \partial G \setminus \Gamma \subset \partial \tilde{G} \quad \text{and} \quad \tilde{G} \setminus G \text{ contains some nonempty open subset.} \quad (2.6)$$

Let $G_0 \subset \subset \tilde{G} \setminus G$ be an open subdomain. We know that there is a $\psi \in C^4(\tilde{G})$ satisfying (see [29, Lemma 5.1] for example)

$$\begin{cases} \psi > 0 & \text{in } \tilde{G}, \\ \psi = 0 & \text{on } \partial \tilde{G}, \\ |\nabla \psi| > 0 & \text{in } G \subset \tilde{G} \setminus G_0. \end{cases} \quad (2.7)$$

Since $G' \subset \subset G$, we can choose a sufficiently large $N > 0$ such that

$$G' \subset \left\{ x : x \in \tilde{G}, \psi(x) > \frac{4}{N} |\psi|_{L^\infty(\tilde{G})} \right\} \cap G. \quad (2.8)$$

Further, letting $\rho = \varepsilon/\sqrt{2} > 0$, there exists a positive number c such that

$$c\rho^2 < |\psi|_{L^\infty(\tilde{G})} < 2c\rho^2. \quad (2.9)$$

Then, define

$$\phi(t, x) = \psi(x) - c(t - t_0)^2, \quad \alpha(t, x) = e^{\mu\phi(t, x)} \quad (2.10)$$

for a fixed $t_0 \in [\sqrt{2}\rho, T - \sqrt{2}\rho]$, and denote $\beta_k = \beta_k(\mu) = e^{\mu \left[\frac{k}{N} |\psi|_{L^\infty(\tilde{G})} - \frac{c\rho^2}{N} \right]}$, $k = 1, 2, 3, 4$. Set

$$Q_k = \{(t, x) : x \in \tilde{G}, \alpha(x, t) > \beta_k\}, \quad k = 1, 2, 3, 4. \quad (2.11)$$

Clearly, Q_k is independent of μ . Moreover, $\psi(x) > \frac{4}{N} |\psi|_{L^\infty(\tilde{G})}$ for any $(t, x) \in \left(t_0 - \frac{\rho}{\sqrt{N}}, t_0 + \frac{\rho}{\sqrt{N}}\right) \times G'$, and thus

$$\psi(x) - c(t - t_0)^2 > \frac{4}{N} |\psi|_{L^\infty(\tilde{G})} - \frac{c\rho^2}{N}.$$

Hence, we see $\alpha(t, x) > \beta_4$ in $(t, x) \in \left(t_0 - \frac{\rho}{\sqrt{N}}, t_0 + \frac{\rho}{\sqrt{N}}\right) \times G'$, and thus $Q_4 \supset \left(t_0 - \frac{\rho}{\sqrt{N}}, t_0 + \frac{\rho}{\sqrt{N}}\right) \times G'$. On the other hand, for any $(t, x) \in Q_1$,

$$\psi(x) - c(t - t_0)^2 > \frac{1}{N} |\psi|_{L^\infty(\tilde{G})} - \frac{c\rho^2}{N}.$$

This yields

$$|\psi|_{L^\infty(\tilde{G})} - \frac{1}{N}|\psi|_{L^\infty(\tilde{G})} + \frac{c\rho^2}{N} > c(t - t_0)^2.$$

Together with (2.9) we have

$$2\left(1 - \frac{1}{N}\right)c\rho^2 + \frac{c\rho^2}{N} > c(t - t_0)^2.$$

Therefore, we conclude

$$\left(t_0 - \frac{\rho}{\sqrt{N}}, t_0 + \frac{\rho}{\sqrt{N}}\right) \times G' \subset Q_1 \subset (t_0 - \sqrt{2}\rho, t_0 + \sqrt{2}\rho) \times \overline{G}. \quad (2.12)$$

Next, for any $(t, x) \in \partial Q_1$, we know $x \in \overline{G}$ and $\alpha(t, x) \geq \beta_1$. In above set, if $x \in G$, $\alpha(t, x) = \beta_1$, and if $x \in \partial G$, it must hold $x \in \Gamma$. Indeed, if $x \in \partial G \setminus \Gamma$, from $\partial G \setminus \Gamma \subset \partial \tilde{G}$ we get $\psi(x) = 0$. On the other hand, since $\alpha(t, x) \geq \beta_1$, we know

$$\psi(x) - c(t - t_0)^2 = -c(t - t_0)^2 \geq \frac{1}{N}|\psi|_{L^\infty(\tilde{G})} - \frac{c\rho^2}{N}.$$

And thus

$$0 \leq c(t - t_0)^2 \leq \frac{1}{N}(c\rho^2 - |\psi|_{L^\infty(\tilde{G})}),$$

which contradicts (2.9). Therefore, we have

$$\partial Q_1 = \Sigma_1 \cup \Sigma_2, \quad (2.13)$$

where

$$\Sigma_1 \subset [0, T] \times \Gamma, \quad \Sigma_2 = \{(t, x) \in [0, T] \times G : x \in G, \alpha(t, x) = \beta_1\}. \quad (2.14)$$

Let η be a smooth cutoff on $[0, T] \times \overline{G}$, supported in Q_2 relative to this cylinder, such that $0 \leq \eta \leq 1$ and $\eta = 1$ in Q_3 . For any y solving (1.1), let $z = \eta y$, then z solves

$$\begin{cases} dz - \sum_{i,j=1}^n (a^{ij} z_i)_j dt = (\langle a_1, \nabla z \rangle + a_2 z + \tilde{f} + \eta f) dt + a_3 z dW(t) & \text{in } Q_1, \\ z = \frac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma_2. \end{cases} \quad (2.15)$$

Here $\tilde{f} = - \sum_{i,j=1}^n (a_j^{ij} \eta_i y + a^{ij} \eta_{ij} y + 2a^{ij} \eta_i y_j) - \langle a_1, \nabla \eta \rangle y + \eta_t y \in L^2_{\mathbb{F}}(0, T; L^2(G))$ and $\tilde{f}$ is supported in $Q_2 \setminus Q_3$.

Applying Lemma 2.1 to (2.15) with $v = \theta z$, $b^{ij} = a^{ij}$, $\ell = \lambda \alpha$ and $\Psi = 2 \sum_{i,j=1}^n a^{ij} \ell_{ij}$, integrating

it in G and taking mean value, noting that z is supported in Q_1 , we see

$$\begin{aligned}
& 2\mathbb{E} \int_{Q_1} \theta \left[- \sum_{i,j=1}^n (a^{ij} v_i)_j + \mathcal{A}v \right] \left[dz - \sum_{i,j=1}^n (a^{ij} z_i)_j dt \right] dx + 2\mathbb{E} \int_{Q_1} \sum_{i,j=1}^n (a^{ij} v_i dv)_j dx \\
& + 2\mathbb{E} \int_{Q_1} \sum_{i,j=1}^n \left[\sum_{i',j'=1}^n \left(2a^{ij} a^{i'j'} \ell_{i'} v_i v_{j'} - a^{ij} a^{i'j'} \ell_i v_{i'} v_{j'} \right) + \Psi a^{ij} v_i v - a^{ij} \left(\mathcal{A} \ell_i + \frac{\Psi_i}{2} \right) v^2 \right]_j dx dt \\
& \geq 2 \sum_{i,j=1}^n \mathbb{E} \int_{Q_1} c^{ij} v_i v_j dx dt + \mathbb{E} \int_{Q_1} \mathcal{B} v^2 dx dt + \mathbb{E} \int_{Q_1} \left| - \sum_{i,j=1}^n (a^{ij} v_i)_j + \mathcal{A}v \right|^2 dx dt \\
& - \mathbb{E} \int_{Q_1} \theta^2 \sum_{i,j=1}^n a^{ij} (dz_i + \ell_i dz) (dz_j + \ell_j dz) dx - \mathbb{E} \int_{Q_1} \theta^2 \mathcal{A} (dz)^2 dx,
\end{aligned} \tag{2.16}$$

where

$$\begin{cases} \mathcal{A} = - \sum_{i,j=1}^n (a^{ij} \ell_i \ell_j - a_j^{ij} \ell_i - a^{ij} \ell_{ij}) - \Psi, \\ \mathcal{B} = 2 \left[A \Psi - \sum_{i,j=1}^n (A a^{ij} \ell_i)_j \right] - A_t - \sum_{i,j=1}^n (a^{ij} \Psi_j)_i - \ell_t^2, \\ c^{ij} = \sum_{i',j'=1}^n \left[2a^{ij'} (a^{i'j} \ell_{i'})_{j'} - (a^{ij} a^{i'j'} \ell_{i'})_{j'} \right] - \frac{a_t^{ij}}{2} + \Psi a^{ij}. \end{cases} \tag{2.17}$$

Now, we estimate $\mathcal{A}$, $\mathcal{B}$ and c^{ij} .

$$\begin{aligned}
\mathcal{A} &= - \sum_{i,j=1}^n (a^{ij} \ell_i \ell_j - a_j^{ij} \ell_i - a^{ij} \ell_{ij}) - \Psi \\
&= -\lambda^2 \mu^2 \alpha^2 \sum_{i,j=1}^n a^{ij} \psi_i \psi_j + \lambda \mu \alpha \sum_{i,j=1}^n a_j^{ij} \psi_i - \lambda \mu^2 \alpha \sum_{i,j=1}^n a^{ij} \psi_i \psi_j - \lambda \mu \alpha \sum_{i,j=1}^n a^{ij} \psi_{ij} \\
&= -\lambda^2 \mu^2 \alpha^2 \sum_{i,j=1}^n a^{ij} \psi_i \psi_j + \lambda \alpha O(\mu^2).
\end{aligned} \tag{2.18}$$

In order to estimate $\mathcal{B}$, we first do some computations. For Ψ , we have

$$\Psi = 2 \sum_{i,j=1}^n a^{ij} (\lambda \mu^2 \alpha \psi_i \psi_j + \lambda \mu \alpha \psi_{ij}) = 2 \lambda \mu^2 \alpha \sum_{i,j=1}^n a^{ij} \psi_i \psi_j + \lambda \alpha O(\mu). \tag{2.19}$$

Next, we have

$$\ell_{i'j'} = \lambda \mu^3 \alpha \psi_{i'} \psi_{j'} \psi_j + \lambda \alpha O(\mu^2), \quad \ell_{i'j'ij} = \lambda \mu^4 \alpha \psi_{i'} \psi_{j'} \psi_i \psi_j + \lambda \alpha O(\mu^3).$$

Therefore, we get

$$\Psi_j = 2 \sum_{i',j'=1}^n (a^{i'j'} \ell_{i'j'})_j = 2 \sum_{i',j'=1}^n (a_j^{i'j'} \ell_{i'j'} + a^{i'j'} \ell_{i'j'j}) = 2 \lambda \mu^3 \alpha \sum_{i',j'=1}^n a^{i'j'} \psi_{i'} \psi_{j'} \psi_j + \lambda \alpha O(\mu^2),$$

and

$$\Psi_{ij} = 2 \sum_{i',j'=1}^n (a_{ij}^{i'j'} \ell_{i'j'} + a_{i'j'}^{ij} \ell_{i'j'ij} + a_i^{i'j'} \ell_{i'j'j} + a_j^{i'j'} \ell_{i'j'i}) = 2\lambda\mu^4\alpha \sum_{i',j'=1}^n a^{i'j'} \psi_{i'} \psi_{j'} \psi_i \psi_j + \lambda\alpha O(\mu^3).$$

Hence, we find

$$- \sum_{i,j=1}^n (a^{ij} \Psi_j)_i = - \sum_{i,j=1}^n (a_i^{ij} \Psi_j + a^{ij} \Psi_{ji}) = -2\lambda\mu^4\alpha \left(\sum_{i,j=1}^n a^{ij} \psi_i \psi_j \right)^2 + \lambda\alpha O(\mu^3). \quad (2.20)$$

Further, from (2.18) and (2.19), we have

$$\mathcal{A}\Psi = -2\lambda^3\mu^4\alpha^3 \left(\sum_{i,j=1}^n a^{ij} \psi_i \psi_j \right)^2 + \lambda^3\alpha^3 O(\mu^3) + \lambda^2\alpha^2 O(\mu^4). \quad (2.21)$$

From the definition of A , we find

$$\begin{aligned} \mathcal{A}_j &= - \sum_{i',j'=1}^n (a_j^{i'j'} \ell_{i'} \ell_{j'} + 2a^{i'j'} \ell_{i'} \ell_{j'j} - a_{j'j}^{i'j'} \ell_{i'} - a_{j'}^{i'j'} \ell_{i'j} + a_j^{i'j'} \ell_{i'j'} + a^{i'j'} \ell_{i'j'j}) \\ &= - \sum_{i',j'=1}^n (2a^{i'j'} \ell_{i'} \ell_{j'j} + a^{i'j'} \ell_{i'j'j}) + (\lambda\alpha + \lambda^2\alpha^2) O(\mu^2) \\ &= -2\lambda^2\mu^3\alpha^2 \sum_{i',j'=1}^n a^{i'j'} \psi_{i'} \psi_{j'} \psi_j + O_\mu(\lambda) + \lambda^2\alpha^2 O(\mu^2). \end{aligned}$$

Hence, we see

$$\sum_{i,j=1}^n \mathcal{A}_j a^{ij} \ell_i = -2\lambda^3\mu^4\alpha^3 \left(\sum_{i,j=1}^n a^{ij} \psi_i \psi_j \right)^2 + O_\mu(\lambda^2) + \lambda^3\alpha^3 O(\mu^3),$$

which leads to

$$\begin{aligned} \sum_{i,j=1}^n (\mathcal{A} a^{ij} \ell_i)_j &= \sum_{i,j=1}^n \mathcal{A}_j a^{ij} \ell_i + \mathcal{A} \sum_{i,j=1}^n (a_j^{ij} \ell_i + a^{ij} \ell_{ij}) \\ &= -3\lambda^3\mu^4\alpha^3 \left(\sum_{i,j=1}^n a^{ij} \psi_i \psi_j \right)^2 + O_\mu(\lambda^2) + \lambda^3\alpha^3 O(\mu^3). \end{aligned} \quad (2.22)$$

Further, we have

$$\begin{aligned} \mathcal{A}_t &= - \sum_{i,j=1}^n \left(a^{ij} \ell_i \ell_j - a_j^{ij} \ell_i + a^{ij} \ell_{ij} \right)_t \\ &= - \sum_{i,j=1}^n \left[a^{ij} (\ell_i \ell_j)_t - a_j^{ij} \ell_{it} + a^{ij} \ell_{ijt} \right] + \lambda^2\alpha^2 O(\mu^2) + \lambda\alpha O(\mu^2) + \lambda\alpha T O(\mu^3) \\ &= \lambda^2\alpha^2 T O(\mu^3) + \lambda^2\alpha^2 O(\mu^2) + \lambda\alpha O(\mu^2) + \lambda\alpha T O(\mu^3). \end{aligned} \quad (2.23)$$

Finally, it holds

$$\ell_t^2 = \lambda^2\mu^2\alpha^2\phi_t^2 = O_\mu(\lambda^2). \quad (2.24)$$

From the definition of B (see (2.17)), and combining (2.20)–(2.24), we have

$$\begin{aligned}\mathcal{B} &= -4\lambda^3\mu^4\alpha^3\left(\sum_{i,j=1}^n a^{ij}\psi_i\psi_j\right)^2 + 6\lambda^3\mu^4\alpha^3\left(\sum_{i,j=1}^n a^{ij}\psi_i\psi_j\right)^2 - 2\lambda\mu^4\alpha\left(\sum_{i,j=1}^n a^{ij}\psi_i\psi_j\right)^2 \\ &\quad + \lambda^3\alpha^3O(\mu^3) + O_\mu(\lambda^2) \\ &= 2\lambda^3\mu^4\alpha^3\left(\sum_{i,j=1}^n a^{ij}\psi_i\psi_j\right)^2 + \lambda^3\alpha^3O(\mu^3) + O_\mu(\lambda^2).\end{aligned}$$

Hence, we know

$$\mathcal{B} \geq 2s_0^2\lambda^3\mu^4\alpha^3|\nabla\psi|^4 + \lambda^3\alpha^3O(\mu^3) + O_\mu(\lambda^2). \quad (2.25)$$

Now we estimate c^{ij} . By direct computation, we have

$$\begin{aligned}\sum_{i,j=1}^n c^{ij}v_iv_j &= \sum_{i,j=1}^n \left\{ \sum_{i',j'=1}^n \left[2a^{ij'}a^{i'j}\ell_{i'j'} + a^{ij}a^{i'j'}\ell_{i'j'} + 2a^{ij'}a^{i'j}\ell_{i'} - (a^{ij}a^{i'j'})_{j'}\ell_{i'} \right] - \frac{a_t^{ij}}{2} \right\} v_iv_j \\ &= \sum_{i,j=1}^n \left\{ \sum_{i',j'=1}^n \left[2\lambda\mu^2\alpha a^{ij'}a^{i'j}\psi_{i'}\psi_{j'} + \lambda\mu^2\alpha a^{ij}a^{i'j'}\psi_{i'}\psi_{j'} + \lambda\alpha O(\mu) \right] + O(1) \right\} v_iv_j \\ &= 2\lambda\mu^2\alpha\left(\sum_{i,j=1}^n a^{ij}\psi_i\psi_j\right)^2 + \lambda\mu^2\alpha\left(\sum_{i,j=1}^n a^{ij}\psi_i\psi_j\right)\left(\sum_{i,j=1}^n a^{ij}v_iv_j\right) \\ &\quad + \lambda\alpha|\nabla v|^2O(\mu) + O(1)|\nabla v|^2 \\ &\geq [s_0^2\lambda\mu^2\alpha|\nabla\psi|^2 + \lambda\alpha O(\mu) + O(1)]|\nabla v|^2.\end{aligned} \quad (2.26)$$

Owing to that z is a solution to equation (2.15), we find

$$\begin{aligned}& 2\mathbb{E} \int_{Q_1} \theta \left[-\sum_{i,j=1}^n (a^{ij}v_i)_j + \mathcal{A}v \right] \left[dz - \sum_{i,j=1}^n (a^{ij}z_i)_j dt \right] dx \\ &\leq \mathbb{E} \int_{Q_1} \left[-\sum_{i,j=1}^n (a^{ij}v_i)_j + \mathcal{A}v \right]^2 dx dt + \mathbb{E} \int_{Q_1} \theta^2 \left(\langle a_1, \nabla z \rangle + a_2 z + \tilde{f} + \eta f \right)^2 dx dt \\ &\leq \mathbb{E} \int_{Q_1} \left[-\sum_{i,j=1}^n (a^{ij}v_i)_j + \mathcal{A}v \right]^2 dx dt + 3|a_1|_{L^\infty(0,T;L^\infty(G;\mathbb{R}^n))}^2 \mathbb{E} \int_{Q_1} \theta^2 |\nabla z|^2 dx dt \\ &\quad + 3|a_2|_{L^\infty(0,T;L^\infty(G))}^2 \mathbb{E} \int_{Q_1} \theta^2 z^2 dx dt + 6\mathbb{E} \int_{Q_1} \theta^2 (\tilde{f}^2 + \eta^2 f^2) dx dt.\end{aligned} \quad (2.27)$$

It is clear that $\overline{Q_2} \cap \overline{\Sigma_2} = \emptyset$. Hence, $\eta = 0$ on Σ_2 . Recall that $v = \theta\eta y$, $y = g_1$, $\partial_\nu y = g_2$ on Σ_1 and $g_1 \in L_{\mathbb{F}}^2(\Omega; H^1(0,T; L^2(\Gamma))) \cap L_{\mathbb{F}}^2(0,T; H^1(\Gamma))$. Then it holds that

$$\begin{aligned}\mathbb{E} \int_{Q_1} \sum_{i,j=1}^n (a^{ij}v_i dv)_j dx &= \mathbb{E} \int_{\partial Q_1} \sum_{i,j=1}^n a^{ij}v_i \nu^j dv d\Gamma \\ &= \mathbb{E} \int_{\Sigma_1} \sum_{i,j=1}^n a^{ij}v_i \nu^j dv d\Gamma\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \int_{\Sigma_1} \sum_{i,j=1}^n a^{ij} v_i \nu^j (\theta \eta y_t + (\theta \eta)_t y) d\Gamma dt \\
&\leq C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 (|g_2| + |\nabla_{\Gamma} g_1| + \lambda \mu \alpha |g_1|) (|g_{1,t}| + \lambda \mu \alpha |g_1|) d\Gamma dt \\
&\leq C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 (|g_2|^2 + |\nabla_{\Gamma} g_1|^2 + |g_{1,t}|^2 + \lambda^2 \mu^2 \alpha^2 |g_1|^2) d\Gamma dt,
\end{aligned} \tag{2.28}$$

where ∇_{Γ} denotes the tangential gradient on Γ .

By means of $z = \frac{\partial z}{\partial \nu} = 0$ on Σ_2 , we find

$$\begin{aligned}
&\left| \mathbb{E} \int_{Q_1} \sum_{i,j=1}^n \left[\sum_{i',j'=1}^n \left(2a^{ij} a^{i'j'} \ell_{i'} v_i v_{j'} - a^{ij} a^{i'j'} \ell_i v_{i'} v_{j'} \right) + \Psi a^{ij} v_i v - a^{ij} \left(\mathcal{A} \ell_i + \frac{\Psi_i}{2} \right) v^2 \right]_j dx dt \right| \\
&\leq C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 (|g_2|^2 + |\nabla_{\Gamma} g_1|^2 + \lambda^2 \mu^2 \alpha^2 |g_1|^2) d\Gamma dt.
\end{aligned} \tag{2.29}$$

From (2.25) and (2.26), we know that there is a $\mu_0 > 0$ such that for every $\mu \geq \mu_0$, one can find a $\lambda_0(\mu) > 0$ so that for all $\lambda \geq \lambda_0(\mu)$, it holds that

$$\mathbb{E} \sum_{i,j=1}^n \int_{Q_1} c^{ij} v_i v_j dx dt \geq C \lambda \mu^2 \mathbb{E} \int_{Q_1} \alpha |\nabla v|^2 dx dt, \tag{2.30}$$

and

$$\mathbb{E} \int_{Q_1} B v^2 dx dt \geq C \lambda^3 \mu^4 \mathbb{E} \int_{Q_1} \alpha^3 v^2 dx dt. \tag{2.31}$$

Utilizing the fact that z solves (2.15) again, we get

$$\begin{aligned}
&\mathbb{E} \int_{Q_1} \theta^2 \sum_{i,j=1}^n a^{ij} (dz_i + \ell_i dz) (dz_j + \ell_j dz) dx \\
&= \mathbb{E} \int_{Q_1} \theta^2 \sum_{i,j=1}^n a^{ij} \left[(a_3 z)_i (a_3 z)_j + 2 \lambda \mu \alpha \psi_i (a_3 z)_j a_3 z + \lambda^2 \mu^2 \alpha^2 \psi_i \psi_j a_3^2 z^2 \right] dx dt \\
&\leq C |a_3|_{L_{\mathbb{F}}^{\infty}(0,T;W^{1,\infty}(G))}^2 \left(\mathbb{E} \int_{Q_1} \theta^2 (z^2 + |\nabla z|^2) dx dt + \lambda^2 \mu^2 \mathbb{E} \int_{Q_1} \theta^2 \alpha^2 |z|^2 dx dt \right).
\end{aligned} \tag{2.32}$$

By $z_i = \theta^{-1}(v_i - \ell_i v) = \theta^{-1}(v_i - \lambda \mu \alpha \psi_i v)$ and $v_i = \theta(z_i + \ell_i z) = \theta(z_i + \lambda \mu \alpha \psi_i z)$, we get

$$\frac{1}{C} \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \alpha^2 z^2) \leq |\nabla v|^2 + \lambda^2 \mu^2 \alpha^2 v^2 \leq C \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \alpha^2 z^2). \tag{2.33}$$

Combining (2.16), (2.27)–(2.33), we know that there is a $\mu_1 \geq \mu_0$ such that for all $\mu \geq \mu_1$, we can find a $\lambda_1(\mu_1) \geq \lambda_0(\mu_0)$ so that for every $\lambda \geq \lambda_1(\mu_1)$, it holds that

$$\begin{aligned}
&\lambda^3 \mu^4 \mathbb{E} \int_{Q_1} \alpha^3 \theta^2 z^2 dx dt + \lambda \mu^2 \mathbb{E} \int_{Q_1} \alpha \theta^2 |\nabla z|^2 dx dt \\
&\leq C \mathbb{E} \int_{Q_1} \theta^2 (\tilde{f}^2 + f^2) dx dt + C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 |g_2|^2 d\Gamma dt \\
&\quad + C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 (|\nabla_{\Gamma} g_1|^2 + |g_{1,t}|^2 + \lambda^2 \mu^2 \alpha^2 |g_1|^2) d\Gamma dt.
\end{aligned} \tag{2.34}$$

Recalling that $\tilde{f} = -\sum_{i,j=1}^n (a_j^{ij} \eta_i y + a^{ij} \eta_{ij} y + 2a^{ij} \eta_i y_j) - \langle a_1, \nabla \eta \rangle y + \eta_t y$, from (2.34), we find that

$$\begin{aligned} & \lambda^3 \mu^4 \mathbb{E} \int_{Q_1} \alpha^3 \theta^2 z^2 dx dt + \lambda \mu^2 \mathbb{E} \int_{Q_1} \alpha \theta^2 |\nabla z|^2 dx dt \\ & \leq C \mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|y|^2 + |\nabla y|^2) dx dt + C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 |g_2|^2 d\Gamma dt \\ & \quad + C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 (|\nabla_{\Gamma} g_1|^2 + |g_{1,t}|^2 + \lambda^2 \mu^2 \alpha^2 |g_1|^2) d\Gamma dt + C \mathbb{E} \int_Q \theta^2 f^2 dx dt, \end{aligned} \quad (2.35)$$

which, together with $z = y$ in Q_3 , implies that

$$\begin{aligned} & \lambda^3 \mu^4 \mathbb{E} \int_{Q_3} \alpha^3 \theta^2 y^2 dx dt + \lambda \mu^2 \mathbb{E} \int_{Q_3} \alpha \theta^2 |\nabla y|^2 dx dt \\ & \leq C \mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|y|^2 + |\nabla y|^2) dx dt + C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 |g_2|^2 d\Gamma dt \\ & \quad + C \lambda \mu \mathbb{E} \int_0^T \int_{\Gamma} \alpha \theta^2 (|\nabla_{\Gamma} g_1|^2 + |g_{1,t}|^2 + \lambda^2 \mu^2 \alpha^2 |g_1|^2) d\Gamma dt + C \mathbb{E} \int_Q \theta^2 f^2 dx dt. \end{aligned} \quad (2.36)$$

From the choice of Q_3 and (2.8), we know that

$$\begin{aligned} & \lambda^3 \mu^4 \mathbb{E} \int_{Q_3} \alpha^3 \theta^2 y^2 dx dt + \lambda \mu^2 \mathbb{E} \int_{Q_3} \alpha \theta^2 |\nabla y|^2 dx dt \\ & \geq e^{2\lambda\beta_4} \mathbb{E} \int_{t_0 - \frac{\rho}{\sqrt{N}}}^{t_0 + \frac{\rho}{\sqrt{N}}} \int_{G'} (\lambda^3 y^2 + \lambda |\nabla y|^2) dx dt. \end{aligned} \quad (2.37)$$

From the definition of Q_2 and Q_3 , we find

$$\mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|y|^2 + |\nabla y|^2) dx dt \leq e^{2\lambda\beta_3} \mathbb{E} \int_{Q_1} (|y|^2 + |\nabla y|^2) dx dt. \quad (2.38)$$

Now, we fix $\mu = \mu_1$ in β_3, β_4 and θ . Let $\gamma = \max_{(t,x) \in \overline{G} \times (0,T)} \alpha(t, x)$. From (2.36)–(2.38), we obtain that

$$\begin{aligned} & e^{2\lambda\beta_4} \mathbb{E} \int_{t_0 - \frac{\rho}{\sqrt{N}}}^{t_0 + \frac{\rho}{\sqrt{N}}} \int_{G'} (\lambda^3 y^2 + \lambda |\nabla y|^2) dx dt \\ & \leq C \lambda^3 e^{2\lambda\gamma} (|f|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 + |g_1|_{L_{\mathbb{F}}^2(0,T;H^1(\Gamma))}^2 + |g_{1,t}|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma))}^2 + |g_2|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma))}^2) \\ & \quad + C e^{2\lambda\beta_3} \mathbb{E} \int_{Q_1} (|y|^2 + |\nabla y|^2) dx dt, \end{aligned}$$

which implies that for all $\lambda \geq \lambda_1$, it holds that

$$\begin{aligned} & \mathbb{E} \int_{t_0 - \frac{\rho}{\sqrt{N}}}^{t_0 + \frac{\rho}{\sqrt{N}}} \int_{G'} (y^2 + |\nabla y|^2) dx dt \\ & \leq C e^{-2\lambda(\beta_4 - \beta_3)} \mathbb{E} \int_{Q_1} (|y|^2 + |\nabla y|^2) dx dt \\ & \quad + C e^{2\lambda\gamma} (|f|_{L_{\mathbb{F}}^2(0,T;L^2(G))}^2 + |g_1|_{L_{\mathbb{F}}^2(0,T;H^1(\Gamma))}^2 + |g_{1,t}|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma))}^2 + |g_2|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma))}^2). \end{aligned} \quad (2.39)$$

Put $h = \rho/\sqrt{N}$ and $L = T - 2\varepsilon > 0$. Take the centers $t_i = \varepsilon + 2ih$ for $i = 0, \dots, \lfloor L/(2h) \rfloor$, and also take the center $T - \varepsilon$ if it is not already in this list. Their intervals $(t_i - h, t_i + h)$ cover $[\varepsilon, T - \varepsilon]$ up to endpoints. All centers lie in the permitted range, and their number is fixed independently of λ and the data. Summing (2.39) and bounding every cutoff-domain integral by the full-domain norm gives

$$\begin{aligned} \mathbb{E} \int_{\varepsilon}^{T-\varepsilon} \int_{G'} (y^2 + |\nabla y|^2) dx dt &\leq C e^{-2\lambda(\beta_4 - \beta_3)} |y|_{L^2_{\mathbb{F}}(0, T; H^1(G))}^2 \\ &\quad + C e^{2\lambda\gamma} \left(|f|_{L^2_{\mathbb{F}}(0, T; L^2(G))}^2 + |g_1|_{L^2_{\mathbb{F}}(0, T; H^1(\Gamma))}^2 \right. \\ &\quad \left. + |g_1|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}^2 + |g_2|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))}^2 \right). \end{aligned} \quad (2.40)$$

We now balance the terms on the right hand side of (2.40) via choosing $\lambda = \lambda(\delta)$ such that $e^{\lambda\gamma}\delta = e^{-\lambda(\beta_4 - \beta_3)}$. This implies that

$$\lambda = \frac{-\ln \delta}{\gamma + \beta_4 - \beta_3}.$$

Hence, for $\delta \in (0, \delta_0)$, where the number δ_0 is so small that $\frac{-\ln \delta}{\gamma + \beta_4 - \beta_3} \geq \lambda_1$, we have (2.2) with $\beta = \frac{\beta_4 - \beta_3}{\gamma + \beta_4 - \beta_3}$. $\square$

Remark 2.1. The inequality (2.34) is called the Carleman estimate for (1.1).

3 Regularization Method

For $u \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^2(G))$, set

$$\begin{aligned} (Pu)(t, x) &\triangleq u(t, x) - u(0, x) \\ &\quad - \int_0^t \left\{ \sum_{i,j=1}^n (a^{ij}(s, x) u_i(s, x))_j + [\langle a_1(s, x), \nabla u(s, x) \rangle + a_2(s, x) u(s, x)] \right\} ds \\ &\quad - \int_0^t a_3(s, x) u(s, x) dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Let

$$\begin{aligned} \mathcal{H} &\triangleq \{u \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^2(G)) : Pu \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G))), \\ &\quad u|_{(0, T) \times \Gamma} = g_1, \partial_{\nu} u|_{(0, T) \times \Gamma} = g_2\}. \end{aligned}$$

For this section we additionally assume that the global admissible set $\mathcal{H}$ is nonempty and that a lift $F \in \mathcal{H}$ is given. This is an additional feasibility assumption: the local H^2 solution space in Definition 1.1 alone does not imply global $H^2(G)$ regularity. A solution with global $L^2_{\mathbb{F}}(0, T; H^2(G))$ regularity and the stated continuous trajectories belongs to $\mathcal{H}$.

Denote $\hat{f}(t) = \int_0^t f(s) ds$. Clearly, we have

$$|\hat{f}|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} \leq C |f|_{L^2_{\mathbb{F}}(0, T; L^2(G))}. \quad (3.1)$$

Given a function $F \in \mathcal{H}$, the Tikhonov functional is now constructed as

$$J_{\gamma}(u) = |Pu - \hat{f}|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma |u - F|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2, \quad (3.2)$$

where $u \in \mathcal{H}$ and $\gamma \in (0, 1)$.

We have the following result.

Theorem 3.1. *For every $\gamma \in (0, 1)$, there exists a unique minimizer $u_\gamma \in \mathcal{H}$ of the functional $J_\gamma(u)$ in (3.2).*

Proof. Let

$$\mathcal{H}_0 \triangleq \{u \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^2(G)) : Pu \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G))), \\ u|_{(0, T) \times \Gamma} = 0, \partial_\nu u|_{(0, T) \times \Gamma} = 0\}.$$

Fix $\gamma \in (0, 1)$. We define the inner products as follows:

$$\langle \varphi, \psi \rangle_{\mathcal{H}_0} = \langle P\varphi, P\psi \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle \varphi, \psi \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))} \quad (3.3)$$

where $\varphi, \psi \in \mathcal{H}_0$. This graph space is complete. Indeed, let u_m be Cauchy in this norm, with $u_m \rightarrow u$ in $L^2_{\mathbb{F}}(0, T; H^2(G))$ and $Pu_m \rightarrow h$ in $L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))$. Write $Lu = \sum_{i,j} (a^{ij}u_i)_j + \langle a_1, \nabla u \rangle + a_2u$ and $Bu(t) = \int_0^t Lu ds + \int_0^t a_3 u dW$. The deterministic coefficient bounds, Cauchy-Schwarz inequality and the martingale maximal inequality imply $Bu_m \rightarrow Bu$ in $L^2(\Omega; C([0, T]; L^2(G)))$. Also $Pu_m \rightarrow h$ in that space by the time Sobolev embedding. Since

$$u_m(0) = T^{-1} \int_0^T (u_m - Bu_m - Pu_m) dt,$$

the initial values converge in $L^2(\Omega; L^2(G))$ to an $\mathcal{F}_0$ -measurable limit u_0 . Thus $u = u_0 + Bu + h$ has an adapted continuous representative. As $h(0) = 0$, its initial value is u_0 and $Pu = h$. Continuity of the spatial trace maps from $H^2(G)$ preserves both zero boundary traces. Hence $u \in \mathcal{H}_0$ and $\mathcal{H}_0$ is a Hilbert space.

For $u \in \mathcal{H}$, let $v = u - F$. Then $v \in \mathcal{H}_0$. By (3.2), we should minimize the following functional

$$\bar{J}_\gamma(v) = |Pv + PF - \hat{f}|^2_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma |v|^2_{L^2_{\mathbb{F}}(0, T; H^2(G))}, \quad v \in \mathcal{H}_0. \quad (3.4)$$

If $v_\gamma \in \mathcal{H}_0$ is a minimizer of the functional (3.4), then $u_\gamma = v_\gamma + F$ is a minimizer of the functional (3.2). On the other hand, if u_γ is a minimizer of the functional (3.2), then $v_\gamma = u_\gamma - F \in \mathcal{H}_0$ is a minimizer of the functional (3.4).

By the variational principle, any minimizer v_γ of the functional (3.4) should satisfy the following condition

$$\begin{aligned} & \langle Pv_\gamma, Ph \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle v_\gamma, h \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))} \\ &= \langle Ph, \hat{f} - PF \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}, \quad \forall h \in \mathcal{H}_0. \end{aligned} \quad (3.5)$$

From (3.3), it can be rewritten as

$$\langle v_\gamma, h \rangle_{\mathcal{H}_0} = \langle Ph, \hat{f} - PF \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}, \quad \forall h \in \mathcal{H}_0. \quad (3.6)$$

From (3.1), it holds that

$$|\langle Ph, \hat{f} - PF \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}| \leq C(|PF|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + |f|_{L^2_{\mathbb{F}}(0, T; L^2(G))})|h|_{\mathcal{H}_0}. \quad (3.7)$$

Hence, the right hand side of (3.6) is a bounded linear functional on $\mathcal{H}_0$. By Riesz representation theorem, there exists an element $w_\gamma \in \mathcal{H}_0$ such that

$$\langle Ph, \hat{f} - PF \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} = \langle w_\gamma, h \rangle_{\mathcal{H}_0}, \quad \forall h \in \mathcal{H}_0.$$

This and (3.6) imply that

$$\langle v_\gamma, h \rangle_{\mathcal{H}_0} = \langle w_\gamma, h \rangle_{\mathcal{H}_0}, \quad \forall h \in \mathcal{H}_0.$$

Hence, the minimizer $v_\gamma = w_\gamma$. □

Remark 3.1. In the proof of Theorem 3.1, we utilized solely the variational principle and Riesz's theorem, without invoking the Carleman estimate. However, we make use of this estimate in Theorem 3.2, where we establish the rate of convergence of minimizers u_γ to the precise solution, provided that certain conditions are met.

Assume additionally that (1.1) has an exact solution

$$y^* \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^2(G))$$

with the exact data

$$\begin{cases} f^* \in L^2_{\mathbb{F}}(0, T; L^2(G)), \\ y^*|_{\Gamma} = g_1^* \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(\Gamma))) \cap L^2_{\mathbb{F}}(0, T; H^1(\Gamma)), \\ \partial_\nu y^*|_{\Gamma} = g_2^* \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma)). \end{cases}$$

By Theorem 2.1, the exact solution y^* is unique. Because of the existence of y^* , there also exists an exact function F^* with the same global regularity, $PF^* \in L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))$, and exact traces g_1^*, g_2^* , such that

$$|F^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))} \leq C|y^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))}. \quad (3.8)$$

Here is an example of such a function F^* . Let the function $\rho \in C^2(\overline{Q})$ be such that $\rho(x) = 1$ in a small neighborhood

$$\rho(t, x) = \begin{cases} 1, & (t, x) \in N_\sigma((0, T) \times \Gamma) = \{(t, x) \in Q : \text{dist}((t, x), (0, T) \times \Gamma) < \sigma\}, \\ 0, & x \in Q \setminus N_{2\sigma}((0, T) \times \Gamma), \end{cases}$$

where $\sigma > 0$ is a sufficiently small number. Then F^* can be constructed as $F^*(t, x) = \rho(t, x)y^*(t, x)$. Let $\delta > 0$ be a sufficiently small number characterizing the error in the data. We assume that

$$\begin{aligned} |f^* - f|_{L^2_{\mathbb{F}}(0, T; L^2(G))} &\leq \delta, & |g_1^* - g_1|_{L^2_{\mathbb{F}}(0, T; H^1(\Gamma))} &\leq \delta, & |g_{1,t}^* - g_{1,t}|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))} &\leq \delta, \\ |g_2^* - g_2|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma))} &\leq \delta, \end{aligned} \quad (3.9)$$

and

$$|PF^* - PF|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + |F^* - F|_{L^2_{\mathbb{F}}(0, T; H^2(G))} \leq \delta. \quad (3.10)$$

Theorem 3.2. (Convergence rate). Assume (3.9) and (3.10), and let the regularization parameter $\gamma = \delta^{2\alpha}$, where $\alpha \in (0, 1]$ is a constant. Let $G', \varepsilon, \delta_0$, and β be the same as in Theorem 2.1. Then there exists a sufficiently small number $\delta_1 \in (0, 1)$ and a constant $C > 0$ such that if $\delta_1 \in (0, \delta_0^{1/\alpha})$, then

$$|y_\gamma - y^*|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} \leq C(1 + |y^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))})\delta^{\alpha\beta}, \quad \forall \delta \in (0, \delta_1), \quad (3.11)$$

where $y_{\gamma(\delta)}$ is the minimizer of the functional (3.2).

Proof. Let $v^* = y^* - F^*$. Then $v^* \in \mathcal{H}_0$ and $Pv^* = \hat{f}^* - PF^*$. Hence,

$$\begin{aligned} &\langle Pv^*, Ph \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle v^*, h \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))} \\ &= \langle Ph, \hat{f}^* - PF^* \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle v^*, h \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))}, \quad \forall h \in \mathcal{H}_0. \end{aligned} \quad (3.12)$$

Subtract identity (3.5) from identity (3.12) and denote $\tilde{v}_\gamma = v^* - v_\gamma$, $\tilde{f} = \hat{f}^* - \hat{f}$ and $\tilde{F} = F^* - F$. Then

$$\begin{aligned} & \langle P\tilde{v}_\gamma, Ph \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle \tilde{v}_\gamma, h \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))} \\ &= \langle Ph, \tilde{f} - P\tilde{F} \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle v^*, h \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))}, \quad \forall h \in \mathcal{H}_0. \end{aligned}$$

Setting here $h \triangleq \tilde{v}_\gamma$, we obtain

$$\begin{aligned} & |P\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma |\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\ &= \langle P\tilde{v}_\gamma, \tilde{f} - P\tilde{F} \rangle_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))} + \gamma \langle v^*, \tilde{v}_\gamma \rangle_{L^2_{\mathbb{F}}(0, T; H^2(G))}. \end{aligned} \quad (3.13)$$

Applying the Cauchy-Schwarz inequality to (3.13), we obtain

$$\begin{aligned} & |P\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma |\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\ &\leq \frac{1}{2} |P\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \frac{1}{2} |\tilde{f} - P\tilde{F}|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \frac{\gamma}{2} |v^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \\ &\quad + \frac{\gamma}{2} |\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2. \end{aligned} \quad (3.14)$$

Hence, by (3.9) and (3.10)

$$|P\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 + \gamma |\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \leq C\delta^2 + \gamma |v^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2. \quad (3.15)$$

Since $\gamma = \delta^{2\alpha}$, where $\alpha \in (0, 1]$, then $\delta^2 \leq \gamma$. Hence, (3.15) implies that

$$|\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2 \leq C(1 + |v^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2), \quad (3.16)$$

$$|P\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\Omega; H^1(0, T; L^2(G)))}^2 \leq C(1 + |v^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))}^2)\delta^{2\alpha}. \quad (3.17)$$

Let $w_\gamma = \tilde{v}_\gamma(1 + |v^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))})^{-1}$. Then (3.16), (3.17) and Theorem 2.1 imply that

$$|w_\gamma|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} \leq C\delta^{\alpha\beta}, \quad \forall \delta \in (0, \delta_1).$$

Therefore,

$$|\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} \leq C(1 + |v^*|_{L^2_{\mathbb{F}}(0, T; H^2(G))})\delta^{\alpha\beta}, \quad \forall \delta \in (0, \delta_1). \quad (3.18)$$

Next, since $\tilde{v}_\gamma = (y^* - y_\gamma) + (F - F^*)$, by (3.9) and (3.10),

$$|F^* - F|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} \leq \delta,$$

then

$$\begin{aligned} |\tilde{v}_\gamma|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} &\geq |y_\gamma - y^*|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} - |F^* - F|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} \\ &\geq |y_\gamma - y^*|_{L^2_{\mathbb{F}}(\varepsilon, T-\varepsilon; H^1(G'))} - \delta. \end{aligned} \quad (3.19)$$

Since numbers $\beta, \delta \in (0, 1)$ and since $\alpha \in (0, 1]$, then $\delta^{\alpha\beta} > \delta$. Thus, using (3.8), (3.18) and (3.19), we obtain (3.11). $\square$

4 Numerical Approximations

In this section, we aim to numerically solve the ill-posed Cauchy problem of the stochastic parabolic differential equation given by (1.1). For the sake of simplicity, we set $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $f = 0$ and $T = 1$ in the system for all numerical tests to follow. Since an explicit expression for the exact solution is unavailable, we resort to numerically solving the initial-boundary value problem:

$$\begin{cases} dy - \sum_{i,j=1}^n (a^{ij} y_i)_j dt = [\langle a_1, \nabla y \rangle + a_2 y] dt + a_3 y dW(t) & \text{in } (0, 1) \times G, \\ y(0, x) = y_0(x) & \text{in } G, \\ y(t, x) = g_1(t, x) & \text{on } (0, 1) \times \partial G, \end{cases} \quad (4.1)$$

by employing the finite difference method with time discretized via the Euler-Maruyama method [10] to obtain the Cauchy data on $(0, 1) \times \Gamma$. The archived boundary-fitting reconstructions below use deterministic kernels. They are not a discretization of (3.2), and the convergence rate in Section 3 does not apply to these plots. The plots are retained as historical numerical illustrations; they have not been recomputed here.

The following three examples describe the archived illustrations, subject to the limitation stated above.

Example 4.1. Let $G = (0, 1)$ and $\Gamma = 1$. Suppose that

- (a) $y_0(x) = x(1 - x)$, $g_1(t, x) = 0$.
- (b) $y_0(x) = \begin{cases} 4x, & x \in [0, 0.25) \\ -\frac{4}{3}x + \frac{4}{3}, & x \in [0.25, 1] \end{cases}$, $g_1(t, x) = 0$.

The simplest time-discrete approximation is the stochastic version of the Euler approximation, also known as the Euler-Maruyama method [10]. We simply describe the process of solving the initial-boundary value problem (4.1) in Example 4.1 by the Euler-Maruyama method in the following. Use the consistent convention $y_{j,k} = y(x_j, t_k)$ with $x_j = (j-1)h$, $t_k = (k-1)\tau$, $j = 1, \dots, m+1$, $k = 1, \dots, n+1$, $h = 1/m$, and $\tau = 1/n$. It was shown that not all heuristic time-discrete approximations of stochastic differential equations converge to the solution process in a useful sense as the time step size τ tends to zero [10]. The explicit Euler-Maruyama update used in the description is

$$\frac{y_{j,k+1} - y_{j,k}}{\tau} = \frac{y_{j-1,k} - 2y_{j,k} + y_{j+1,k}}{h^2} + \frac{W(t_{k+1}) - W(t_k)}{\tau} y_{j,k}, \quad 2 \leq j \leq m, 1 \leq k \leq n.$$

where the initial and boundary value are given by

$$y_{j,1} = y_0(x_j), \quad 1 \leq j \leq m+1, \quad y_{1,k} = g_1(0, t_k), \quad y_{m+1,k} = g_1(1, t_k), \quad 2 \leq k \leq n+1.$$

The archived resolution is reported as $m = 15$ and $n = 450$. Under the consistent grid convention above, $r = \tau/h^2 = 1/2$. For the homogeneous Dirichlet grid, a Laplacian eigenmode has one-step mean-square factor $(1 - 4r \sin^2(k\pi/(2m)))^2 + \tau$; for these parameters its maximum is $\cos^2(\pi/15) + 1/450 < 1$. This checks the stated update at the stated resolution, not the provenance or accuracy of the archived plots. By solving above algebraic system, we obtain distribution of y at meshed grids of the initial-boundary value problem (4.1). In the process of solving Cauchy problem (1.1) numerically, we using $y_{m,k}$ and $y_{m+1,k}$ instead of the Cauchy data at $x = 1$.

In the following, we describe the archived kernel-based boundary-fitting problem. Its residual and penalty differ from those in (3.2).

For the following mild representation, assume that the parabolic coefficients are deterministic and admit a deterministic evolution kernel $\varphi(x, t, \xi, s)$. Its homogeneous equation is

$$y_t - \sum_{i,j=1}^n (a^{ij} y_i)_j - \langle a_1, \nabla y \rangle - a_2 y = 0 \quad \text{in } (0, T) \times \mathbb{R}^d,$$

then the mild solution $y(x, t)$ of

$$\begin{cases} dy - \sum_{i,j=1}^n (a^{ij} y_i)_j dt = [\langle a_1, \nabla y \rangle + a_2 y] dt + a_3 y dW(t) & \text{in } (0, T) \times \mathbb{R}^n, \\ y(x, 0) = y_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (4.2)$$

can be written as

$$y(x, t) = \int_0^t \int_{\mathbb{R}^n} \varphi(x, t, \xi, s) a_3(s, \xi) y(s, \xi) d\xi dW(s) + \int_{\mathbb{R}^n} \varphi(x, t, \xi, 0) y_0(\xi) d\xi. \quad (4.3)$$

The stochastic convolution in (4.3) contains the unknown solution. This representation is not asserted for a general random principal coefficient satisfying (H1). In the numerical setting $a_3 = 1$ and deterministic heat operator, Itô's formula instead gives the useful transformation $y(t, x) = e^{W(t)-t/2} v(t, x)$, where $v_t = \Delta v$ with the correspondingly transformed boundary data.

Assume that

$$\tilde{y}(x, t) = \sum_{l=1}^N \lambda_l \Phi_l(x, t), \quad l = 1, \dots, N. \quad (4.4)$$

Here Φ_l are the basis functions, N is the number of source points, and λ_l are unknown coefficients to be determined.

The archived deterministic-kernel ansatz takes $\Phi_l(x, t) = \varphi(x - \xi_l, t - \tau_l)$, $l = 1, \dots, N$, where φ is the fundamental solution of the deterministic parabolic equation, (ξ_l, τ_l) are source points. The suitable choice of the source points can ensure that $\tilde{y}$ is analytic in the domain $(0, T) \times G$ and make the algorithm effective and accurate. However, the optimal rule for the location of source points is still open. In recent work, we occupy uniformly distributed points on $DT \times [-R, R + 1]$ below $t = 0$, where DT and R are post-prior determined, as the source points. This choice is given in [5] and related works, and performs better in comparison with other schemes.

The ansatz (4.4) satisfies the deterministic heat equation, but generally not (4.1): its paths have zero quadratic variation, whereas a nonzero solution of $dy = \Delta y dt + y dW$ has a nonzero martingale part. The coefficients below are fitted to the recorded boundary data only. No adapted stochastic-solution property is claimed for this fit. From the process of solving the initial boundary value problem we know that, (x_{m+1}, t_k) and (x_m, t_k) , $k = 1, \dots, n$ can be set as the collocation points, and the problem converts to solving unknowns $\lambda = [\lambda_1, \dots, \lambda_N]^T$ from the linear system

$$A\lambda = b, \quad (4.5)$$

where

$$A = \begin{bmatrix} \varphi(x_{m+1} - \xi_l, t - \tau_l) \\ \varphi(x_m - \xi_l, t - \tau_l) \end{bmatrix}_{2n \times N},$$

and

$$b = [y(x_{m+1}, t_k), y(x_m, t_k)]_{2n}^T,$$

Here $A\lambda - b$ is a boundary-data residual. It is not the interior stochastic residual $Pu - \hat{f}$ in (3.2); likewise the coefficient penalty $\|\lambda\|^2$ is not the $L_t^2 H_x^2$ penalty of that functional. Thus the following finite-dimensional regularization has no convergence guarantee from Theorem 3.2. Choosing the regularization parameter γ by using the L-curve method, that is, a regularized solution near the “corner” of the L-curve [12]

$$L = \{(\log(\|\lambda_\gamma\|^2), \log(\|A\lambda_\gamma - b\|^2)), \gamma > 0\},$$

and denoting the regularized solution of linear system (4.5) by λ_γ^* , leads to the approximated solution

$$\tilde{y}_\gamma^*(x, t) = \sum_{l=1}^N \lambda_{\gamma, l}^* \Phi_l(x, t) \quad (4.6)$$

of the archived boundary-fitting problem.

To illustrate the comparison of the exact solution and its approximation $\tilde{y}_\gamma^*$ and in order to avoid “inverse crime”, we choose $M = m + n$ and compute $y(x, t)$ by finite difference method again such that $y(x, t)$ and $\tilde{y}_\gamma^*$ be defined on the same grid.

Figure 4.1 shows the numerical solution of the initial-boundary problem and the approximation solutions of the Cauchy problem of Example 4.1(a) with different noisy levels. Since the data is given at $x = 1$, the numerical solution seems worse as x tends to 0, this is a descriptive feature of the archived fit and is not a verification of the convergence estimate in Section 3. Furthermore, the convergence rate always holds in the temporal interval $[\varepsilon, T - \varepsilon]$, $\varepsilon > 0$ in section 3. However, from this figure, we find that the proposed method also works well at $t = 0$.

For nonzero denominators, the relative spatial error at a fixed time is

$$E_t(t) = \frac{\|\tilde{y}_\gamma^*(\cdot, t) - y(\cdot, t)\|_{L^2(G)}}{\|y(\cdot, t)\|_{L^2(G)}}. \quad (4.7)$$

The labels $E(x)$ and $E(t)$ in the archived figures are retained as historical labels. The available archive does not specify their actual summation and normalization, so they cannot be identified with (4.7) or used to reproduce the reported error curves.

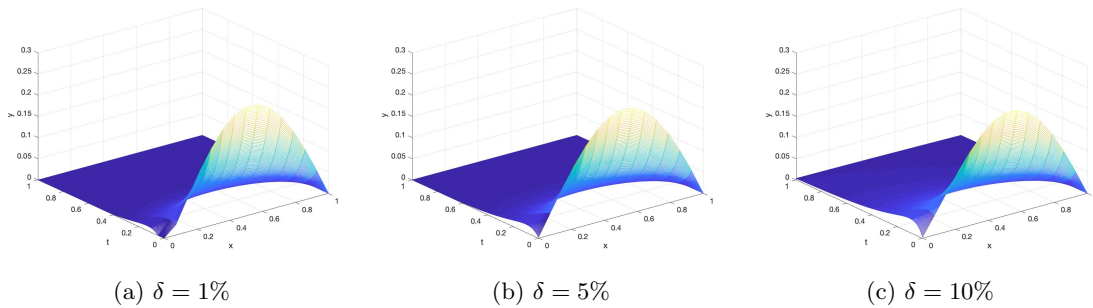


Figure 4.1: Approximated solution with different noisy levels of Example 4.1(a).

As in the study of stochastic equations, large numbers of sample paths should be checked in the numerical experiments for simulating the expectation of the solution. Thus, we do the tests with different sample paths. It is interesting from Figure 4.2 that when the number of sample paths is great than 10, the results seems no better off. Thus in the following, we only consider do the experiments with number of sample paths $\# = 10$.

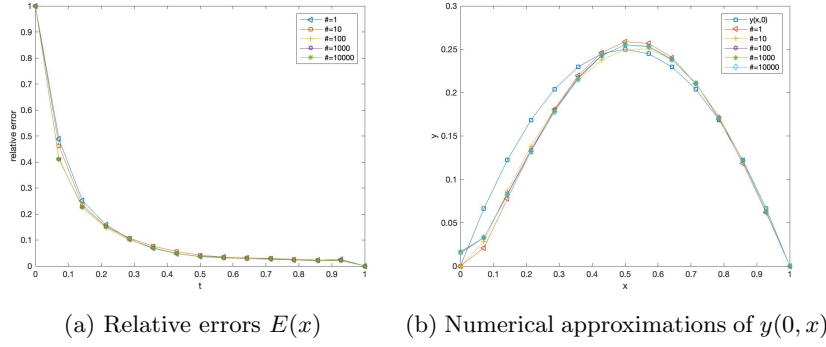


Figure 4.2: Numerical results by expectation of solutions of different numbers of sample paths.

Now we consider the problem in Example 4.1(b). In this case, y_0 is a piecewise smooth function. Figure 4.3 shows the approximations of boundary value $y(t, 0)$ in (a) and the approximations of initial value $y(0, x)$ with different noise levels in (b). Moreover, to illustrate the effectiveness of the proposed method, the change of relative error along lines x and t are also given in Figure 4.3(c) and (d), respectively.

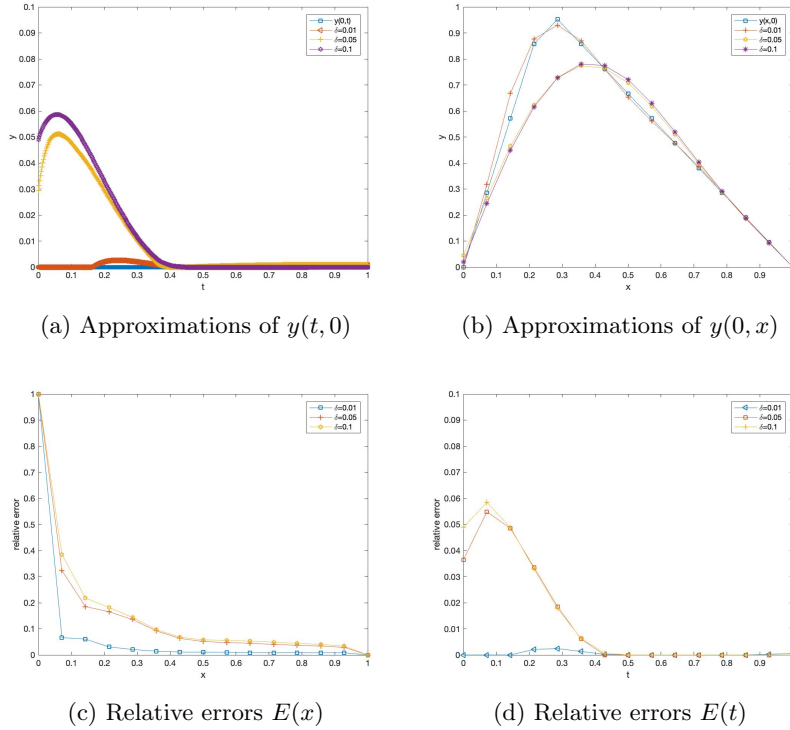


Figure 4.3: Numerical results of Example 4.1(b) with different noisy level data δ .

We mention that the Cauchy problem of the deterministic parabolic equation has been considered in [20], that the numerical algorithm based on Carleman weight function proposed can stably invert the solution near the known boundary conditions, more precisely, solutions for $t \in [0.5, 1]$ for 1-D example. With comparison of the results of Example 4.1, our method works well for recovering

the solution for $t \in [0.1, 1]$ for the stochastic parabolic equation. We believe this algorithm also works well for deterministic parabolic equation, and can be an improvement of previous works. Furthermore, the proposed method also works well for 2-D examples in both rectangular and discal domains.

We explore the effectiveness of the proposed numerical methods in following two examples in bounded domains in $\mathbb{R}^2$, as well as the influence of parameter in the kernel-based learning method. We always let $\delta = 3\%$ unless particularly stated.

Example 4.2. Suppose that $G = [-1, 1] \times [-1, 1]$, and $\Gamma_1 = \{x_1 = 1, x_2 \in [-1, 1]\}$, $\Gamma_3 = \partial G \setminus \{x_1 = -1, x_2 \in [-1, 1]\}$. The symbol Γ_2 below is retained only as a historical experiment label: its boundary subset was not completely recorded in the archive. Accordingly, the panels and parameter comparisons bearing this label are not reproducible boundary-observation tests. Let

$$(a) \quad y_0(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2) + 2, \quad g_1(t, x_1, x_2) = 2.$$

$$(b) \quad y_0(x_1, x_2) = \begin{cases} 3, & (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.15^2 \\ 3, & (x_1 - 0.5)^2 + (x_2 + 0.5)^2 \leq 0.15^2 \\ 3, & (x_1 + 0.5)^2 + (x_2 - 0.5)^2 \leq 0.15^2 \\ 3, & (x_1 + 0.5)^2 + (x_2 + 0.5)^2 \leq 0.15^2 \\ 1, & \text{otherwise.} \end{cases}, \quad g_1(t, x_1, x_2) = 1.$$

We first consider the optimal choices of parameters R and DT in case that the Cauchy data are given on Γ_2 in the kernel-based approximation processes. We set $DT = -0.1$ and observe relative errors E with the change of R . It can be seen from figure 4.4 that $R \in [2.5, 3.5]$, the methods performs well. Numerical results in Figure 4.5 illustrate that any $DT \in [-0.2, -0.1]$ is a reasonable choice. Thus we fix $R = 3.5, DT = -0.1$ in the following computing.

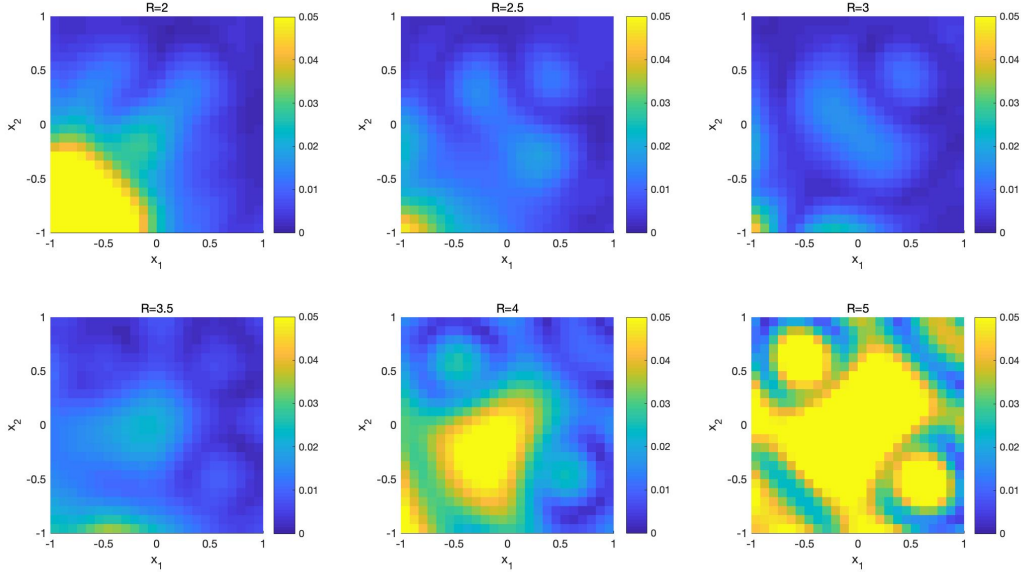


Figure 4.4: The relative error $E(x_1, x_2)$ for different R .

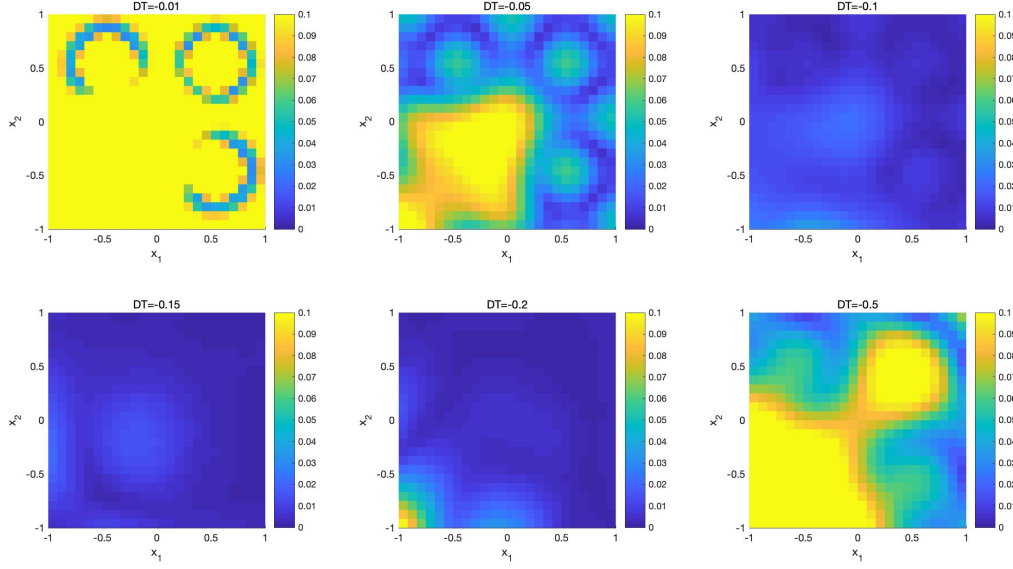


Figure 4.5: The relative error $E(x_1, x_2)$ for different DT .

Now we show the numerical approximations and relative errors for different noise level in Figure 4.6.

Figure 4.7 records archived fits using data on the indicated boundary subsets $\Gamma_j, j = 1, 2, 3$. These empirical comparisons do not verify the stochastic conditional stability or regularization theorem.

In Example 4.2, G is a rectangular domain that the initial-boundary problem could be computed well by the finite difference method. However, in case that G is a general bounded domain, we should improve the finite difference method by heterogeneous grids. To avoid the complicated analysis for the numerical solution of initial-boundary problem, we solve the problem by kernel-based learning theory, by treating the fundamental solution as kernels.

Example 4.3. Suppose that $G = \{r^2 \leq 1\}$, where $r \triangleq \sqrt{x_1^2 + x_2^2}$, $\tan \theta = x_2/x_1$, and $\Gamma_\Theta = \{\theta \in [0, \Theta], r = 1\}$. Let $y_0(x_1, x_2) = \begin{cases} 3, & 0.2 \leq x_1 \leq 0.6, 0.2 \leq x_2 \leq 0.6 \\ 1, & \text{otherwise,} \end{cases}$, $g_1(t, x_1, x_2) = 1$.

We show the numerical results when $\Theta = \frac{\pi}{3}$ with varies δ and results with different partial of boundary for $\Theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$ for $\delta = 3\%$ in Figure 4.8 and 4.9, respectively.

The figures above are retained as archived boundary-fitting illustrations. The deterministic-kernel trial space, boundary residual and coefficient penalty differ from the stochastic equation and functional analyzed in this paper. A stochastic-consistent adapted discretization, a precise specification of the reconstructed random object, and a separate convergence analysis remain necessary before these experiments can serve as a numerical validation of the theory. No new experiment or error rate is claimed here.

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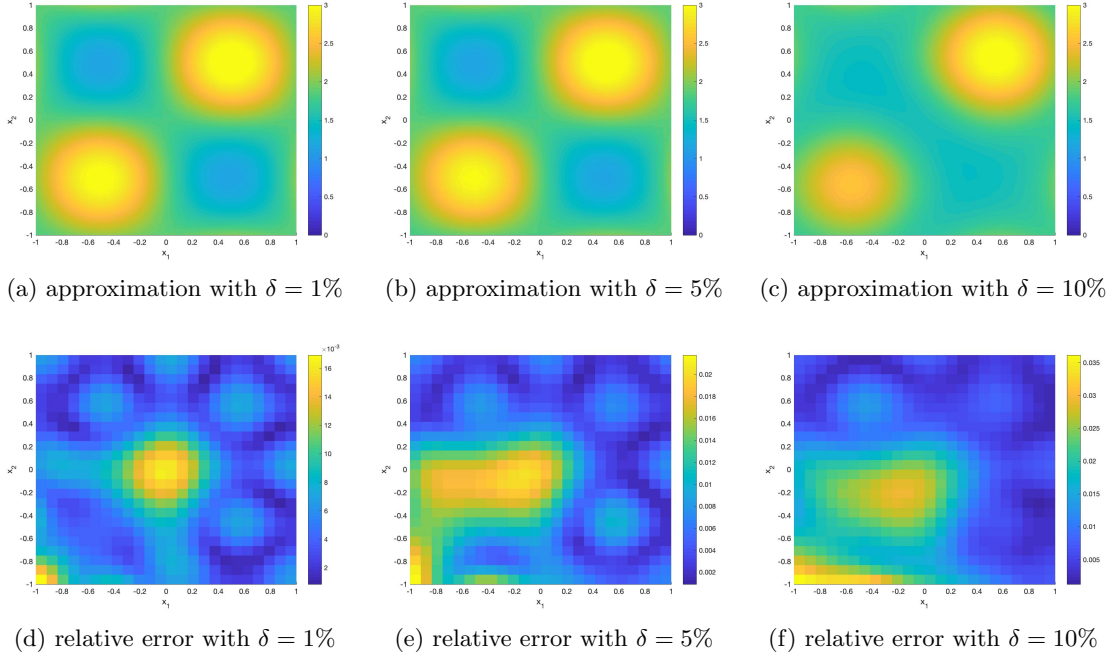


Figure 4.6: The approximations for $y(0, x_1, x_2)$ and relative errors $E(x_1, x_2)$ for different noise levels for Example 4.2(a).

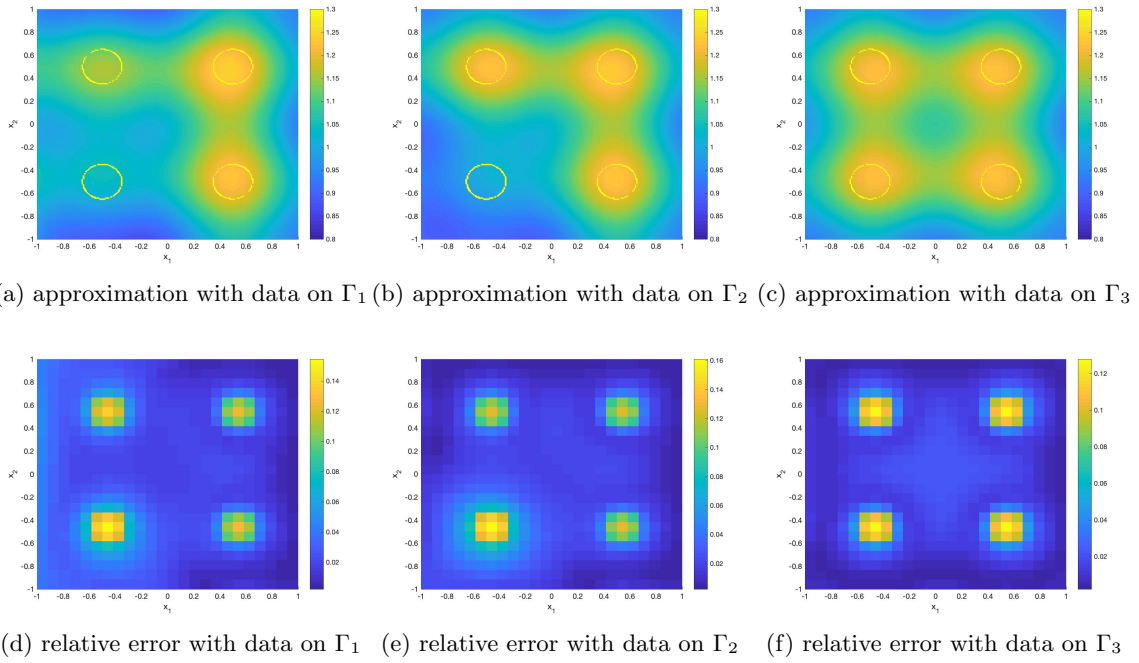


Figure 4.7: Approximations and relative errors for Example 4.2(b) with Cauchy data be given on different partial of boundary.

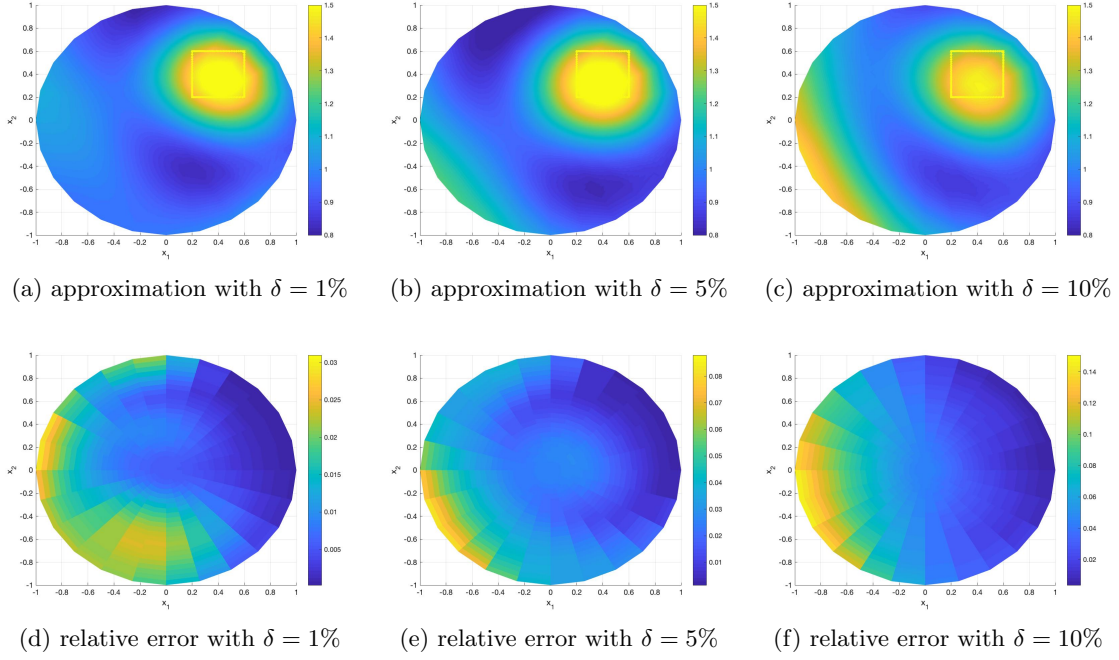


Figure 4.8: Numerical results with different noisy level for Example 4.3.

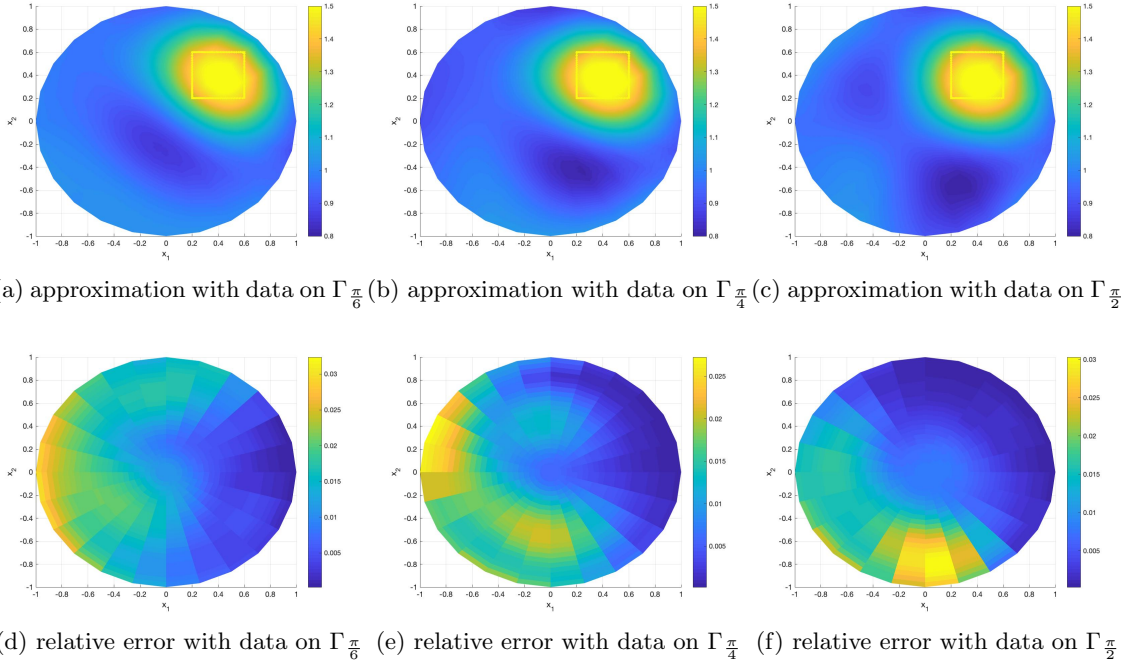


Figure 4.9: Numerical results with Cauchy data on different partial boundary for Example 4.3.

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